

Space-time residual-based a posteriori estimators for the $\mathbf{A} - \varphi$ magnetodynamic formulation of the Maxwell system

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Abstract

In this paper, an a posteriori residual error estimator is proposed for the A/φ magnetodynamic Maxwell system given in its potential and space/time formulation and solved by a Finite Element method. The reliability as well as the efficiency of the estimator are established for several norms. Then, numerical tests are performed, allowing to illustrate the obtained theoretical results.

Key Words : Maxwell equations, potential formulation, space-time a posteriori estimators, finite element method.

AMS (MOS) subject classification 35Q61; 65N30; 65N15; 65N50.

1 Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^3$ be an open connected bounded polyhedral domain, with a lipschitz boundary Γ that is also connected. In this paper, we consider the Maxwell system given in $\Omega \times [0, T]$ by :

$$\operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}, \quad (1)$$

$$\operatorname{curl} \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \quad (2)$$

with initial and boundary conditions to be specified. Here, \mathbf{E} stands for the electrical field, \mathbf{H} for the magnetic field, \mathbf{B} for the magnetic flux density, \mathbf{J} for the current flux density (or eddy current) and \mathbf{D} for the displacement flux density. In the low frequency regime, the quasistatic approximation can be applied, which consists in neglecting the temporal variation of the displacement flux density with respect to the current density [1], so that the propagation phenomena are not taken into account. Consequently, equation (2) becomes :

$$\operatorname{curl} \mathbf{H} = \mathbf{J}. \quad (3)$$

The current density \mathbf{J} can be decomposed in two terms such that $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e$, where \mathbf{J}_s is a known distribution current density, generally generated by a coil, while \mathbf{J}_e represents the unknown eddy current. Both equations (1) and (3) are linked by the material constitutive laws :

$$\mathbf{B} = \mu \mathbf{H}, \quad (4)$$

$$\mathbf{J}_e = \sigma \mathbf{E}, \quad (5)$$

where μ stands for the magnetic permeability and σ for the electrical conductivity of the material. Figure 1 displays two possible domain configurations we are interested in. The domain configuration is composed of an open connected conductor domain $\Omega_c \subset \Omega$ which boundary $B = \partial\Omega_c$ is supposed to be lipschitz and also connected and such that $B \cap \Gamma = \emptyset$. In Ω_c , the electrical conductivity σ is not equal to zero so that eddy currents can be created. The domain $\Omega_e = \Omega \setminus \overline{\Omega_c}$ is defined as the part of Ω where the electrical conductivity σ is identically equal to zero. Boundary

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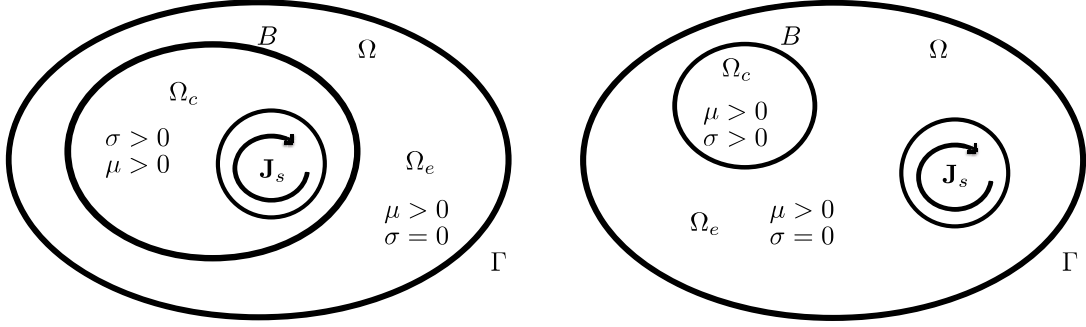


Figure 1: Domains configuration: with respectively $\text{supp } \mathbf{J}_s \subset \Omega_c$ and $\text{supp } \mathbf{J}_s \cap \Omega_c = \emptyset$.

conditions associated with the previous system are given by $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{J}_e \cdot \mathbf{n} = 0$ on B , where \mathbf{n} denotes the unit outward normal to Ω and Ω_c respectively. In the conductor domain Ω_c , the electromagnetic equations can be solved by only considering the electrical field, leading to the classical \mathbf{E} formulation :

$$\text{curl } \mu^{-1} \text{curl } \mathbf{E} + \sigma \partial_t \mathbf{E} = 0.$$

The same approach can be carried out with the magnetic field \mathbf{H} . In that case, we obtain the so-called \mathbf{H} formulation :

$$\text{curl } \sigma^{-1} \text{curl } \mathbf{H} + \mu \partial_t \mathbf{H} = 0.$$

Unfortunately, these two formulations can only be considered in the conductor domain Ω_c since the electrical conductivity σ and the eddy current only exist in Ω_c . Consequently, in order to solve a problem with the quasistatic approximation, a formulation which is able to take into account the eddy current in Ω_c and which verifies in Ω_e Maxwell's equations must be developed. That can be obtained by using the potential formulations often used for electromagnetic problems [20]. From the fact that $\text{div } \mathbf{B} = 0$ in Ω and that its boundary is connected, by Theorem 3.12 of [2], a magnetic vector potential \mathbf{A} can be introduced such that:

$$\mathbf{B} = \text{curl } \mathbf{A} \text{ in } \Omega, \quad (6)$$

with the boundary condition $\mathbf{A} \times \mathbf{n} = 0$ on Γ allowing to guarantee $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ . Like \mathbf{B} , the vector potential \mathbf{A} exists in the whole domain Ω . To ensure the uniqueness of the solution, it is then necessary to impose a gauge condition. The most popular one is $\text{div } \mathbf{A} = 0$ (so-called the Coulomb gauge). Moreover, from equations (1) and (6), an electrical scalar potential φ can be introduced in Ω_c so that the electrical field takes the form :

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi \text{ in } \Omega_c. \quad (7)$$

Like the vector potential, it must be gauged so the averaged value of the potential φ on Ω_c is taken equal to zero to obtain uniqueness of the solution. From (4),(5),(6) and (7), equation (3) leads to the so-called $\mathbf{A} - \varphi$ formulation :

$$\text{curl } (\mu^{-1} \text{curl } \mathbf{A}) + \sigma (\partial_t \mathbf{A} + \nabla \varphi) = \mathbf{J}_s. \quad (8)$$

The great interest of this formulation relies in its effectivity in both domain Ω_c and Ω_e . Indeed, in Ω_e , where σ is zero, the second term vanishes and the $\mathbf{A} - \varphi$ formulation becomes the classical \mathbf{A} formulation used in the magnetostatic case.

We are here interested in the numerical resolution of (8) by the Finite Element Method in the context of electromagnetic problems [7, 8, 21]. More particularly, we have in mind to derive an *a posteriori* residual error estimator, in order to determine the numerical parameters (namely, the space mesh refinement and the time step) to be used in a space-time adaptivity context.

Concerning the harmonic formulation of some Maxwell problems, several contributions have been proposed for the last decade. In that case, since there are no more time derivatives to be considered, one only has to deal with the spatial variable.

Some explicit residual error estimators have been successively derived. In [3], the eddy current formulation was considered in a smooth context, generalized to piecewise constant coefficients in [24] or to lipschitz domains in [28].

The robustness of the estimates was addressed in [14], and the dependance of the constants arising in the upper and lower bounds with respect to the polynomial degree of the ansatz space was investigated in [10]. An adaptive algorithm was proposed in [11], which was proven to converge in the sense of the reduction of the energy norm of the error. In the low-frequency framework, an adaptive algorithm was also proposed in [12] which was proven to be efficient for singular solutions. Some works devoted to the potential formulations were also performed in [15, 33]. An estimator for a coupling of the Boundary and Finite element methods was introduced in [18], as well as in [27] in the context of a Discontinuous Galerkin Method. Very recently, an adaptive $h - p$ finite element algorithm was proposed for time-harmonic Maxwell's equations [17].

Other kinds of estimators have also been developed for Maxwell problems such as implicit [16] or reconstructed [19] estimators for harmonic problems, equilibrated [9] or reconstructed [22] estimators for the \mathbf{E} -formulation, as well as hierarchical one for the magnetoquasistatic approximation [13].

This paper is devoted to a space-time explicit residual *a posteriori* estimator derivation for the A/φ formulation given by (8). Here the goal is to start from the work developed for other parabolic-type equations [4, 5, 6, 25, 26, 31], and to adapt it to the case of a magnetodynamic problem. We follow the same philosophy as the one in [4, 25], which consists in splitting the error in a "time" one and in a "spatial" one, allowing to obtain some corresponding "time" and "spatial" error indicators. Our contribution can also be compared to [32], devoted to the H/Ψ formulation. In our work, both potentials (vector and scalar) are kept during all the analysis, and the support of \mathbf{J}_s can intersect Ω_c . Moreover, the upper bound (reliability) as well as the lower bound (efficiency) are obtained for the same space/time error.

Let us finish this introduction by some notation used in the whole paper. On a given domain D , the $L^2(D)$ norm is denoted by $\|\cdot\|_D$, and the corresponding $L^2(D)$ inner product by $(\cdot, \cdot)_D$. The usual norm and semi-norm on $H^1(D)$ are respectively denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$. In the case $D = \Omega$, the index Ω is dropped. Recall that $H_0^1(D)$ is the subspace of $H^1(D)$ with vanishing trace on ∂D . The notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 , which are independent of the quantities a and b under consideration, as well as of the coefficients μ , σ and the discrete parameters h and τ (see below) such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively. When needed, C denotes a generic constant which is not necessarily the same throughout the paper.

The paper is organized as follows. In section 2, different formulations of the problem are presented : the continuous one, the semi-discrete one and the fully-discrete one. Then, section 3 is devoted to the different errors and estimators definition. The reliability of the proposed *a posteriori* estimator is proved in section 4, and its efficiency in section 5. Finally, section 6 presents some numerical tests to underline the theoretical predictions.

2 Formulations of the problem

In this section, the three formulations we are going to deal with are introduced : the continuous formulation, the semi-discrete formulation in time, and the fully discrete formulation in time and space. For each of them, the question of their well-posedness is addressed, and some properties on the corresponding solutions are underlined.

2.1 Continuous formulation

Assuming that $\operatorname{div} \mathbf{J}_s = 0$, the $\mathbf{A} - \varphi$ formulation of the magnetodynamic problem can be written

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}) + \sigma(\partial_t \mathbf{A} + \nabla \varphi) = \mathbf{J}_s \text{ in } \Omega, \quad (9)$$

$$\operatorname{div}(\sigma(\partial_t \mathbf{A} + \nabla \varphi)) = 0 \text{ in } \Omega_c, \quad (10)$$

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma, \quad (11)$$

$$\sigma(\partial_t \mathbf{A} + \nabla \varphi) \cdot \mathbf{n} = 0 \text{ on } B, \quad (12)$$

$$A(t=0, \cdot) = 0 \text{ in } \Omega_c. \quad (13)$$

We suppose that $\mu \in L^\infty(\Omega)$ and that there exists $\mu_0 \in \mathbb{R}_+^*$ such that $\mu > \mu_0$ in Ω . We also assume that $\sigma \in L^\infty(\Omega)$, $\sigma|_{\Omega_c} \equiv 0$, and that there exists $\sigma_0 \in \mathbb{R}_+^*$ such that $\sigma > \sigma_0$ in Ω_c . At last, we recall the Gauge conditions. Like mentioned in section 1, we choose the Coulomb one $\operatorname{div} \mathbf{A} = 0$ in Ω , and we ask for the averaged value of φ in Ω_c to be equal to zero.

We now define :

$$\begin{aligned}
X(\Omega) = H_0(\text{curl}, \Omega) &= \left\{ \mathbf{A} \in \mathbf{L}^2(\Omega); \text{curl } \mathbf{A} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{A} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\}, \\
X^0(\Omega) &= \left\{ \mathbf{A} \in X(\Omega); (\mathbf{A}, \nabla \xi) = 0 \forall \xi \in H_0^1(\Omega) \right\}, \\
H(\text{div} = 0, \Omega) &= \left\{ \mathbf{A} \in \mathbf{L}^2(\Omega); (\mathbf{A}, \nabla \xi) = 0 \forall \xi \in H_0^1(\Omega) \right\}, \\
\widetilde{H}^1(\Omega_c) &= \left\{ \varphi \in H^1(\Omega_c); \int_{\Omega_c} \varphi \, d\mathbf{x} = 0 \right\},
\end{aligned}$$

where $X(\Omega)$ is equipped with its usual norm :

$$\|\mathbf{A}\|_{X(\Omega)}^2 = \|\mathbf{A}\|^2 + \|\text{curl } \mathbf{A}\|^2.$$

We finally denote $V = X^0(\Omega) \times \widetilde{H}^1(\Omega_c)$, and $X^0(\Omega)'$ stands for the dual space associated with $X^0(\Omega)$.

The variational formulation associated with (9)-(13) is consequently given by : Find $\mathbf{A} \in L^2(0, T; X^0(\Omega))$ and $\varphi \in L^2(0, T; \widetilde{H}^1(\Omega_c))$ such that $\partial_t \mathbf{A} \in L^2(0, T; X^0(\Omega)')$, $\mathbf{A}(0, \cdot) \equiv 0$ in Ω_c and such that for all $\mathbf{A}' \in X^0(\Omega)$ and $\varphi' \in \widetilde{H}^1(\Omega_c)$, we have :

$$(\mu^{-1} \text{curl } \mathbf{A}, \text{curl } \mathbf{A}') + (\sigma (\partial_t \mathbf{A} + \nabla \varphi), \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s, \mathbf{A}'). \quad (14)$$

Using the theory of Showalter on degenerated parabolic problem [29, Theorem V4.B], and appropriated energy estimates, the following existence result for problem (14) is proved in Theorem 2.1 of [23].

Theorem 2.1. Let us assume that $\mathbf{J}_s \in H^1(0, T; H(\text{div} = 0, \Omega))$ and set $\mathbf{J}_{s,0} = \mathbf{J}_s(t=0)$. Assume that

$$\mathbf{J}_{s,0} \cdot \mathbf{n} = 0 \text{ on } B,$$

and that there exists $\mathbf{A}_0 \in X^0(\Omega)$ satisfying

$$\mathbf{A}_0 = 0 \text{ in } \Omega_c,$$

and

$$(\mu^{-1} \text{curl } \mathbf{A}_0, \text{curl } \mathbf{A}') = (\mathbf{J}_{s,0}, \mathbf{A}')_{\Omega_c}, \forall \mathbf{A}' \in X(\Omega) \text{ such that } \text{div } \mathbf{A}' \in L^2(\Omega).$$

Then problem (14) has a unique solution (\mathbf{A}, φ) in $H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c))$ with $\mathbf{A}(t=0) = \mathbf{A}_0$.

Due to the divergence free property of \mathbf{J}_s , we further notice that the test functions in (14) can be ungauged.

Lemma 2.2. Under the assumptions of Theorem 2.1, the unique solution $(\mathbf{A}, \varphi) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c))$ of (14) also satisfies

$$(\mu^{-1} \text{curl } \mathbf{A}, \text{curl } \mathbf{A}') + (\sigma (\partial_t \mathbf{A} + \nabla \varphi), \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s, \mathbf{A}'), \forall (\mathbf{A}', \varphi') \in X(\Omega) \times H^1(\Omega_c). \quad (15)$$

Proof: In (14), we first take $\mathbf{A}' \equiv 0$ to deduce that

$$(\sigma (\partial_t \mathbf{A} + \nabla \varphi), \nabla \varphi')_{\Omega_c} = 0, \forall \varphi' \in H^1(\Omega_c).$$

In a second step taking any $\psi \in H_0^1(\Omega)$, as \mathbf{J}_s is divergence free we get

$$(\sigma (\partial_t \mathbf{A} + \nabla \varphi), \nabla \psi)_{\Omega_c} = (\mathbf{J}_s, \nabla \psi).$$

We conclude by using the Helmholtz decomposition of $\mathbf{A}' \in X(\Omega)$ into $\mathbf{A}' = \mathbf{B}' + \nabla \psi$ with $\psi \in H_0^1(\Omega)$ and $\mathbf{B}' \in X^0(\Omega)$. ■

2.2 Semi-discrete formulation in time

In order to discretize in time equations (9)-(13), a partition of the interval $[0, T]$ into subintervals $[t_{m-1}, t_m[$ is introduced, $1 \leq m \leq N$, such that $0 = t_0 < t_1 < \dots < t_N = T$. We denote by τ_m the length $t_m - t_{m-1}$, we set $\tau = \max_{1 \leq m \leq N} \tau_m$, and we define the time regularity parameter σ_τ by :

$$\sigma_\tau = \max_{2 \leq m \leq N} \frac{\tau_m}{\tau_{m-1}}. \quad (16)$$

Assuming that \mathbf{J}_s is continuous in time, we denote \mathbf{J}_s^m the value of $\mathbf{J}_s(t_m)$, and $\pi_\tau \mathbf{J}_s$ the piecewise constant interpolation in time of \mathbf{J}_s given by $\pi_\tau \mathbf{J}_s(t) = \mathbf{J}_s^m$ for $t \in]t_{m-1}, t_m]$, $1 \leq m \leq N$. We denote by \mathbf{A}^m and φ^m the approximations of $\mathbf{A}(t_m)$ and $\varphi(t_m)$ respectively. The semi-discrete problem in time issued from Euler's implicit scheme is then given by :

$$\begin{aligned} \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}^m) + \sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) &= \mathbf{J}_s^m \text{ in } \Omega, \\ \operatorname{div} \left(\sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \right) &= 0 \text{ in } \Omega_c, \\ \mathbf{A}^m \times \mathbf{n} &= 0 \text{ on } \Gamma, \\ \sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \cdot \mathbf{n} &= 0 \text{ on } B, \\ A^0 &= 0 \text{ in } \Omega_c. \end{aligned} \quad (17)$$

The corresponding weak formulation consists in looking for $(\mathbf{A}^m, \varphi^m) \in V$, $0 \leq m \leq N$ such that $\mathbf{A}^0 = \mathbf{A}(0, \cdot) \equiv 0$ in Ω_c and such that for all m , $1 \leq m \leq N$, we have :

$$a_m((\mathbf{A}^m, \varphi^m), (\mathbf{A}', \varphi')) = l_m((\mathbf{A}', \varphi')) \quad \forall (\mathbf{A}', \varphi') \in V, \quad (18)$$

with a_m and l_m the bilinear and linear forms respectively defined by :

$$\begin{aligned} a_m((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) &= (\mu^{-1} \operatorname{curl} \mathbf{A}, \operatorname{curl} \mathbf{A}') + \left(\frac{\sigma}{\tau_m} (\mathbf{A} + \tau_m \nabla \varphi), \mathbf{A}' + \tau_m \nabla \varphi' \right)_{\Omega_c}, \\ l_m((\mathbf{A}', \varphi')) &= (\mathbf{J}_s^m, \mathbf{A}') + \left(\frac{\sigma}{\tau_m} \mathbf{A}^{m-1}, \mathbf{A}' + \tau_m \nabla \varphi' \right)_{\Omega_c}. \end{aligned}$$

Now we address the question of the well-posedness of problem (18).

Theorem 2.3. Problem (18) has a unique solution $(\mathbf{A}^m, \varphi^m) \in V$, $1 \leq m \leq N$.

Proof: The proof is in a point similar to the one of [15], Lemma 2.1. With a recurrence argument, it is mainly based on the fact that the bilinear form a_m is coercive on V , allowing the use of the Lax-Milgram lemma. \blacksquare

As before, due to the divergence free property of \mathbf{J}_s , we can show that the test functions in (18) can be ungauged.

Lemma 2.4. Let $(\mathbf{A}^m, \varphi^m) \in V$, $1 \leq m \leq N$ be the solution of problem (18) with $\mathbf{A}^0 = \mathbf{A}(0, \cdot) \equiv 0$ in Ω_c . Then we have for all m , $1 \leq m \leq N$:

$$a_m((\mathbf{A}^m, \varphi^m), (\mathbf{A}', \varphi')) = l_m((\mathbf{A}', \varphi')), \quad \forall (\mathbf{A}', \varphi') \in X(\Omega) \times \widetilde{H}^1(\Omega_c).$$

Proof: This result is proved in the same manner than in Lemma 2.2 (see also [15, Lemma 2.2]). \blacksquare

2.3 Fully discrete formulation

At each computational time t_m , $0 \leq m \leq N$, is associated a conforming mesh \mathcal{T}_{hm} made of tetrahedra, each element T of \mathcal{T}_{hm} belonging either to Ω_c or to Ω_e . We denote h_T the diameter of the element T and ρ_T the diameter of its largest inscribed ball. We suppose that for any element T , the ratio h_T/ρ_T is bounded by a constant $\alpha > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}_{hm}} h_T$. Moreover, \mathcal{T}_{hm} , \mathcal{N}_{hm} , \mathcal{N}_{hm}^{int} , \mathcal{E}_{hm} , \mathcal{E}_{hm}^{int} , \mathcal{F}_{hm} and \mathcal{F}_{hm}^{int} respectively denote the set of tetrahedra, nodes, internal nodes, edges, internal edges, faces and internal faces of \mathcal{T}_{hm} . We denote h_F the diameter of the face F . Finally, the conductivity σ and the permeability μ are supposed to be constant on each tetrahedron :

$$\begin{aligned} \sigma_T &= \sigma|_T \quad \forall T \in \mathcal{T}_{hm}, \quad \sigma_{\min} = \min_{T \in \mathcal{T}_{hm}, T \in \Omega_c} \sigma_T, \quad \sigma_{\max} = \max_{T \in \mathcal{T}_{hm}, T \in \Omega_c} \sigma_T, \\ \mu_T &= \mu|_T \quad \forall T \in \mathcal{T}_{hm}, \quad \mu_{\min} = \min_{T \in \mathcal{T}_{hm}} \mu_T, \quad \mu_{\max} = \max_{T \in \mathcal{T}_{hm}} \mu_T. \end{aligned}$$

The approximation space V_h is defined by $V_h = X_h^0(\Omega) \times \tilde{\Theta}_h(\Omega_c)$, where :

$$\begin{aligned} X_h(\Omega) &= \left\{ \mathbf{A}_h \in X(\Omega); \mathbf{A}_h|_T \in \mathcal{ND}_1(T), \forall T \in \mathcal{T}_{hm} \right\}, \\ \mathcal{ND}_1(T) &= \left\{ \mathbf{A}_h : T \rightarrow \mathbb{R}^3 : \mathbf{x} \rightarrow \mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \right\}, \\ \Theta_h^0(\Omega) &= \left\{ \xi_h \in H_0^1(\Omega); \xi_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_{hm} \right\}, \\ X_h^0(\Omega) &= \left\{ \mathbf{A}_h \in X_h(\Omega); (\mathbf{A}_h, \nabla \xi_h) = 0 \quad \forall \xi_h \in \Theta_h^0(\Omega) \right\}, \\ \tilde{\Theta}_h(\Omega_c) &= \left\{ \varphi_h \in \widetilde{H}^1(\Omega_c); \varphi_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_{hm} \right\}. \end{aligned}$$

The fully-discrete formulation consists in looking for $(\mathbf{A}_h^m, \varphi_h^m) \in V_h$, $0 \leq m \leq N$ such that $\mathbf{A}_h^0 = \mathbf{A}(0, \cdot) \equiv 0$ in Ω_c and such that for all m , $1 \leq m \leq N$, we have :

$$a_m((\mathbf{A}_h^m, \varphi_h^m), (\mathbf{A}_h', \varphi_h')) = l_m((\mathbf{A}_h', \varphi_h')) \quad \forall (\mathbf{A}_h', \varphi_h') \in V_h. \quad (19)$$

Theorem 2.5. Problem (19) has a unique solution $(\mathbf{A}_h^m, \varphi_h^m) \in V_h$, $1 \leq m \leq N$.

Proof: The proof is similar to the one of Theorem 2.3 in the semi-discrete case, using this time a discrete Friedrichs inequality instead of a continuous one (see [21] lemma 7.20 page 185). ■

We have moreover a similar result than the one given in Lemma 2.4.

Lemma 2.6. Let $(\mathbf{A}_h^m, \varphi_h^m) \in V_h$, $1 \leq m \leq N$ be the solution of problem (19) with $\mathbf{A}_h^0 = \mathbf{A}(0, \cdot) \equiv 0$ in Ω_c . Then we have for all m , $1 \leq m \leq N$:

$$a_m((\mathbf{A}_h^m, \varphi_h^m), (\mathbf{A}_h', \varphi_h')) = l_m((\mathbf{A}_h', \varphi_h')), \quad \forall (\mathbf{A}_h', \varphi_h') \in X_h(\Omega) \times \widetilde{\Theta}_h(\Omega_c).$$

Proof: The proof is similar to the one of Lemma 2.2, by using this time a discrete Helmholtz decomposition of $\mathbf{A}_h' \in X_h$ into $\mathbf{A}_h' = \mathbf{B}_h' + \nabla \psi_h$ with $\mathbf{B}_h' \in X_h^0(\Omega)$ and $\psi_h \in \Theta_h^0(\Omega)$ (see [15] page 9). ■

3 Definitions of the errors and of the estimators

3.1 Errors

We first build an interpolation in time of the solution of (18) by :

$$\begin{cases} \mathbf{A}_\tau(t) = \frac{t - t_{m-1}}{\tau_m} \mathbf{A}^m + \frac{t_m - t}{\tau_m} \mathbf{A}^{m-1} & t_{m-1} < t \leq t_m, \\ \varphi_\tau(t) = \varphi^m & t_{m-1} < t \leq t_m. \end{cases} \quad (20)$$

We proceed similarly for the solution of (19) by :

$$\begin{cases} \mathbf{A}_{h\tau}(t) = \frac{t - t_{m-1}}{\tau_m} \mathbf{A}_h^m + \frac{t_m - t}{\tau_m} \mathbf{A}_h^{m-1} & t_{m-1} < t \leq t_m, \\ \varphi_{h\tau}(t) = \varphi_h^m & t_{m-1} < t \leq t_m. \end{cases}$$

The errors we are interested in are first the time one, given by :

$$\mathbf{e}_{A,\tau}(t) = \mathbf{A}(t) - \mathbf{A}_\tau(t), \quad e_{\varphi,\tau}(t) = \varphi(t) - \varphi_\tau(t),$$

as well as the spatial one, given by :

$$\mathbf{e}_{A,h\tau}(t) = \mathbf{A}_\tau(t) - \mathbf{A}_{h\tau}(t), \quad e_{\varphi,h\tau}(t) = \varphi_\tau(t) - \varphi_{h\tau}(t).$$

3.2 Estimators

For all m , $1 \leq m \leq N$, the temporal a posteriori error estimator η_τ^m is defined by :

$$\eta_\tau^m = \left(\frac{\tau_m}{3} \right)^{1/2} \|\mu^{-1/2} \operatorname{curl}(\mathbf{A}_h^m - \mathbf{A}_h^{m-1})\|. \quad (21)$$

The spatial a posteriori error estimator is then defined by :

$$\eta_h^m = \left((\eta_{\Omega;1}^m)^2 + (\eta_{\Omega;2}^m)^2 + (\eta_{\Omega;3}^m)^2 + (\eta_{J;1}^m)^2 + (\eta_{J;2}^m)^2 + (\eta_{J;3}^m)^2 \right)^{1/2},$$

with

$$\begin{aligned} (\eta_{\Omega;i}^m)^2 &= \sum_{T \in \mathcal{T}_{hm}} (\eta_{T;i}^m)^2, \quad i = 1, 2, 3, \\ (\eta_{J;i}^m)^2 &= \sum_{F \in \mathcal{F}_{hm}^{int}} (\eta_{F;i}^m)^2, \quad i = 1, 2, 3, \end{aligned}$$

where, for all $T \in \mathcal{T}_{hm}$,

$$\eta_{T;1}^m = h_T \left\| \mathbf{J}_{s,h}^m - \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) - \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \right\|_T, \quad (22)$$

$$\eta_{T;2}^m = h_T \left\| \operatorname{div} \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \right) \right\|_T, \quad (23)$$

$$\eta_{T;3}^m = h_T \left\| \operatorname{div} \left(\sigma \left(\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p \right) \right) \right\|_T, \quad (24)$$

$$\eta_{F;1}^m = h_F^{1/2} \left\| [\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m]_F \right\|_F, \quad (25)$$

$$\eta_{F;2}^m = h_F^{1/2} \left\| \left[\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot \mathbf{n} \right]_F \right\|_F, \quad (26)$$

$$\eta_{F;3}^m = h_F^{1/2} \left\| [\sigma \left(\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p \right) \cdot \mathbf{n}]_F \right\|_F. \quad (27)$$

Here, \mathbf{n} stands for the unit normal to the face F and $[u]_F$ denotes the jump of the quantity u through the face F , namely :

$$[u(\mathbf{x})]_F = \begin{cases} \lim_{\epsilon \rightarrow 0+} u(\mathbf{x} + \epsilon \mathbf{n}) - u(\mathbf{x} - \epsilon \mathbf{n}) & \text{if } F \in \mathcal{F}_{hm}^{int}, \\ 0 & \text{if } F \in \mathcal{F}_{hm} \setminus \mathcal{F}_{hm}^{int}. \end{cases}$$

Finally the oscillating term is defined by $(\xi^m)^2 = \sum_{T \in \mathcal{T}_{hm}} (\xi_T^m)^2$, where for all $T \in \mathcal{T}_{hm}$

$$\xi_T^m = h_T \left\| \mathbf{J}_s^m - \mathbf{J}_{s,h}^m \right\|_T, \quad (28)$$

$\mathbf{J}_{s,h}^m$ being the Raviart-Thomas finite element approximation of \mathbf{J}_s^m on the mesh defined by

$$\int_F \mathbf{J}_{s,h}^m \cdot \mathbf{n} \, d\gamma(\mathbf{x}) = \int_F \mathbf{J}_s^m \cdot \mathbf{n} \, d\gamma(\mathbf{x}) \quad \text{for all } F \subset \partial T, \text{ with } T \in \mathcal{T}_{hm}.$$

Recall that the divergence free property of \mathbf{J}_s^m implies the same property for $\mathbf{J}_{s,h}^m$.

Let us remark that in the above expressions, φ_h^m does not make sense in $T \subset \Omega_e$, and consequently we should replace φ_h^m by one fixed extension, but since this extension is multiplied by σ which is zero on such a T , the expression is in any case zero on such a tetrahedron and therefore we prefer to use this slight abuse of notations.

Concerning the spatial estimator, the terms (22), (23), (25) and (26) can easily be set in correspondence with our PDE system, namely the element contributions $\eta_{\Omega;1}^n$ and $\eta_{\Omega;2}^n$ represent the residual associated with the equations (9)-(10), and the jump contributions $\eta_{j;1}^n$ and $\eta_{j;2}^n$ are related to the regularity of the considered obtained functions. Compared to [15], the new contributions are in fact $\eta_{\Omega;3}^n$ and $\eta_{j;3}^n$. They are related to the unstationary nature of the problem and consist in the residual and jump terms corresponding to the time integration of (10) leading to

$$\operatorname{div} \left(\sigma \left(\mathbf{A}(t, \mathbf{x}) + \nabla \int_0^t \varphi \right) \right) = 0, \quad (29)$$

reminding that $\mathbf{A}(0, \cdot) \equiv 0$ in Ω_c .

The global error estimator η^n at time t_n ($1 \leq n \leq N$) is finally given by :

$$\eta^n = \left(\sum_{m=1}^n (\eta_\tau^m)^2 + \tau_m (\eta_h^m)^2 \right)^{1/2}. \quad (30)$$

4 Reliability of the estimator

4.1 Reliability of the time discretization

Lemma 4.1. For any $\mathbf{v}^m \in X(\Omega)$, $0 \leq m \leq N$, let us define its corresponding linear interpolation \mathbf{v}_τ as

$$\mathbf{v}_\tau(t) = \frac{t - t_{m-1}}{\tau_m} \mathbf{v}^m + \frac{t_m - t}{\tau_m} \mathbf{v}^{m-1} \quad \text{for } t_{m-1} < t \leq t_m.$$

Then we have :

$$\begin{aligned} \frac{\tau_1}{6} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^0 \right\|^2 + \sum_{m=1}^n \left(\frac{\tau_m}{6} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\|^2 \right) &\leq \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}_\tau \right\|^2 dt \\ &\leq \frac{\tau_1}{2} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^0 \right\|^2 + \left(\frac{1 + \sigma_\tau}{2} \right) \sum_{m=1}^{n-1} \left(\tau_m \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\|^2 \right) + \frac{\tau_n}{2} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^n \right\|^2, \end{aligned} \quad (31)$$

where σ_τ is the time regularity parameter defined in (16).

Proof: A simple calculation leads to :

$$\begin{aligned} \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}_\tau \right\|^2 dt &= \int_{\Omega} \mu^{-1} \frac{\tau_m}{3} \left(|\operatorname{curl} \mathbf{v}^m|^2 + |\operatorname{curl} \mathbf{v}^{m-1}|^2 + \operatorname{curl} \mathbf{v}^m \cdot \operatorname{curl} \mathbf{v}^{m-1} \right) d\mathbf{x} \\ &\geq \frac{\tau_m}{3} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\|^2 + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^{m-1} \right\|^2 - \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\| \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^{m-1} \right\| \right). \end{aligned} \quad (32)$$

Since

$$a^2 + b^2 - ab \geq \frac{a^2 + b^2}{2}, \quad \forall a, b \in \mathbb{R},$$

we get :

$$\int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}_\tau \right\|^2 dt \geq \frac{\tau_m}{6} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\|^2 + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^{m-1} \right\|^2 \right). \quad (33)$$

Then, summing from $m = 1$ to n , the left inequality of (31) is established.

To prove the right one, we simply apply the estimate $a \cdot b \leq \frac{a^2 + b^2}{2}$ valid for all $a, b \in \mathbb{R}$ to the first row of (32) :

$$\int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}_\tau \right\|^2 dt \leq \frac{\tau_m}{2} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^m \right\|^2 + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{v}^{m-1} \right\|^2 \right),$$

and by summing from $m = 1$ to N we conclude. ■

Now, from (20) we remark that :

$$\partial_t \mathbf{A}_\tau = \frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} \text{ for any } t \text{ in }]t_{m-1}, t_m].$$

Thanks to Lemma 2.4, we can write that for all $\mathbf{A}' \in X(\Omega)$ and $\varphi' \in \widetilde{H}^1(\Omega_c)$, we have :

$$(\mu^{-1} \operatorname{curl} \mathbf{A}^m, \operatorname{curl} \mathbf{A}') + (\sigma(\partial_t \mathbf{A}_\tau + \nabla \varphi^m), \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s^m, \mathbf{A}'). \quad (34)$$

By subtracting (34) to (15) we get the temporal residual equation : for all $\mathbf{A}' \in X(\Omega)$ and $\varphi' \in \widetilde{H}^1(\Omega_c)$, we have :

$$\begin{aligned} (\mu^{-1} \operatorname{curl} \mathbf{e}_{A,\tau}, \operatorname{curl} \mathbf{A}') + (\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}), \mathbf{A}' + \nabla \varphi')_{\Omega_c} \\ = (\mathbf{J}_s - \pi_\tau \mathbf{J}_s, \mathbf{A}') + (\mu^{-1} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau), \operatorname{curl} \mathbf{A}'). \end{aligned} \quad (35)$$

This allows to show that the error in time is controlled by the estimator in time and the error in space, up to high order terms :

Theorem 4.2. Under the assumptions of Theorem 2.1, we have :

$$\begin{aligned} & \left\| \sigma^{1/2} (\mathbf{e}_{A,\tau}(t_n) + \nabla \int_0^{t_n} e_{\varphi,\tau}(s) ds) \right\|_{\Omega_c}^2 + \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(s) \right\|^2 ds \\ & \leq 2 \sum_{m=1}^n (\eta_\tau^m)^2 + 4 \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(s) \right\|^2 ds + C \mu_{\max} \|\mathbf{J}_s - \pi_\tau \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2, \end{aligned} \quad (36)$$

where C is independent of the time steps τ_m , $1 \leq m \leq N$.

Proof: The proof is similar to the one proposed in the context of the heat equation, see e.g. Theorem 4.1 in [25] (time upper error bound).

Step 1. For a given $t \in]t_{m-1}, t_m]$, we choose in the residual equation (35) the test functions $\mathbf{A}' = \mathbf{e}_{A,\tau}(t)$ and $\varphi' = \int_0^t e_{\varphi,\tau}(s, \cdot) ds$, so that :

$$\begin{aligned} & (\mu^{-1} \operatorname{curl} \mathbf{e}_{A,\tau}(t), \operatorname{curl} \mathbf{e}_{A,\tau}(t)) + \left(\sigma(\partial_t \mathbf{e}_{A,\tau}(t) + \nabla e_{\varphi,\tau}(t)), \mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s, \cdot) ds \right)_{\Omega_c} \\ & = (\mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t), \mathbf{e}_{A,\tau}) + (\mu^{-1} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)), \operatorname{curl} \mathbf{e}_{A,\tau}(t)). \end{aligned}$$

Now, the gauge condition on \mathbf{A} and \mathbf{A}_τ allows to use the Friedrichs inequality :

$$\|\mathbf{e}_{A,\tau}(t)\|_{X(\Omega)} \lesssim \|\operatorname{curl} \mathbf{e}_{A,\tau}(t)\|,$$

so that by the successive use of Cauchy-Schwarz and Young inequalities, we get :

$$\begin{aligned} & \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|^2 + \frac{1}{2} \frac{d}{dt} \left\| \sigma^{1/2} (\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s, \cdot) ds) \right\|_{\Omega_c}^2 \\ & \leq \frac{1}{2} \left(C \sqrt{\mu_{\max}} \|\mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t)\|_{X(\Omega)'} + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)) \right\| \right)^2 + \frac{1}{2} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|^2 \\ & \leq C \mu_{\max} \|\mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t)\|_{X(\Omega)'}^2 + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)) \right\|^2 + \frac{1}{2} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|^2, \end{aligned}$$

that is to say, by using the fact that $\mathbf{e}_{A,\tau}(0) = 0$:

$$\begin{aligned} & \frac{d}{dt} \left(\left\| \sigma^{1/2} (\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s, \cdot) ds) \right\|_{\Omega_c}^2 + \int_0^t \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(s) \right\|_{\Omega}^2 ds \right) \\ & \leq C \mu_{\max} \|\mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t)\|_{X(\Omega)'}^2 + 2 \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)) \right\|^2. \end{aligned}$$

By integrating this last relation over $[0, t_n]$, we get :

$$\begin{aligned} & \left\| \sigma^{1/2} (\mathbf{e}_{A,\tau}(t_n) + \nabla \int_0^{t_n} e_{\varphi,\tau}(s) ds) \right\|_{\Omega_c}^2 + \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(s) \right\|_{\Omega}^2 ds \\ & \leq C \mu_{\max} \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2 + 2 \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(s)) \right\|_{\Omega}^2 ds. \end{aligned} \quad (37)$$

Step 2. Now, it remains to bound the last term in (37).

From the definition (20) of \mathbf{A}_{τ} and the triangular inequality, we have for any $t \in]t_{m-1}, t_m]$:

$$\begin{aligned} \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(s)) \right\|_{\Omega}^2 ds & \leq \frac{\tau_m}{3} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_h^m) \right\|_{\Omega}^2 \\ & + \frac{\tau_m}{3} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^{m-1} - \mathbf{A}_h^{m-1}) \right\|_{\Omega}^2 \\ & + \frac{\tau_m}{3} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}_h^m - \mathbf{A}_h^{m-1}) \right\|_{\Omega}^2 \\ & \leq 2 \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(s) \right\|_{\Omega}^2 ds + \frac{\tau_m}{3} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}_h^m - \mathbf{A}_h^{m-1}) \right\|_{\Omega}^2, \end{aligned}$$

where the last inequality follows from (33) with $\mathbf{v}_{\tau} = \mathbf{e}_{A,h\tau}$. By summing from $m = 1$ to $m = n$, and from definition (21) of η_{τ}^m , we easily obtain :

$$\sum_{m=1}^n \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(s)) \right\|_{\Omega}^2 ds \leq 2 \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(s) \right\|_{\Omega}^2 ds + \sum_{m=1}^n (\eta_{\tau}^m)^2. \quad (38)$$

Step 3. (36) directly follows from (37) and (38). ■

Theorem 4.3. Under the same assumptions as in Theorems 2.1 and 4.2, the following estimation holds:

$$\begin{aligned} & \left\| \sigma^{1/2} \partial_t (\mathbf{e}_{A,\tau} + \nabla \int_0^t e_{\varphi,\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2 \\ & \leq 8 (\sigma_{\min} \mu_{\min})^{-1} \left(\sum_{m=1}^n (\eta_{\tau}^m)^2 + 3 \int_0^{t_n} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(s) \right\|_{\Omega}^2 ds \right) \\ & + 2 \sigma_{\min}^{-1} (C \mu_{\min}^{-1} \mu_{\max} + 1) \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2 \end{aligned}$$

Proof: The residual equation (35) with $\varphi' = 0$, joined to the fact that $\sigma_{\min}^{1/2} \leq \sigma^{1/2}$, $\mu^{-1/2} \leq \mu_{\min}^{-1/2}$ and the use of the Cauchy-Schwarz inequality give for all $\mathbf{A}' \in X(\Omega)$:

$$\begin{aligned} & \sigma_{\min}^{1/2} \left(\sigma^{1/2} \partial_t (\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s) ds), \mathbf{A}' \right)_{\Omega_c} \\ & \leq \|\mathbf{J}_s(t) - \pi_{\tau} \mathbf{J}_s(t)\|_{X(\Omega)'} \|\mathbf{A}'\|_{X(\Omega)} + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(t)) \right\|_{\Omega} \|\operatorname{curl} \mathbf{A}'\| + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|_{\Omega} \|\operatorname{curl} \mathbf{A}'\| \\ & \leq \mu_{\min}^{-1/2} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|_{\Omega} + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(t)) \right\|_{\Omega} \right) \|\mathbf{A}'\|_{X(\Omega)} + \|\mathbf{J}_s(t) - \pi_{\tau} \mathbf{J}_s(t)\|_{X(\Omega)'} \|\mathbf{A}'\|_{X(\Omega)}. \end{aligned}$$

Since this relation holds for all $\mathbf{A}' \in X(\Omega)$, by using the dual norm of $X(\Omega)$ we get :

$$\begin{aligned} & \left\| \sigma^{1/2} \partial_t (\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s) ds) \right\|_{X(\Omega)'} \\ & \leq \sigma_{\min}^{-1/2} \mu_{\min}^{-1/2} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|_{\Omega} + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_{\tau}(t)) \right\|_{\Omega} \right) + \sigma_{\min}^{-1/2} \|\mathbf{J}_s(t) - \pi_{\tau} \mathbf{J}_s(t)\|_{X(\Omega)'} . \end{aligned}$$

Squaring this last inequality and using two times the estimate $(a+b)^2 \leq 2(a^2 + b^2)$ valid for all $a, b \in \mathbb{R}$, we get :

$$\begin{aligned}
& \left\| \sigma^{1/2} \partial_t(\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s) ds) \right\|_{X(\Omega)'}^2 \\
& \leq 2 \sigma_{\min}^{-1} \mu_{\min}^{-1} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\| + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)) \right\| \right)^2 + 2 \sigma_{\min}^{-1} \left\| \mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t) \right\|_{X(\Omega)'}^2 \\
& \leq 4 \sigma_{\min}^{-1} \mu_{\min}^{-1} \left(\left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|^2 + \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(t)) \right\|^2 \right) + 2 \sigma_{\min}^{-1} \left\| \mathbf{J}_s(t) - \pi_\tau \mathbf{J}_s(t) \right\|_{X(\Omega)'}^2 .
\end{aligned}$$

Then an integration over $[t_{m-1}, t_m]$ and summing for $m = 1, \dots, n$ give :

$$\begin{aligned}
& \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \left\| \sigma^{1/2} \partial_t(\mathbf{e}_{A,\tau}(t) + \nabla \int_0^t e_{\varphi,\tau}(s) ds) \right\|_{X(\Omega)'}^2 dt \\
& \leq 4 \sigma_{\min}^{-1} \mu_{\min}^{-1} \sum_{m=1}^n \left(\int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(s) \right\|^2 ds + \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_\tau(s)) \right\|^2 ds \right) \\
& \quad + 2 \sigma_{\min}^{-1} \left\| \mathbf{J}_s - \pi_\tau \mathbf{J}_s \right\|_{L^2(0, t_n; X(\Omega)')}^2 .
\end{aligned} \tag{39}$$

The first term in (39) is estimated by inequality (36) and the second term by the relation (38), so that we can conclude. \blacksquare

4.2 Reliability of the space discretization

For any m , $1 \leq m \leq N$, we denote by $\widetilde{\varphi}^m \in H_0^1(\Omega)$ and $\widetilde{\varphi}_h^m \in H_0^1(\Omega)$ the extensions over Ω of $\varphi^m \in \widetilde{H}^1(\Omega_c)$ and $\varphi_h^m \in \widetilde{\Theta}_h(\Omega_c)$, respectively given by :

$$\widetilde{\varphi}^m = \begin{cases} \varphi^m & \text{in } \Omega_c \\ \varphi_e^m & \text{in } \Omega_e \end{cases} \quad \widetilde{\varphi}_h^m = \begin{cases} \varphi_h^m & \text{in } \Omega_c \\ \varphi_{h_e}^m & \text{in } \Omega_e \end{cases}$$

where $\varphi_e^m \in H^1(\Omega_e)$ and $\varphi_{h_e}^m \in H^1(\Omega_e)$ are respectively defined by :

$$\begin{cases} \Delta \varphi_e^m = 0 & \text{in } \Omega_e, \\ \varphi_e^m = \varphi^m & \text{on } B, \\ \varphi_e^m = 0 & \text{on } \Gamma, \end{cases} \quad \begin{cases} \Delta \varphi_{h_e}^m = 0 & \text{in } \Omega_e, \\ \varphi_{h_e}^m = \varphi_h^m & \text{on } B, \\ \varphi_{h_e}^m = 0 & \text{on } \Gamma. \end{cases}$$

We also define the error $\widetilde{e}_{\varphi,h\tau}(t_m) = \widetilde{\varphi}^m - \widetilde{\varphi}_h^m \in H_0^1(\Omega)$, as well as the spatial error \mathbf{E}^m defined by :

$$\mathbf{E}^m = \mathbf{e}_{A,h\tau}(t_m) + \sum_{p=1}^m \tau_p \nabla \widetilde{e}_{\varphi,h\tau}(t_p) \in H_0(\operatorname{curl}; \Omega). \tag{40}$$

Theorem 4.4. For any m , $1 \leq m \leq N$, the spatial error \mathbf{E}^m admits the following Helmholtz decomposition :

$$\mathbf{E}^m = \nabla \hat{\varphi}^m + \mathbf{e}_\perp^m, \tag{41}$$

where $\hat{\varphi}^m \in H_0^1(\Omega)$ and $\mathbf{e}_\perp^m \in X^0(\Omega)$. Moreover, \mathbf{e}_\perp^m admits the decomposition :

$$\mathbf{e}_\perp^m = \nabla \phi^m + \mathbf{w}^m,$$

where $\phi^m \in H_0^1(\Omega)$ and $\mathbf{w}_c^m = \mathbf{w}_{|\Omega_c}^m \in H^1(\Omega_c)^3$, $\mathbf{w}_e^m = \mathbf{w}_{|\Omega_e}^m \in H^1(\Omega_e)^3$. Finally,

$$\mathbf{E}^m = \mathbf{w}^m + \nabla(\hat{\varphi}^m + \phi^m), \tag{42}$$

and the following inequalities hold :

$$\| \mathbf{e}_\perp^m \|_{X(\Omega)} \lesssim \| \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \|, \tag{43}$$

$$(\| \mathbf{w}_c^m \|_{1,\Omega_c}^2 + \| \mathbf{w}_e^m \|_{1,\Omega_e}^2)^{1/2} + |\phi^m|_1 \lesssim \| \mathbf{e}_\perp^m \|_{X(\Omega)}. \tag{44}$$

Proof: The proof is very similar to the one of Theorem 3.1 in [15] devoted to the harmonic formulation of the same problem. Here, we only highlight the main difference, which lies in the estimation (43): in the harmonic case the term $\|\mathbf{e}_\perp^m\|_{X(\Omega)}$ is bounded by the sum of the magnetic energy and the electric energy (see (3.5) of [15]). Here, using another argument, the corresponding quantity is bounded only by the magnetic energy (this is needed in the proof of Theorem 4.9, see below). \mathbf{e}_\perp^m is built exactly in the same way as in [15], using \mathbf{E}^m instead of $j\omega\mathbf{e}_A + \nabla\tilde{e}_\varphi$. Consequently, we can easily obtain that (see (3.5) of [15]) :

$$\|\mathbf{e}_\perp^m\|_{X(\Omega)}^2 \lesssim \|\mathbf{e}_\perp^m\|_{\Omega_c}^2 + \|\operatorname{curl}\mathbf{e}_\perp^m\|^2. \quad (45)$$

Moreover, let us recall that by construction we have :

$$\begin{cases} \operatorname{div}\mathbf{e}_\perp^m &= 0 & \text{in } \Omega_c, \\ \mathbf{e}_\perp^m|_{\Omega_c} \cdot \mathbf{n} &= 0 & \text{on } B, \end{cases}$$

so that $\mathbf{e}_\perp^m \in H(\operatorname{curl}, \Omega_c) \cap H_0(\operatorname{div}, \Omega_c)$ where $H_0(\operatorname{div}, \Omega_c) = \{\mathbf{A} \in \mathbf{L}^2(\Omega_c) : \operatorname{div}\mathbf{A} \in \mathbf{L}^2(\Omega_c) \text{ and } \mathbf{A} \cdot \mathbf{n} = 0 \text{ on } B\}$. By the compact embedding of $H(\operatorname{curl}, \Omega_c) \cap H_0(\operatorname{div}, \Omega_c)$ into $\mathbf{L}^2(\Omega_c)$ and the fact that $\operatorname{div}\mathbf{e}_\perp^m = 0$ in Ω_c , (45) becomes :

$$\|\mathbf{e}_\perp^m\|_{X(\Omega)}^2 \lesssim \|\operatorname{curl}\mathbf{e}_\perp^m\|_{\Omega_c}^2 + \|\operatorname{curl}\mathbf{e}_\perp^m\|^2 \lesssim \|\operatorname{curl}\mathbf{e}_\perp^m\|^2 = \|\operatorname{curl}\mathbf{e}_{A,h\tau}(t_m)\|,$$

so that (43) holds. ■

Now, in the same spirit than in [25], we give four technical Lemmas which will be used in the following. Our objective is to obtain an upper bound for the spatial error (see Theorem 4.9).

Lemma 4.5. The error \mathbf{E}^m defined in (40) satisfies the following Galerkin orthogonality relation : $\forall (\mathbf{A}'_h, \varphi'_h) \in X_h(\Omega) \times \widetilde{\Theta}_h(\Omega_c)$, we have :

$$\int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{A}'_h \, d\mathbf{x} = \int_{\Omega_c} \sigma \frac{\mathbf{E}^{m-1} - \mathbf{E}^m}{\tau_m} \cdot (\mathbf{A}'_h + \tau_m \nabla \varphi'_h) \, d\mathbf{x}. \quad (46)$$

Proof: Let us first remark that from Lemma 2.4 and Lemma 2.6, we have :

$$a_m((\mathbf{e}_{A,h\tau}(t_m), e_{\varphi,h\tau}(t_m)), (\mathbf{A}'_h, \varphi'_h)) = \int_{\Omega_c} \sigma \frac{\mathbf{e}_{A,h\tau}(t_{m-1})}{\tau_m} \cdot (\mathbf{A}'_h + \tau_m \nabla \varphi'_h) \, d\mathbf{x},$$

that is to say :

$$\begin{aligned} & \int_{\Omega_c} \sigma \left(\frac{\mathbf{e}_{A,h\tau}(t_m)}{\tau_m} + \nabla e_{\varphi,h\tau}(t_m) \right) \cdot (\mathbf{A}'_h + \tau_m \nabla \varphi'_h) \, d\mathbf{x} \\ & + \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{A}'_h \, d\mathbf{x} = \int_{\Omega_c} \sigma \frac{\mathbf{e}_{A,h\tau}(t_{m-1})}{\tau_m} \cdot (\mathbf{A}'_h + \tau_m \nabla \varphi'_h) \, d\mathbf{x}. \end{aligned}$$

From the definition (40) of \mathbf{E}^m , we have :

$$\frac{\mathbf{E}^{m-1} - \mathbf{E}^m}{\tau_m} = \frac{\mathbf{e}_{A,h\tau}(t_{m-1}) - \mathbf{e}_{A,h\tau}(t_m)}{\tau_m} - \nabla e_{\varphi,h\tau}(t_m) \quad \text{in } \Omega_c,$$

so that (46) holds. ■

Lemma 4.6. For all $\mathbf{v} \in X(\Omega)$ we have :

$$\begin{aligned} & \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x} \\ & = \int_{\Omega} \mathbf{J}_s^m \cdot \mathbf{v} \, d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \cdot \mathbf{v} \, d\mathbf{x} \\ & + \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathbf{v} \, d\gamma(\mathbf{x}) - \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathbf{v} \, d\mathbf{x}. \end{aligned} \quad (47)$$

Proof: The definition of $\mathbf{e}_{A,h\tau}(t_m) = \mathbf{A}^m - \mathbf{A}_h^m$ and an integration by parts give :

$$\begin{aligned} & \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x} \\ &= \int_{\Omega} (\mu^{-1} \operatorname{curl} \mathbf{A}^m) \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x} + \sum_{T \in \mathcal{T}_{h,m}} \left(\int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathbf{v} \, d\gamma(\mathbf{x}) - \int_T \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathbf{v} \, d\mathbf{x} \right), \end{aligned}$$

and, from Lemma 2.4 with $\varphi' = 0$), we obtain (47). \blacksquare

As in [15] §3.2-3.3, we introduce now the usual Clément-like interpolants associated with a given fixed mesh $\mathcal{T}_{h,m}$ at time t_m , $m = 1, \dots, N$. The standard Clément interpolation operators are defined by :

$$\begin{aligned} \mathbf{I}_{\text{Cl},\Omega}^0 &: H_0^1(\Omega) \rightarrow \Theta_h^0(\Omega), \, v \mapsto \mathbf{I}_{\text{Cl},\Omega}^0 v = \sum_{\mathbf{x} \in \mathcal{N}_{h,m}^{\text{int}}} \frac{1}{|\omega_{\mathbf{x}}|} \left(\int_{\omega_{\mathbf{x}}} v \right) \varphi_{\mathbf{x}}, \\ \mathbf{I}_{\text{Cl},\Omega_c} &: H^1(\Omega_c) \rightarrow \Theta_h(\Omega_c), \, v \mapsto \mathbf{I}_{\text{Cl},\Omega_c} v = \sum_{\mathbf{x} \in \mathcal{N}_{h,m}^{\text{int}} \cap \Omega_c} \frac{1}{|\omega_{\mathbf{x}} \cap \Omega_c|} \left(\int_{\omega_{\mathbf{x}} \cap \Omega_c} v \right) \varphi_{\mathbf{x}}, \end{aligned}$$

where $\omega_{\mathbf{x}}$ is the set of tetrahedra containing the node \mathbf{x} and $\varphi_{\mathbf{x}}$ is the \mathbb{P}_1 nodal basis function associated with the node $\mathbf{x} \in \mathcal{N}_{h,m}^{\text{int}}$. Moreover, we denote $\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 v}$ an extension of $\mathbf{I}_{\text{Cl},\Omega_c} v$ over Ω such that $\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 v} \in \Theta_h^0(\Omega)$. For any edge $E \in \mathcal{E}_{h,m}$ we fix one of its adjacent faces $F_E \in \mathcal{F}_{h,m}$; and the standard vectorial Clément-type interpolation operator is defined by:

$$\mathcal{P}_{\text{Cl},\Omega}^0 : PH^1(\Omega)^3 \cap X(\Omega) \rightarrow X_h(\Omega), \, \mathbf{v} \mapsto \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} = \sum_{E \in \mathcal{E}_{h,m}} \left((\mathbf{v} \times \mathbf{n}_{F_E}) \cdot \mathbf{f}_E^{F_E} \right) \mathbf{w}_E,$$

where $PH^1(\Omega)$ denotes the set of functions which belong to $H^1(\Omega_c) \cap H^1(\Omega_e)$, $\mathbf{w}_E \in X_h(\Omega)$ is the basis function associated with the edge $E \in \mathcal{E}_{h,m}$ and defined by the condition :

$$\int_{E'} \mathbf{w}_E \cdot \mathbf{t}_{E'} = \delta_{E,E'} \quad \forall E' \in \mathcal{E}_{h,m},$$

with \mathbf{t}_E the unit vector directed along E , and, finally, the functions $\mathbf{f}_{E'}^{F_E}$ are determined by the condition :

$$\int_{F_E} (\mathbf{w}_{E'} \times \mathbf{n}_{F_E}) \cdot \mathbf{f}_{E'}^{F_E} = \delta_{E',E''} \quad \forall E', E'' \in \mathcal{E}_{h,m} \cup \partial F_E.$$

These interpolant operators fulfill the following estimations (see Lemmas 3.1 and 3.2 of [15]) : for any $v^0 \in H_0^1(\Omega)$, $v \in H^1(\Omega)$ and $\mathbf{v} \in PH^1(\Omega)^3 \cap X(\Omega)$ we have :

$$\sum_{T \in \mathcal{T}_{h,m}} h_T^{-2} \|v^0 - \mathbf{I}_{\text{Cl},\Omega}^0 v^0\|_T^2 + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} h_F^{-1} \|v^0 - \mathbf{I}_{\text{Cl},\Omega}^0 v^0\|_F^2 \lesssim \|\nabla v^0\|^2, \quad (48)$$

$$\sum_{T \in \mathcal{T}_{h,m}} h_T^{-2} \|v - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 v}\|_T^2 + \sum_{F \in \mathcal{F}_{h,m}} h_F^{-1} \|v - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 v}\|_F^2 \lesssim \|\nabla v\|^2, \quad (49)$$

$$\sum_{T \in \mathcal{T}_{h,m}} h_T^{-2} \|\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}\|_T^2 + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} h_F^{-1} \|\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}\|_F^2 \lesssim \|\nabla_P \mathbf{v}\|^2, \quad (50)$$

where $\|\nabla_P \mathbf{v}\|^2 = \|\nabla \mathbf{v}\|_{\Omega_c}^2 + \|\nabla \mathbf{v}\|_{\Omega_e}^2$.

Lemma 4.7. For all $\mathbf{v} \in X(\Omega)$ we have :

$$\begin{aligned} & \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x} \\ &= \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} \\ &+ \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\gamma(\mathbf{x}) - \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} \\ &- \int_{\Omega_c} \sigma \left(\frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right) \cdot \mathbf{v} \, d\mathbf{x}. \end{aligned}$$

Moreover, using the Helmholtz decomposition (42) and considering $\mathbf{v} = \mathbf{w}^m \in H^1(\Omega_c)^3 \cap H^1(\Omega_e)^3 \cap X(\Omega)$, we have :

$$\left\| \sigma^{1/2} \mathbf{E}^m \right\|_{\Omega_c}^2 + \tau_m \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{w}^m d\mathbf{x} \quad (51)$$

$$= (\sigma \mathbf{E}^{m-1}, \mathbf{E}^m)_{\Omega_c} \quad (52)$$

$$+ (\sigma (\mathbf{E}^m - \mathbf{E}^{m-1}), \nabla(\hat{\varphi}^m + \phi^m) - \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m}))_{\Omega_c} \quad (53)$$

$$+ \tau_m \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \quad (54)$$

$$- \tau_m \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \quad (55)$$

$$- \tau_m \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \quad (56)$$

$$+ \tau_m \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} \int_F [\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m]_F \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\gamma(\mathbf{x}). \quad (57)$$

Proof: First identity. We apply Lemma 4.6 :

$$\begin{aligned} & \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{v} d\mathbf{x} \\ &= \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} \\ &+ \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\gamma(\mathbf{x}) - \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} \\ &+ \int_{\Omega} \mathbf{J}_s^m \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} \\ &+ \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\gamma(\mathbf{x}) - \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} \\ &+ \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x}. \end{aligned}$$

First we notice that

$$\begin{aligned} & \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} \\ &= - \int_{\Omega_c} \sigma \left(\frac{\mathbf{e}_{A,h\tau}(t_m) - \mathbf{e}_{A,h\tau}(t_{m-1})}{\tau_m} + \nabla e_{\varphi,h\tau}(t_m) \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x} \\ &= - \int_{\Omega_c} \sigma \left(\frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) d\mathbf{x}, \end{aligned}$$

second, using Lemma 2.4 with $\mathbf{A}' = \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}$ and $\varphi' = 0$, we have

$$\int_{\Omega} \mathbf{J}_s^m \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} = \int_{\Omega} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \operatorname{curl} \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x},$$

and thirdly elementwise integrations by parts yield

$$\begin{aligned} & \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\gamma(\mathbf{x}) - \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x} \\ &= - \sum_{T \in \mathcal{T}_{h,m}} \int_T \mu^{-1} \operatorname{curl} \mathbf{A}_h^m \cdot \operatorname{curl} \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} d\mathbf{x}. \end{aligned}$$

Hence we get :

$$\begin{aligned}
& \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{v} \, d\mathbf{x} \\
&= \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} \\
&- \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} + \sum_{T \in \mathcal{T}_{h,m}} \int_{\partial T} (\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\gamma(\mathbf{x}) \\
&- \int_{\Omega_c} \sigma \left(\frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right) \cdot (\mathbf{v} - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}) \, d\mathbf{x} + \int_{\Omega} \mu^{-1} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_h^m) \cdot \operatorname{curl} \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} \, d\mathbf{x}.
\end{aligned}$$

By applying Lemma 4.5 with $\mathbf{A}'_h = \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v}$ and $\varphi'_h = 0$ we have :

$$\int_{\Omega} (\mu^{-1} \operatorname{curl} (\mathbf{A}^m - \mathbf{A}_h^m)) \cdot \operatorname{curl} \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} \, d\mathbf{x} = - \int_{\Omega_c} \sigma \left(\frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right) \cdot \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{v} \, d\mathbf{x},$$

and we conclude.

Second identity. From the first equation we have :

$$\begin{aligned}
& \tau_m \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \operatorname{curl} \mathbf{w}^m \, d\mathbf{x} \\
&= \tau_m \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) \, d\mathbf{x} - \tau_m \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) \, d\mathbf{x} \\
&- \tau_m \sum_{T \in \mathcal{T}_{h,m}} \int_T \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) \, d\mathbf{x} \\
&+ \tau_m \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} \int_F [\mathbf{n} \times \mu^{-1} \operatorname{curl} \mathbf{A}_h^m]_F \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) \, d\gamma(\mathbf{x}) - (\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \mathbf{w}^m)_{\Omega_c}.
\end{aligned}$$

Nevertheless,

$$-(\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \mathbf{w}^m)_{\Omega_c} = -(\sigma \mathbf{E}^m, \mathbf{E}^m)_{\Omega_c} + (\sigma \mathbf{E}^{m-1}, \mathbf{E}^m)_{\Omega_c} + (\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \mathbf{E}^m - \mathbf{w}^m)_{\Omega_c}.$$

From the Helmholtz decomposition (42), it appears that in Ω_c :

$$\mathbf{E}^m - \mathbf{w}^m = \nabla(\hat{\varphi}^m + \phi^m).$$

The conclusion follows by observing that :

$$(\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m} - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m))_{\Omega_c} = 0$$

because of Lemma 4.5 with $\mathbf{A}'_h = 0$. ■

Lemma 4.8. Let $\hat{\varphi}^m \in H_0^1(\Omega)$ for $m \in \{0, \dots, N\}$ be defined in Theorem 4.4. Then we have in Ω_c :

$$\|\nabla \hat{\varphi}^m\|_{\Omega_c} \lesssim \sigma_{\min}^{-1} (\eta_{\Omega;3}^m + \eta_{J;3}^m), \tag{58}$$

where $\eta_{\Omega;3}^m$ and $\eta_{J;3}^m$ are respectively defined in (24) and (27).

Proof:

By using the Helmholtz decomposition (41) and as from Lemma 4.5 (with $\mathbf{A}'_h = 0$ and $\varphi'_h = \hat{\varphi}^m$) with a recurrence argument we have :

$$\int_{\Omega_c} \sigma \mathbf{E}^m \cdot \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m}) \, d\mathbf{x} = 0,$$

we deduce :

$$\begin{aligned}
\| \nabla \hat{\varphi}^m \|_{\Omega_c}^2 &= \int_{\Omega_c} \nabla \hat{\varphi}^m \cdot (\mathbf{E}^m - \mathbf{e}_\perp^m) d\mathbf{x} \\
&\leq \sigma_{\min}^{-1} \int_{\Omega_c} \sigma \mathbf{E}^m \cdot \nabla (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m) d\mathbf{x} - \int_{\Omega_c} \mathbf{e}_\perp^m \cdot \nabla \hat{\varphi}^m d\mathbf{x} \\
&\leq \sigma_{\min}^{-1} \left(- \int_{\Omega_c} \sigma (\mathbf{A}^m + \sum_{p=1}^m \tau_p \nabla \varphi^p) \cdot \nabla (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m) d\mathbf{x} \right.
\end{aligned} \tag{59}$$

$$+ \sum_{T \in \mathcal{T}_h^m, T \subset \Omega_c} \int_T \text{div} (\sigma (\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p)) (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m) d\mathbf{x} \tag{60}$$

$$- \sum_{F \in \mathcal{F}_h^{int}, F \subset \Omega_c} \int_F \left[\sigma (\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p) \cdot \mathbf{n} \right]_F (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m) d\gamma(\mathbf{x}), \tag{61}$$

where we have integrated by parts and used the fact that $\text{div} \mathbf{e}_\perp^c = 0$ in Ω_c and $\mathbf{e}_\perp^c \cdot \mathbf{n} = 0$ on B (see the construction of \mathbf{e}_\perp^c in the proof of Theorem 3.1 of [15]), so that :

$$\int_{\Omega_c} \mathbf{e}_\perp^m \cdot \nabla \hat{\varphi}^m d\mathbf{x} = - \int_{\Omega_c} \text{div} (\mathbf{e}_\perp^m) \hat{\varphi}^m d\mathbf{x} - \int_{\partial\Omega_c} \mathbf{e}_\perp^m \cdot \mathbf{n} \nabla \hat{\varphi}^m d\gamma(\mathbf{x}) = 0.$$

Defining $\varphi' = \hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m$, we remark that :

$$- \int_{\Omega_c} \sigma (\mathbf{A}^m + \sum_{p=1}^m \tau_p \nabla \varphi^p) \cdot \varphi' d\mathbf{x} = - \sum_{q=1}^m \int_{\Omega_c} \sigma (\mathbf{A}^q - \mathbf{A}^{q-1} + \tau_q \nabla \varphi^q) \cdot \varphi' d\mathbf{x} = 0, \tag{62}$$

where the last deduction is due to the semi-discrete weak formulation (18) with $\mathbf{A}' = 0$ applied for all the discrete time steps $q = 1, \dots, m$.

The use of the continuous and the discrete Cauchy-Schwarz inequalities to the reminding terms (60) and (61), combined with the definitions (24) and (27) of some parts of the estimator, and the use of the stability result on the standard Clément interpolate (49), lead to :

$$\begin{aligned}
\| \nabla \hat{\varphi}^m \|_{\Omega_c}^2 &\leq \sigma_{\min}^{-1} \sum_{T \in \mathcal{T}_h^m, T \subset \Omega_c} h_T \| \text{div} (\sigma (\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p)) \|_T h_T^{-1} \| \hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m \| \\
&+ \sigma_{\min}^{-1} \sum_{F \in \mathcal{F}_h^{int}, F \subset \Omega_c} h_F^{1/2} \| [\sigma (\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \varphi_h^p) \cdot \mathbf{n}]_F \| h_F^{-1/2} \| \hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m \| \\
&\lesssim \sigma_{\min}^{-1} (\eta_{\Omega;3}^m + \eta_{J;3}^m) \| \nabla \hat{\varphi}^m \|_{\Omega_c},
\end{aligned}$$

so that (58) holds. ■

Theorem 4.9. Let $n \in \{0, \dots, N\}$. Then we have the following upper bound :

$$\begin{aligned}
&\left\| \sigma^{1/2} \left(\mathbf{e}_{A,h\tau}(t_n) + \sum_{m=1}^n \tau_m \nabla e_{\varphi,h\tau}(t_m) \right) \right\|_{\Omega_c}^2 + \sum_{m=1}^n \tau_m \left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m) \right\|_{\Omega}^2 \\
&\lesssim \max(\sigma_{\min}^{-1}, \mu_{\max}) \sum_{m=1}^n \tau_m ((\eta_h^m)^2 + (\xi^m)^2)
\end{aligned} \tag{63}$$

Proof: In the two first steps, the error (51) is estimated at time t_m : an upper bound is proposed for the terms in the right-hand-side of the second relation associated with Lemma 4.7. In particular, the first and the second

steps are devoted to the estimation of the terms (53), (54), (55) (56) and (57). In the third step, the estimation is extended to all times t_m , $0 \leq m \leq n$.

Step 1. The continuous and the discrete Cauchy-Schwarz inequalities combined to the definitions (22), (25) of some parts of the estimator and (28) of the error space approximation of the data lead to an estimation of (54), (55), (56) and (57) :

$$\begin{aligned}
& \int_{\Omega} \mathbf{J}_s^m \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& - \sum_{T \in \mathcal{T}_{h,m}} \int_T \text{curl} (\mu^{-1} \text{curl} \mathbf{A}_h^m) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& - \sum_{T \in \mathcal{T}_{h,m}} \int_T \text{curl} (\mu^{-1} \text{curl} \mathbf{A}_h^m) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} \int_F [\mathbf{n} \times \mu^{-1} \text{curl} \mathbf{A}_h^m]_F \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\gamma(\mathbf{x}) \\
& = \sum_{T \in \mathcal{T}_{h,m}} \int_T (\mathbf{J}_s^m - \pi_h \mathbf{J}_s^m) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& + \sum_{T \in \mathcal{T}_{h,m}} \int_T \left(\pi_h \mathbf{J}_s^m - \text{curl} (\mu^{-1} \text{curl} \mathbf{A}_h^m) - \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \widetilde{\varphi}_h^m \right) \right) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} [\mathbf{n} \times \mu^{-1} \text{curl} \mathbf{A}_h^m]_F \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\gamma(\mathbf{x}) \\
& \leq \sum_{T \in \mathcal{T}_{h,m}} h_T \|\mathbf{J}_s^m - \pi_h \mathbf{J}_s^m\|_T h_T^{-1} \|\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m\|_T \\
& + \sum_{T \in \mathcal{T}_{h,m}} h_T \left\| \pi_h \mathbf{J}_s^m - \text{curl} (\mu^{-1} \text{curl} \mathbf{A}_h^m) - \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \widetilde{\varphi}_h^m \right) \right\|_T h_T^{-1} \|\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m\|_T \\
& + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} h_F^{1/2} \|[\mathbf{n} \times \mu^{-1} \text{curl} \mathbf{A}_h^m]_F\|_F h_F^{-1/2} \|\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m\|_F \\
& \leq \left(\left(\sum_{T \in \mathcal{T}_{h,m}} (\eta_{T;1}^m)^2 \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_{h,m}} (\xi_T^m)^2 \right)^{1/2} \right) \left(\sum_{T \in \mathcal{T}_{h,m}} h_T^{-2} \|\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m\|_T^2 \right)^{1/2} \\
& + \left(\sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} (\eta_{F;1}^m)^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} h_F^{-1} \|\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m\|_F^2 \right)^{1/2}.
\end{aligned}$$

From the stability result on the vectorial Cl  ment interpolant (50), there exists $C > 0$, which does not depend on T , F and \mathbf{w}^m , such that the previous estimation becomes :

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_{h,m}} \int_T (\mathbf{J}_s^m - \text{curl} (\mu^{-1} \text{curl} \mathbf{A}_h^m)) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& - \int_{\Omega_c} \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\mathbf{x} \\
& + \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}} \int_F [\mathbf{n} \times \mu^{-1} \text{curl} \mathbf{A}_h^m]_F \cdot (\mathbf{w}^m - \mathcal{P}_{\text{Cl},\Omega}^0 \mathbf{w}^m) d\gamma(\mathbf{x}) \\
& \lesssim \|\nabla_p \mathbf{w}^m\| (\xi^m + \eta_{\Omega;1}^m + \eta_{J;1}^m) \\
& \lesssim \mu_{\max}^{1/2} \|\mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m)\| (\xi^m + \eta_{\Omega;1}^m + \eta_{J;1}^m) \\
& \quad \text{(from (44) and (43) in theorem 4.4)}
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\lesssim \mu_{\max}^{1/2} \left(\xi^m + \eta_{\Omega;1}^m + \eta_{J;1}^m \right)^2 + \frac{1}{4} \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2 \\
&\quad (\text{thanks to Young's inequality } ab \leq a^2 + \frac{b^2}{4}, \forall a, b \in \mathbb{R}) \\
&\lesssim \mu_{\max} \left((\xi^m)^2 + (\eta_{\Omega;1}^m)^2 + (\eta_{J;1}^m)^2 \right) + \frac{1}{4} \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2 \\
&\quad (\text{thanks to the relation } (a+b+c)^2 \leq 3(a^2+b^2+c^2), \forall a, b, c \in \mathbb{R}).
\end{aligned}$$

Step 2. Here we evaluate (53): we explicitly write the errors $\mathbf{e}_{A,h\tau}(t_m)$ and $e_{\varphi,h\tau}(t_m)$ in order to split the temporal and spatial contributions $(\mathbf{A}^m, \varphi^m)$ and $(\mathbf{A}_h^m, \varphi_h^m)$. Then we apply Green's formula, so that (53) takes the following form :

$$\begin{aligned}
&(\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \nabla(\hat{\varphi}^m + \phi^m) - \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m} - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m))_{\Omega_c} \\
&= \tau_m \int_{\Omega_c} \sigma \left(\frac{\mathbf{e}_{A,h\tau}(t_m) - \mathbf{e}_{A,h\tau}(t_{m-1})}{\tau_m} + \nabla e_{\varphi,h\tau}(t_m) \right) \cdot \nabla(\hat{\varphi}^m + \phi^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m} - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m) d\mathbf{x} \\
&= \tau_m \int_{\Omega_c} \left(\sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \right) \cdot \nabla(\hat{\varphi}^m + \phi^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m} - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m) d\mathbf{x} \tag{65} \\
&+ \tau_m \sum_{T \in \mathcal{T}_{h,m}, T \subset \Omega_c} \int_T \operatorname{div} \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \right) (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m}) d\mathbf{x} \\
&- \tau_m \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}, F \subset \Omega_c} \int_F \left[\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right) \cdot \mathbf{n} \right]_F (\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m}) d\gamma(\mathbf{x}) \\
&+ \tau_m \sum_{T \in \mathcal{T}_{h,m}, T \subset \Omega_c} \int_T \operatorname{div} \left(\sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \right) (\phi^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m) d\mathbf{x} \\
&- \tau_m \sum_{F \in \mathcal{F}_{h,m}^{\text{int}}, F \subset \Omega_c} \int_F \left[\sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right) \cdot \mathbf{n} \right]_F (\phi^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m) d\gamma(\mathbf{x}).
\end{aligned}$$

The term (65) is equal to zero thanks to the semi-discrete weak formulation (18) with $\mathbf{A}' = 0$, as shown in the proof of the Lemma 4.8 (see the relation (62)). Using the Cauchy-Schwarz inequality and the definitions (23) and (26) of some parts of the estimator, the right-hand side of the previous identity can be estimated as follows :

$$\begin{aligned}
&(\sigma(\mathbf{E}^m - \mathbf{E}^{m-1}), \nabla(\hat{\varphi}^m + \phi^m) - \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m} - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m))_{\Omega_c} \\
&\leq \tau_m \left[\eta_{\Omega;2}^m \left(\sum_{T \in \mathcal{T}_{h,m}, T \subset \Omega_c} h_T^{-2} \|\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m}\|_T^2 \right)^{1/2} + \eta_{J;2}^m \left(\sum_{F \in \mathcal{F}_{h,m}^{\text{int}}, F \subset \Omega_c} h_F^{-1} \|\hat{\varphi}^m - \widetilde{\mathbf{I}_{\text{Cl},\Omega}^0 \hat{\varphi}^m}\|_F^2 \right)^{1/2} \right. \\
&+ \eta_{\Omega;2}^m \left(\sum_{T \in \mathcal{T}_{h,m}} h_T^{-2} \|\phi^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m\|_T^2 \right)^{1/2} + \eta_{J;2}^m \left(\sum_{F \in \mathcal{F}_{h,m}^{\text{int}}, F \subset \Omega_c} h_F^{-1} \|\phi^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m\|_F^2 \right)^{1/2} \left. \right] \\
&\lesssim \tau_m (\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\nabla \hat{\varphi}^m\|_{\Omega_c} + \tau_m (\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\nabla \phi^m\|, \tag{66}
\end{aligned}$$

the last line coming from the usual stability estimates (48) and (49). Applying (58) for the first term of the right-hand-side of this last inequality, and (44) and (43) for the second term, we get :

$$(\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\nabla \hat{\varphi}^m\|_{\Omega_c} \lesssim \sigma_{\min}^{-1} (\eta_{\Omega;2}^m + \eta_{J;2}^m) (\eta_{\Omega;3}^m + \eta_{J;3}^m), \tag{67}$$

$$(\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\nabla \phi^m\|_{\Omega} \lesssim \mu_{\max}^{1/2} (\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|. \tag{68}$$

Then coming back to (66) and using the Young inequalities $ab \leq a^2 + \frac{b^2}{4}$ for (68) and $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ for (67),

and afterwards the relation $(a+b)^2 \leq 2(a^2+b^2)$, we can conclude that :

$$\begin{aligned}
& (\sigma (\mathbf{E}^m - \mathbf{E}^{m-1}), \nabla(\hat{\varphi}^m + \phi^m) - \nabla(\widetilde{\mathbf{I}_{\text{Cl},\Omega}^0} \hat{\varphi}^m - \mathbf{I}_{\text{Cl},\Omega}^0 \phi^m))_{\Omega_c} \\
& \leq C \tau_m (\sigma_{\min}^{-1} (\eta_{\Omega;2}^m + \eta_{J;2}^m)^2 + \sigma_{\min}^{-1} (\eta_{\Omega;3}^m + \eta_{J;3}^m)^2 + \mu_{\max} (\eta_{\Omega;2}^m + \eta_{J;2}^m)^2) \\
& + \frac{\tau_m}{4} \|\mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2 \\
& \leq C \max(\sigma_{\min}^{-1}, \mu_{\max}) \tau_m ((\eta_{\Omega;2}^m)^2 + (\eta_{J;2}^m)^2 + (\eta_{\Omega;3}^m)^2 + (\eta_{J;3}^m)^2) + \frac{\tau_m}{4} \|\mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2.
\end{aligned}$$

Step 3. In this step, the results obtained in steps 1 and 2 are used to estimate the second equation arising in Lemma 4.7, and the Young inequality is applied to (52) :

$$\begin{aligned}
& \left\| \sigma^{1/2} \mathbf{E}^m \right\|_{\Omega_c}^2 + \tau_m \int_{\Omega} \mu^{-1} \text{curl} \mathbf{e}_{A,h\tau}(t_m) \cdot \text{curl} \mathbf{w} \, d\mathbf{x} \\
& \leq \frac{1}{2} \left\| \sigma^{1/2} \mathbf{E}^{m-1} \right\|_{\Omega_c}^2 + \frac{1}{2} \left\| \sigma^{1/2} \mathbf{E}^m \right\|_{\Omega_c}^2 \\
& + C \max(\sigma_{\min}^{-1}, \mu_{\max}) \tau_m ((\eta_{\Omega;1}^m)^2 + (\eta_{J;1}^m)^2 + (\eta_{\Omega;2}^m)^2 + (\eta_{J;2}^m)^2 + (\eta_{\Omega;3}^m)^2 + (\eta_{J;3}^m)^2 + (\xi^m)^2) \\
& + \tau_m \frac{1}{2} \|\mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2.
\end{aligned}$$

Having in mind the definition (22) of the spatial estimator η_h^m , we get :

$$\begin{aligned}
& \left\| \sigma^{1/2} \mathbf{E}^m \right\|_{\Omega_c}^2 + \tau_m \|\mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m)\|^2 \\
& \leq \left\| \sigma^{1/2} \mathbf{E}^{m-1} \right\|_{\Omega_c}^2 + C \max(\sigma_{\min}^{-1}, \mu_{\max}) \tau_m ((\eta_h^m)^2 + (\xi^m)^2),
\end{aligned}$$

and summing for $m = 1, \dots, n$ yields (63). ■

Theorem 4.10. Under the same assumptions than in Theorem 4.9, the following estimation holds:

$$\begin{aligned}
& \left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,h\tau} + \nabla e_{\varphi,h\tau}) \right\|_{L^2(0,t_n; X(\Omega)')}^2 \\
& \lesssim \max(\sigma_{\min}^{-1}, (\sigma_{\min} \mu_{\min})^{-1}) \max(1, \sigma_{\min}^{-1}, \mu_{\max}) \sum_{m=1}^n \tau_m ((\eta_h^m)^2 + (\xi^m)^2).
\end{aligned}$$

Proof: Thanks to the spatial error definition (see section 3.1), to Lemma 2.4 with $\varphi' = 0$, for any $t \in]t_{m-1}, t_m]$ with $1 \leq m \leq n$, and for any $\mathbf{A}' \in X(\Omega)$, the following relation holds :

$$\left(\sigma \partial_t (\mathbf{e}_{A,h\tau}(t) + \nabla \int_0^t e_{\varphi,h\tau}(s) \, ds), \mathbf{A}' \right)_{\Omega_c} \quad (69)$$

$$\begin{aligned}
& = \left(\sigma \left(\frac{\mathbf{A}^m - \mathbf{A}^{m-1}}{\tau_m} + \nabla \varphi^m \right), \mathbf{A}' \right)_{\Omega_c} - \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m, \mathbf{A}' \right)_{\Omega_c} \\
& = -(\mu^{-1} \text{curl} \mathbf{A}^m, \text{curl} \mathbf{A}') + (\mathbf{J}_s^m, \mathbf{A}') - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m, \mathbf{A}' \right), \mathbf{A}' \right)_{\Omega_c} \\
& \quad - (\mu^{-1} \text{curl} \mathbf{A}_h^m, \text{curl} \mathbf{A}') + (\mu^{-1} \text{curl} \mathbf{A}_h^m, \text{curl} \mathbf{A}') \\
& = -(\mu^{-1} \text{curl} \mathbf{e}_{A,h\tau}(t_m), \text{curl} \mathbf{A}') + (\mathbf{J}_s^m, \mathbf{A}') \\
& \quad - (\mu^{-1} \text{curl} \mathbf{A}_h^m, \text{curl} \mathbf{A}') - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m, \mathbf{A}' \right), \mathbf{A}' \right)_{\Omega_c}. \quad (70)
\end{aligned}$$

Since $\mathbf{A}' \in X(\Omega)$, we use the Helmholtz decomposition for \mathbf{A}' in a similar manner as made in Theorem 4.4 : there exist \mathbf{w} such that $\mathbf{w}_c = \mathbf{w}|_{\Omega_c} \in H^1(\Omega_c)^3$, $\mathbf{w}_e = \mathbf{w}|_{\Omega_e} \in H^1(\Omega_e)^3$, $\hat{\varphi} \in H_0^1(\Omega)$ and $\phi \in H_0^1(\Omega)$ such that :

$$\mathbf{A}' = \mathbf{w} + \nabla(\hat{\varphi} + \phi), \quad (71)$$

and such that $\|\nabla \hat{\varphi}\|_{\Omega_c} \lesssim \|\mathbf{A}'\|$, $\|\nabla \phi\| \lesssim \|\operatorname{curl} \mathbf{A}'\|$ and $\|\nabla_p \mathbf{w}\| \lesssim \|\operatorname{curl} \mathbf{A}'\|$ which, using the relation $\|\operatorname{curl} \mathbf{A}'\| \lesssim \|\mathbf{A}'\|_{X(\Omega)}$, become :

$$\|\nabla \hat{\varphi}\|_{\Omega_c} \lesssim \|\mathbf{A}'\|_{X(\Omega)}, \quad (72)$$

$$\|\nabla \phi\| \lesssim \|\mathbf{A}'\|_{X(\Omega)}, \quad (73)$$

$$\|\nabla_p \mathbf{w}\| \lesssim \|\mathbf{A}'\|_{X(\Omega)}. \quad (74)$$

So remembering the divergence-free property of \mathbf{J}_s^m and that the curl of a gradient is equal to zero, we rewrite (70) as follows :

$$(\mathbf{J}_s^m, \mathbf{A}') - (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m, \operatorname{curl} \mathbf{A}') - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \mathbf{A}' \right)_{\Omega_c} \quad (75)$$

$$= (\mathbf{J}_s^m, \mathbf{w}) - (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m, \operatorname{curl} \mathbf{w}) - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \mathbf{w} \right)_{\Omega_c} \quad (76)$$

$$- \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \nabla(\hat{\varphi} + \phi) \right)_{\Omega_c}. \quad (77)$$

Thanks to Lemma 2.6 with $\mathbf{A}'_h = \mathcal{P}_{\text{Cl}, \Omega}^0 \mathbf{w}$ and $\varphi'_h = 0$, we estimate (76) proceeding as in the Step 1 of the proof of Theorem 4.9 (see the estimation just above inequality (64)) :

$$\begin{aligned} & (\mathbf{J}_s^m, \mathbf{w}) - (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m, \operatorname{curl} \mathbf{w}) - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \mathbf{w} \right)_{\Omega_c} \\ & \lesssim (\eta_{\Omega;1}^m + \eta_{J;1}^m + \xi^m) \|\nabla_p \mathbf{w}\| \lesssim (\eta_{\Omega;1}^m + \eta_{J;1}^m + \xi^m) \|\mathbf{A}'\|_{X(\Omega)}, \end{aligned} \quad (78)$$

where, in the last inequality, we used (74). Thanks to Lemma 2.6 with $\mathbf{A}'_h = 0$ and $\varphi'_h = \widetilde{\mathcal{I}_{\text{Cl}, \Omega}^0 \hat{\varphi}} + \mathcal{I}_{\text{Cl}, \Omega}^0 \phi$, we proceed to the estimation of (77) as made in Step 2 of the proof of Theorem 4.9 (see the estimation just above inequality (66)) :

$$\begin{aligned} & \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \nabla(\hat{\varphi} + \phi) \right)_{\Omega_c} \\ & \lesssim (\eta_{\Omega;2}^m + \eta_{J;2}^m) (\|\nabla \hat{\varphi}\|_{\Omega_c} + \|\nabla \phi\|) \lesssim (\eta_{\Omega;2}^m + \eta_{J;2}^m) \|\mathbf{A}'\|_{X(\Omega)}, \end{aligned} \quad (79)$$

where, in the last inequality, we used (72) and (73).

Combining (78) and (79), (75) can be estimated as follows :

$$\begin{aligned} & (\mathbf{J}_s^m, \mathbf{A}') - (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m, \operatorname{curl} \mathbf{A}') - \left(\sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \varphi_h^m \right), \mathbf{A}' \right)_{\Omega_c} \\ & \lesssim (\eta_{\Omega;1}^m + \eta_{J;1}^m + \eta_{\Omega;2}^m + \eta_{J;2}^m + \xi^m) \|\mathbf{A}'\|_{X(\Omega)}, \end{aligned}$$

and, consequently, (69) can be estimated as follows¹ :

$$\begin{aligned} & (\sigma^{1/2} \partial_t(\mathbf{e}_{A,h\tau} + \nabla \int_0^t e_{\varphi,h\tau}) , \mathbf{A}') \\ & \leq \sigma_{\min}^{-1/2} \mu_{\min}^{-1/2} \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\| \|\operatorname{curl} \mathbf{A}'\| + C \sigma_{\min}^{-1/2} (\eta_{\Omega;1}^m + \eta_{J;1}^m + \eta_{\Omega;2}^m + \eta_{J;2}^m + \xi^m) \|\mathbf{A}'\|_{X(\Omega)} \\ & \lesssim \max(\sigma_{\min}^{-1/2}, (\sigma_{\min} \mu_{\min})^{-1/2}) \left(\eta_{\Omega;1}^m + \eta_{J;1}^m + \eta_{\Omega;2}^m + \eta_{J;2}^m + \xi^m + \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\| \right) \|\mathbf{A}'\|_{X(\Omega)}. \end{aligned}$$

¹Since $\sigma = 0$ in Ω_e , we can extend the domain of the integral $(\sigma \partial_t(\mathbf{e}_{A,h\tau}(t) + \nabla \int_0^t e_{\varphi,h\tau}(s) ds), \mathbf{A}')_{\Omega_c}$ to the whole domain Ω .

Since this last relation holds for any $\mathbf{A}' \in X(\Omega)$ for any $t \in]t_{m-1}, t_m]$, we obtain

$$\begin{aligned} & \left\| \sigma^{1/2} \partial_t (\mathbf{e}_{A,h\tau} + \nabla \int_0^t e_{\varphi,h\tau}) \right\|_{X(\Omega)'} \\ & \lesssim \max(\sigma_{\min}^{-1/2}, (\sigma_{\min} \mu_{\min})^{-1/2}) \left(\eta_{\Omega;1}^m + \eta_{J;1}^m + \eta_{\Omega;2}^m + \eta_{J;2}^m + \xi^m + \left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m) \right\| \right). \end{aligned}$$

Squaring this last inequality, using two times appropriately the relation $(a_1 + \dots + a_n) \leq n(a_1^2 + \dots + a_n^2)$, integrating over $]t_{m-1}, t_m]$ and summing for $m = 1, \dots, n$, we get :

$$\begin{aligned} & \left\| \sigma^{1/2} \partial_t (\mathbf{e}_{A,h\tau} + \nabla \int_0^t e_{\varphi,h\tau}), \mathbf{A}' \right\|_{L^2(0,t_n;X(\Omega)')} \\ & \lesssim \max(\sigma_{\min}^{-1}, (\sigma_{\min} \mu_{\min})^{-1}) \left(\sum_{m=1}^n \tau_m ((\eta_{\Omega;1}^m)^2 + (\eta_{J;1}^m)^2 + (\eta_{\Omega;2}^m)^2 + (\eta_{J;2}^m)^2 + (\xi^m)^2) \right. \\ & \quad \left. + \sum_{m=1}^n \tau_m \left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t_m) \right\|^2 \right). \end{aligned} \tag{80}$$

Applying Theorem 4.9 to the term (80) leads to the conclusion. ■

4.3 Reliability of the whole error

We can state a first result for the error at time t_n , $n \in \{1, \dots, N\}$:

$$e(t_n)^2 = \left\| \sigma^{1/2} (\mathbf{e}_A(t_n) + \nabla \int_0^{t_n} e_{\varphi}(t) dt) \right\|_{\Omega_c}^2 + \left\| \mu^{-1/2} \text{curl} \mathbf{e}_A(t) \right\|_{L^2(0,t_n;\mathbf{L}^2(\Omega))}^2, \tag{81}$$

where $\mathbf{e}_A = \mathbf{A} - \mathbf{A}_{h\tau} = \mathbf{e}_{A,\tau} + \mathbf{e}_{A,h\tau}$ and $e_{\varphi} = \varphi - \varphi_{h\tau} = e_{\varphi,\tau} + e_{\varphi,h\tau}$.

Theorem 4.11. For all $n = 1, \dots, N$, we have :

$$e(t_n)^2 \lesssim C_{\sigma_{\min}, \mu_{\max}} (1 + \sigma_{\tau}) \left((\eta^n)^2 + \sum_{m=1}^n \tau_m (\xi^m)^2 + \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2 \right), \tag{82}$$

where $C_{\sigma_{\min}, \mu_{\max}}$ denotes a constant which only depends on the values of σ_{\min} and μ_{\max} .

Proof: Using the above definitions of \mathbf{e}_A and \mathbf{e}_{φ} , this result is a direct consequence of Theorems 4.2 and 4.9. It is based on some classical Cauchy-Schwarz and Young inequalities, associated to the relation (31) of Lemma 4.1 to move from continuous to discrete integration in time. ■

We now define the whole error at time t_n , $n \in \{1, \dots, N\}$ by :

$$\begin{aligned} E(t_n)^2 &= \left\| \sigma^{1/2} (\mathbf{e}_{A,\tau}(t_n) + \nabla \int_0^{t_n} e_{\varphi,\tau}(t) dt) \right\|_{\Omega_c}^2 + \left\| \sigma^{1/2} (\mathbf{e}_{A,h\tau}(t_n) + \nabla \int_0^{t_n} e_{\varphi,h\tau}(t) dt) \right\|_{\Omega_c}^2 \\ &+ \left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2 + \left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,h\tau} + \nabla e_{\varphi,h\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2 \\ &+ \left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,\tau}(t) \right\|_{L^2(0,t_n;\mathbf{L}^2(\Omega))}^2 + \left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,h\tau}(t) \right\|_{L^2(0,t_n;\mathbf{L}^2(\Omega))}^2. \end{aligned}$$

Theorem 4.12. For all $n = 1, \dots, N$, we have :

$$E(t_n)^2 \lesssim C_{\sigma_{\min}, \mu_{\min}, \mu_{\max}} (1 + \sigma_{\tau}) \left((\eta^n)^2 + \sum_{m=1}^n \tau_m (\xi^m)^2 + \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2 \right),$$

where $C_{\sigma_{\min}, \mu_{\min}, \mu_{\max}}$ denotes a constant which only depends on the values of σ_{\min} , μ_{\min} and μ_{\max} .

Proof: Using Lemma 4.1, this result is a direct consequence of Theorems 4.2, 4.3, 4.9 and 4.10. ■

Let us remark that another definition of the global error should be considered, which will be shown to be useful in the following.

$$\bar{E}(t_n)^2 = E(t_n)^2 + \sum_{m=1}^n \tau_m \left\| \sigma^{1/2} (\mathbf{e}_{A,h\tau}(t_m) + \nabla \int_0^{t_m} e_{\varphi,h\tau}(s) ds) \right\|_{\Omega_c}^2. \quad (83)$$

In that case, we also have a reliability property :

Corollary 4.13. We have:

$$\bar{E}(t_n)^2 \leq C_{\sigma_{\min}, \mu_{\min}, \mu_{\max}} \left(\sum_{m=1}^n (\eta_{\tau}^m)^2 + \tau_m ((\eta_h^m)^2 + (\xi^m)^2) + \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(0,t_n;X(\Omega)')}^2 \right),$$

where $C_{\sigma_{\min}, \mu_{\min}, \mu_{\max}}$ denotes a constant defined in a similar manner as already specified in Theorem 4.12.

Proof: From the definition of $\bar{E}(t_n)$, by Theorem 4.9, the new term arising in (83) can be bounded as follows :

$$\begin{aligned} \sum_{m=1}^n \tau_m \left\| \sigma^{1/2} (\mathbf{e}_{A,h\tau}(t_m) + \nabla \int_0^{t_m} e_{\varphi,h\tau}(t) dt) \right\|_{\Omega_c}^2 \\ \leq C \max(\sigma_{\min}^{-1}, \mu_{\max}) \sum_{m=1}^n \tau_m \sum_{p=1}^m \tau_p ((\eta_h^p)^2 + (\xi^p)^2) \\ \leq C \max(\sigma_{\min}^{-1}, \mu_{\max}) T \sum_{m=1}^n \tau_m ((\eta_h^m)^2 + (\xi^m)^2). \end{aligned}$$

■

5 Efficiency of the estimator

5.1 Efficiency of the time discretization

For the temporal estimator η_{τ}^m (see definition (21)), the following result can be established :

Lemma 5.1.

$$\begin{aligned} \eta_{\tau}^m &\leq \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{e}_{A,h\tau}(t_m)) \right\| + \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{e}_{A,h\tau}(t_{m-1})) \right\| \\ &+ C \max(\mu_{\max}^{1/2}, (\mu_{\max} \sigma_{\min})^{1/2}, \mu_{\min}^{-1/2}) \left[\left\| \mu^{-1/2} \text{curl} \mathbf{e}_{A,\tau} \right\|_{L^2(t_{m-1}, t_m; \mathbf{L}^2(\Omega))} \right. \\ &\left. + \left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}) \right\|_{L^2(t_{m-1}, t_m; X(\Omega)')} + \|\mathbf{J}_s - \pi_{\tau} \mathbf{J}_s\|_{L^2(t_{m-1}, t_m; X(\Omega)')} \right]. \end{aligned}$$

Proof: From the definition of the temporal estimator (21) and by using the triangular inequality, we get :

$$\begin{aligned} \eta_{\tau}^m &= \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}_h^m - \mathbf{A}_h^{m-1}) \right\| \\ &\leq \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}^m - \mathbf{A}_h^m) \right\| + \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}^{m-1} - \mathbf{A}_h^{m-1}) \right\| \\ &+ \left(\frac{\tau_m}{3} \right)^{1/2} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}^m - \mathbf{A}^{m-1}) \right\|. \end{aligned} \quad (84)$$

Since the first two terms in the right-hand-side of this inequality directly represent the magnetic energy norm of the spatial error respectively $\mathbf{e}_{A,h\tau}(t_m)$ and $\mathbf{e}_{A,h\tau}(t_{m-1})$, we have to estimate only the term (84). Reminding the definition (20) of \mathbf{A}_{τ} , a direct calculation gives :

$$\frac{\tau_m}{3} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}^m - \mathbf{A}^{m-1}) \right\|^2 = \int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \text{curl}(\mathbf{A}^m - \mathbf{A}_{\tau}(s)) \right\|^2 ds.$$

Moreover, from the temporal residual equation (35) with $\mathbf{A}' = \mathbf{A}^m - \mathbf{A}_\tau$ and $\varphi' = 0$, we have :

$$\begin{aligned} \left\| \mu^{-1/2} \operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau) \right\|^2 &\leq (\mathbf{J}_s - \pi_\tau \mathbf{J}_s, \mathbf{A}^m - \mathbf{A}_\tau) \\ &+ (\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}), \mathbf{A}^m - \mathbf{A}_\tau)_{\Omega_c} + (\mu^{-1} \operatorname{curl} \mathbf{e}_{A,\tau}, \operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau)) . \end{aligned} \quad (85)$$

Since the gauge condition of the vector potential \mathbf{A}^m , for $m = 1, \dots, n$, implies that $\operatorname{div}(\mathbf{A}^m - \mathbf{A}_\tau) = 0$, $\mathbf{A}^m - \mathbf{A}_\tau$ belongs to $X(\Omega) \cap H(\operatorname{div}; \Omega)$; and by the compact embedding of $X(\Omega) \cap H(\operatorname{div}; \Omega)$ into $\mathbf{L}^2(\Omega)$, we deduce that :

$$\|(\mathbf{A}^m - \mathbf{A}_\tau)\| \lesssim \|\operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau)\| + \|\operatorname{div}(\mathbf{A}^m - \mathbf{A}_\tau)\| \lesssim \|\operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau)\|.$$

As $\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla \tilde{e}_{\varphi,\tau}) \in X(\Omega)'$, the first two terms of the right-hand-side of the inequality (85) can be estimated as follows :

$$\begin{aligned} &(\mathbf{J}_s - \pi_\tau \mathbf{J}_s, \mathbf{A}^m - \mathbf{A}_\tau)_\Omega + (\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}), \mathbf{A}^m - \mathbf{A}_\tau)_{\Omega_c} \\ &\leq \left(\|\mathbf{J}_s - \pi_\tau \mathbf{J}_s\|_{X(\Omega)'} + \|\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau})\|_{X(\Omega)'} \right) \|\mathbf{A}^m - \mathbf{A}_\tau\|_{X(\Omega)} \end{aligned} \quad (86)$$

$$\leq C \mu_{\max}^{1/2} \left(\|\mathbf{J}_s - \pi_\tau \mathbf{J}_s\|_{X(\Omega)'} + \|\sigma(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau})\|_{X(\Omega)'} \right) \left\| \mu^{-1/2} \operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau) \right\|.$$

An integration of (85) over the interval $[t_{m-1}, t_m]$, the Cauchy-Schwarz inequality and the use of (86) give :

$$\begin{aligned} &\int_{t_{m-1}}^{t_m} \left\| \mu^{-1/2} \operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau) \right\|^2 dt \leq \\ &C \max(\mu_{\max}^{1/2}, (\mu_{\max} \sigma_{\min})^{1/2}, \mu_{\min}^{-1/2}) \left(\|\mathbf{J}_s - \pi_\tau \mathbf{J}_s\|_{L^2(t_{m-1}, t_m; X(\Omega)')} + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau} \right\|_{L^2(t_{m-1}, t_m; \mathbf{L}^2(\Omega))} \right. \\ &\left. + \left\| \sigma^{1/2}(\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}) \right\|_{L^2(t_{m-1}, t_m; X(\Omega)')}^2 \right) \left\| \mu^{-1/2} \operatorname{curl}(\mathbf{A}^m - \mathbf{A}_\tau) \right\|_{L^2(t_{m-1}, t_m; \mathbf{L}^2(\Omega))}. \end{aligned}$$

Using this last result to estimate (84), the conclusion follows. ■

5.2 Efficiency of the space discretization

As specified in assumption 5.4 of [25], since we consider an unstationary problem in an mesh adaptive context, in the following, we suppose that for any m , $1 \leq m \leq N$, there exists a conforming triangulation $\tilde{\mathcal{T}}_{h,m}$ such that each element $T \in \mathcal{T}_{h,m}$ or $T \in \mathcal{T}_{h,m-1}$ is the union of elements \tilde{T} of $\tilde{\mathcal{T}}_{h,m}$ such that $h_T \sim h_{\tilde{T}}$. In the following we use the relation :

$$\mathbf{E}^m - \mathbf{E}^{m-1} = \mathbf{e}_{A,h\tau}(t_m) - \mathbf{e}_{A,h\tau}(t_{m-1}) + \tau_m \nabla \widetilde{e_{\varphi,h\tau}}(t_m),$$

which is deduced directly from the definition (40) of \mathbf{E}^m . We further use the so-called bubble functions (see e.g. Remark 1.2 and Lemma 1.3 pages 9 and 10 of [30]) denoted by b_T and b_F , defined respectively on the tetrahedron $T \in \mathcal{T}_{h,m}$ and on the *patch* $\omega_F = T_1 \cup T_2$ where $F = \overline{T_1} \cap \overline{T_2}$, and the extension operator $F_{\text{ext}} : C(\overline{F}) \rightarrow C(\overline{T})$. We will use the well-known properties of the bubble functions :

$$b_T = 0 \text{ on } \partial T, \quad b_F = 0 \text{ on } \partial \omega_F \quad \text{and} \quad \|b_T\|_{\infty, T} = \|b_F\|_{\infty, \omega_F} = 1,$$

and the corresponding inverse inequalities, for instance we refer to inequalities (3.19)-(3.23) of Lemma 3.3 in [15]. To complete the notations introduced in subsection 2.3, we will write :

$$\sigma_{\omega_T, \max} = \max_{K \in \omega_T} \sigma_K \quad \text{and} \quad \mu_{\omega_T, \min} = \min_{K \in \omega_T} \mu_K,$$

where

$$\omega_T = \bigcup_{\mathcal{N}_{h,m}(T) \cap \mathcal{N}_{h,m}(K) \neq \emptyset} K$$

is the *patch* of the tetrahedron $T \in \mathcal{T}_{h,m}$, with $\mathcal{N}_{h,m}(T)$ and $\mathcal{N}_{h,m}(K)$ denoting respectively the sets of the vertices of T and K .

Lemma 5.2. • For any $T \in \mathcal{T}_{hm}$ and $F \subset \partial T$, we have :

$$\eta_{T;1}^m \lesssim \sigma_T^{1/2} h_T \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_T + \mu_T^{-1/2} \|\mu^{1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|_T + \xi_T^m, \quad (87)$$

$$\eta_{F;1}^m \lesssim \max_{K \in \omega_F} \sigma_K^{1/2} h_T \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{\omega_F} + \min_{K \in \omega_F} \mu_K^{-1/2} \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|_{\omega_F} + \xi_{\omega_F}^m. \quad (88)$$

Using the dual norm $X(\Omega)'$, we also get a global in space result :

$$\eta_{T;1}^m \lesssim \sigma_T^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{X(\Omega)'} + \mu_T^{-1/2} \|\mu^{1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|_T + \xi_T^m, \quad (89)$$

$$\sum_{F \subset \partial T} \eta_{F;1}^m \lesssim \sigma_{\omega_T, \max}^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{X(\Omega)'} + \mu_{\omega_T, \min}^{-1/2} \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t_m)\|_{\omega_T} + \xi_{\omega_T}^m. \quad (90)$$

• For any $T \in \mathcal{T}_{hm}$, $T \subset \Omega_c$, and for any $F \subset \partial T$, we have :

$$\eta_{T;2}^m \lesssim \sigma_T^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_T,$$

$$\eta_{F;2}^m \lesssim \max_{K \in \omega_F} \sigma_K^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{\omega_F}.$$

Using the dual norm $X(\Omega)'$, we get a global in space result :

$$\eta_{T;2}^m \lesssim \sigma_T^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{X(\Omega)'},$$

$$\sum_{F \subset \partial T} \eta_{F;2}^m \lesssim \sigma_{\omega_T, \max}^{1/2} \left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{X(\Omega)'}. \quad (91)$$

Proof: The proof is based on a standard application of the bubble functions inverse inequalities: we have conveniently used the techniques in the proofs of Lemmas 4.3, 4.4 and 4.5 of [15] with the ones in the proof of Theorem 5.6 of [25].

In order to manage the different triangulations coming from the nonstationary nature of the problem, for example, for the estimation (87), we arbitrarily fixed $\tilde{T} \in \tilde{\mathcal{T}}_{hm}$, where $\tilde{\mathcal{T}}_{hm}$ is the conforming triangulation in common with \mathcal{T}_{hm} and \mathcal{T}_{hm-1} (as specified just above) and defined :

$$\mathbf{r}_{\tilde{T}}^m = \left(\mathbf{J}_{s,h}^m - \sigma \left(\frac{\mathbf{A}_h^m - \mathbf{A}_h^{m-1}}{\tau_m} + \nabla \widetilde{\varphi_h^m} \right) - \operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}_h^m) \right)_{\tilde{T}},$$

so that $\eta_{\tilde{T};1}^m = h_{\tilde{T}} \|\mathbf{r}_{\tilde{T}}^m\|_{\tilde{T}}$. Having proved inequality (87) for an arbitrary tetrahedron $\tilde{T} \in \tilde{\mathcal{T}}_{hm}$, the assertion on the triangulation $\tilde{\mathcal{T}}_{hm}$ (at the beginning of this paragraph) implies that :

$$(\eta_{T;1}^m)^2 \lesssim \sum_{\tilde{T} \in \tilde{\mathcal{T}}_{hm} : \tilde{T} \subset T} (\eta_{\tilde{T};1}^m)^2.$$

So we can extend the lower upper error bound for all the T belonging to the triangulation \mathcal{T}_{hm} (remarking that, for a regular triangulation, $h_{\tilde{T}} \sim h_T$ for $\tilde{T} \subset T$).

For this reason, from now on, we can directly work on the triangulation \mathcal{T}_{hm} , bearing in mind that we should work, in a first moment, on a suitable triangulation $\tilde{\mathcal{T}}_{hm}$, and, afterwards, extend the results to the triangulation \mathcal{T}_{hm} .

For the estimation (89), the difference of the proof with the analogous L^2 -estimation (87) lies on the extension of the domain of integration from T to all the domain Ω (thanks to the fact that the bubble function on T is zero outside of T), and the use of the relation :

$$\begin{aligned} \|\mathbf{r}_{\tilde{T}}^m b_{\tilde{T}}\|_{X(\Omega)} &\sim \|\mathbf{r}_{\tilde{T}}^m b_{\tilde{T}}\| + \|\operatorname{curl}(\mathbf{r}_{\tilde{T}}^m b_{\tilde{T}})\| \\ &\sim \|\mathbf{r}_{\tilde{T}}^m b_{\tilde{T}}\|_{\tilde{T}} + \|\operatorname{curl}(\mathbf{r}_{\tilde{T}}^m b_{\tilde{T}})\|_{\tilde{T}} \\ &\lesssim (1 + h_{\tilde{T}}^{-1}) \|\mathbf{r}_{\tilde{T}}^m\|_{\tilde{T}}, \end{aligned}$$

where $b_{\tilde{T}}$ denote the bubble function on \tilde{T} . The others dual estimations proceed similarly. \blacksquare

Lemma 5.3. For any $T \in \mathcal{T}_{hm}$ and for any $F \subset \partial T$, we have :

$$\eta_{T;3}^m \lesssim \sigma_T^{1/2} \|\sigma^{1/2} \mathbf{E}^m\|_T, \quad (91)$$

$$\eta_{F;3}^m \lesssim \max_{K \in \omega_F} \sigma_K^{1/2} \|\sigma^{1/2} \mathbf{E}^m\|_{\omega_F}. \quad (92)$$

Proof: For $T \in \mathcal{T}_{hm}$ we define :

$$\mathbf{r}_T^m = (\operatorname{div}(\sigma(\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \widetilde{\varphi_h^p})))|_T, \text{ so that } \eta_{T;3}^m = h_T \|\mathbf{r}_T^m\|_T. \quad (93)$$

From the inverse inequalities (3.19) and (3.20) of Lemma 3.3 of [15], using the fact that

$$\int_T \sigma(\mathbf{A}^m + \sum_{p=1}^m \tau_p \nabla \varphi^p) \cdot \varphi' d\mathbf{x} = 0,$$

derived from the semi-discrete weak formulation (18) with $\mathbf{A}' = 0$ (see (62)), and the property that $b_T = 0$ over ∂T , we can estimate \mathbf{r}_T^m :

$$\begin{aligned} \|\mathbf{r}_T^m\|_T^2 &\lesssim \|\mathbf{r}_T^m b_T^{1/2}\|_T^2 = \int_T \operatorname{div}(\sigma(\mathbf{A}_h^m + \sum_{p=1}^m \tau_p \nabla \widetilde{\varphi_h^p})) \cdot (\mathbf{r}_T^m b_T) d\mathbf{x} \\ &= \int_T \sigma \mathbf{E}^m \cdot \nabla(\mathbf{r}_T^m b_T) d\mathbf{x} \lesssim \sigma_T^{1/2} \|\sigma^{1/2} \mathbf{E}^m\|_T \|\nabla(\mathbf{r}_T^m b_T)\|_T \\ &\lesssim \sigma_T^{1/2} h_T^{-1} \|\sigma^{1/2} \mathbf{E}^m\|_T \|\mathbf{r}_T^m\|_T. \end{aligned} \quad (94)$$

Joining this result to (93), the estimation (91) follows. The estimation (92) is deduced in a similar manner, using the inverse inequalities (3.21), (3.22) and (3.23) of Lemma 3.3 of [15] and the extension operator F_{ext} in order to estimate the integral over $F \in \partial T$, which leads to an integral on the patch ω_F . \blacksquare

Now, a bound of the local spatial indicator can be stated.

Theorem 5.4. For any $T \in \mathcal{T}_{hm}$, we get :

$$\begin{aligned} \eta_{h,T}^m &\lesssim \max(\sigma_{\omega_T, \max}^{1/2}, \mu_{\omega_T, \min}^{-1/2}) \left(\left\| \sigma^{1/2} \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau_m} \right\|_{\omega_T} + \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_{A, h\tau}(t_m)\|_{\omega_T} \right. \\ &\quad \left. + \sigma_{\omega_T, \max}^{1/2} \|\sigma^{1/2} \mathbf{E}^m\|_{\omega_T} \right) + \xi_{\omega_T}^m, \end{aligned}$$

where $\eta_{h,T}^m = \sum_{j=1}^3 (\eta_{T;j}^m)^2 + \sum_{F \subset \partial T, F \in \mathcal{F}_{hm}^{int}} (\eta_{F;1}^m)^2$ denotes the space error estimator on the element T .

Proof: Since $\eta_{h,T}^m \lesssim \eta_{T;1}^m + \eta_{T;2}^m + \eta_{T;3}^m + \sum_{F \subset \partial T, F \in \mathcal{F}_{hm}^{int}} (\eta_{F;1}^m + \eta_{F;2}^m + \eta_{F;3}^m)$, the conclusion is a direct consequence of Lemmas 5.2 and 5.3. \blacksquare

5.3 Efficiency of the whole error

Theorem 5.5. For all $n = 1, \dots, N$, we have :

$$\eta^n = \sum_{m=1}^n (\eta_\tau^m)^2 + \tau_m (\eta_h^m)^2 \lesssim C_{\sigma_{\min}, \sigma_{\max}, \mu_{\min}, \mu_{\max}} \left(\bar{E}(t_n)^2 + \|\mathbf{J}_s - \pi_\tau \mathbf{J}_s\|_{L^2(0, t_n; X(\Omega)')}^2 + \sum_{m=1}^n \tau_m (\xi^m)^2 \right),$$

where $C_{\sigma_{\min}, \sigma_{\max}, \mu_{\min}, \mu_{\max}}$ denotes a constant which only depends on the values of $\sigma_{\min}, \sigma_{\max}, \mu_{\min}$ and μ_{\max} .

Proof: The result is a direct application of Lemma 5.1, the relations (89), (90) of Lemma 5.2, Lemma 5.3 and the use of relations (33) and the left hand side of (31) with $\mathbf{v}_\tau = \mathbf{e}_{A,h\tau}$. Moreover we recall that :

$$\sum_{m=1}^n \tau_m \left\| \sigma^{1/2} \frac{E^m - E^{m-1}}{\tau_m} \right\|_{X(\Omega)'}^2 = \left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,h\tau} + \nabla e_{\varphi,h\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2.$$

So we can write :

$$\begin{aligned} & \sum_{m=1}^n (\eta_\tau^m)^2 + \tau_m (\eta_h^m)^2 \\ & \lesssim \max(\mu_{\max}, \mu_{\max} \sigma_{\min}, \mu_{\min}^{-1}) \left(\left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,\tau} + \nabla e_{\varphi,\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2 + \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,\tau}(t) \right\|_{L^2(0,t_n;\mathbf{L}^2(\Omega))}^2 \right. \\ & \quad \left. + \left\| \mathbf{J}_s - \pi_\tau \mathbf{J}_s \right\|_{L^2(0,t_n;X(\Omega)')}^2 \right) + \max(\mu_{\min}^{-1}, 1) \left\| \mu^{-1/2} \operatorname{curl} \mathbf{e}_{A,h\tau}(t) \right\|_{L^2(0,t_n;\mathbf{L}^2(\Omega))}^2 \\ & \quad + \sigma_{\max} \left(\left\| \sigma^{1/2} (\partial_t \mathbf{e}_{A,h\tau} + \nabla e_{\varphi,h\tau}) \right\|_{L^2(0,t_n;X(\Omega)')}^2 \right. \\ & \quad \left. + \sum_{m=1}^n \tau_m \left\| \sigma^{1/2} (\mathbf{e}_{A,h\tau}(t_n) + \nabla \int_0^{t_n} e_{\varphi,h\tau}(s) ds) \right\|_{\Omega_c}^2 \right) + \sum_{m=1}^n \tau_m (\xi^m)^2. \end{aligned}$$

■

6 Numerical validation

In this section, numerical experiments are performed to underline and confirm some of our theoretical predictions with the use of the software Code_Carmel. It consists of an analytical test, by solving the fully discrete formulation (19) on the time interval $[0, T]$ and on the domain $\Omega = (-1.2, 1.2)^3$, where the inductor domain Ω_c is defined by $\Omega_c = (-1, 1)^3$, as shown in Figure 2. Here we consider $\mu \equiv 1$ in Ω and $\sigma \equiv 0$ in $\Omega \setminus \Omega_c$, and define the analytical

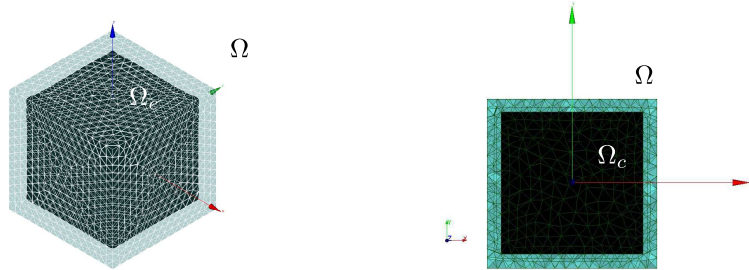


Figure 2: Configuration and regular mesh of the domains Ω and Ω_c .

solution (\mathbf{A}, φ) of the $\mathbf{A} - \varphi$ formulation (8) given by :

$$\mathbf{A}(t, x, y, z) = \sin(2\pi t) \operatorname{curl} \begin{pmatrix} f(x, y, z) \\ 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega \times [0, T],$$

where

$$f(x, y, z) = \begin{cases} (x^2 - 1)^4 (y^2 - 1)^4 (z^2 - 1)^4 & \text{in } \Omega_c, \\ 0 & \text{otherwise,} \end{cases}$$

and $\varphi \equiv 0$ in Ω_c . The source term \mathbf{J}_s is consequently deduced from equation (8). Note further that in that case the support of \mathbf{J}_s is the whole conductor domain Ω_c . For the simulations, we use an uniform discretization in time and a regular mesh family $\mathcal{T}_{h,m}$ of Ω .

We aim to check the expected rates of convergence of the numerical scheme, both in space and time, and also to underline the behavior of the estimators. The parameters corresponding to Test 1, Test 2 and Test 3 are given

	Test 1	Test 2	Test 3
$\sigma_{ \Omega_c}$	1	1	10^3
(h_1, τ_1)	(0.2653, 0.0590)	(0.2653, 0.0304)	(0.1366, 0.125)
(h_2, τ_2)	(0.1874, 0.0416)	(0.1874, 0.0304)	(0.1366, 0.0625)
(h_3, τ_3)	(0.1366, 0.0304)	(0.1366, 0.0304)	(0.1366, 0.0313)

Table 1: Parameters corresponding to the three tests.

in Table 1.

In Test 1, we take $\sigma_{|\Omega_c} = 1$ and we consider three meshes from the coarse one to the more refined one, corresponding to decreasing values of h denoted $h_1 > h_2 > h_3$. The discretization is uniform in time, and the time step τ_i is chosen to be proportionnal to the value of h_i , $1 \leq i \leq 3$. We plot in Figure (3a) the error $e(t_N)$ defined by (81) as a function of h in a log-log scale. We can see that the numerical solution $(\mathbf{A}_h^n, \varphi_h^n)$ converges towards the exact one (\mathbf{A}, φ) at order one, as theoretically expected. Now, in order to illustrate Theorem 4.11, we also compute the estimator η^N defined by (30), and display in Figure (3b) the so-called global effectivity index given by :

$$E_G = \frac{e(t_N)}{\eta^N}.$$

As we can see, the effectivity index converges towards a constant when the couple (h, τ) goes towards zero. This illustrates the reliability of the proposed estimator, having in mind that the other terms arising in the right-hand side of (82) correspond to higher order terms.

In Test 2, we want to illustrate the behavior of the spatial part of the estimator. To do so, we still take $\sigma_{|\Omega_c} = 1$ and the same meshes as the ones used for Test 1, but this time we choose for all computations $\tau = 0.0304$, so that the error in space is significantly larger than the one in time (we observe that while decreasing the time step, the error remains constant). As expected, we observe a convergence at order one in h of the error $e(t_N)$ (see Figure (3c)). Moreover, if we now introduce the space effectivity index given by :

$$E_S = \frac{e(t_N)}{\left(\sum_{m=1}^N \tau_m (\eta_h^m)^2 \right)^{1/2}},$$

we see in Figure (3d) that it converges towards a constant when h goes towards zero, showing in that case the equivalence between the error and the spatial part of the estimator.

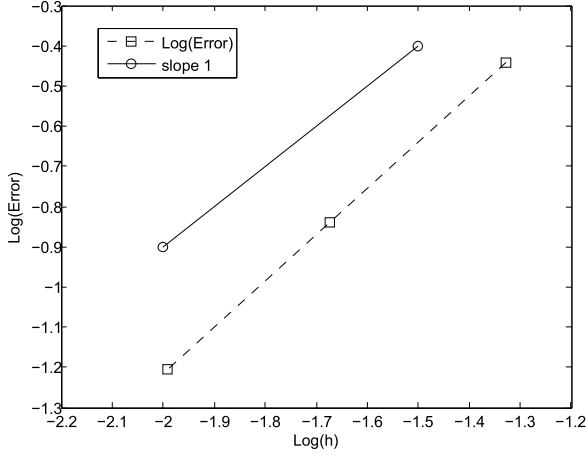
Similarly, in Test 3 we want to illustrate the behavior of the temporal part of the estimator. Hence, we now consider $\sigma_{|\Omega_c} = 10^3$ in order to voluntary increase the error in time, and we take the same mesh for all computations corresponding to $h = 0.1366$. As expected, we observe a convergence at order one in τ of the error $e(t_N)$ (see Figure (3e)). Moreover, if we now introduce the time effectivity index given by :

$$E_T = \frac{e(t_N)}{\left(\sum_{m=1}^N (\eta_\tau^m)^2 \right)^{1/2}},$$

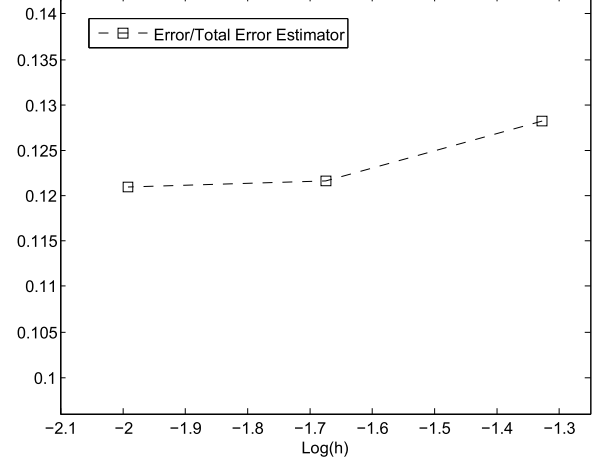
we see in Figure (3f) that it converges towards a constant when τ goes towards zero, showing in that case the equivalence between the error and the temporal part of the estimator.

Acknowledgments

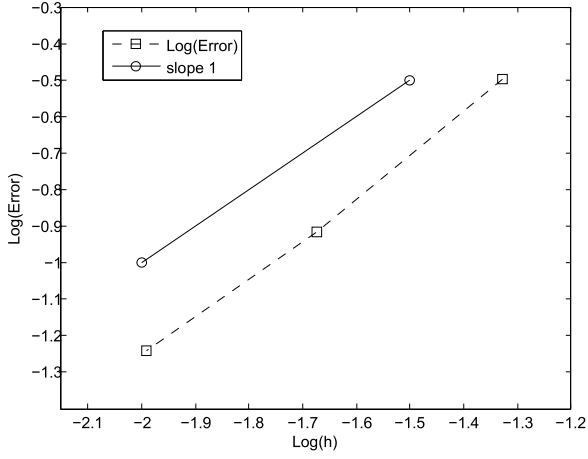
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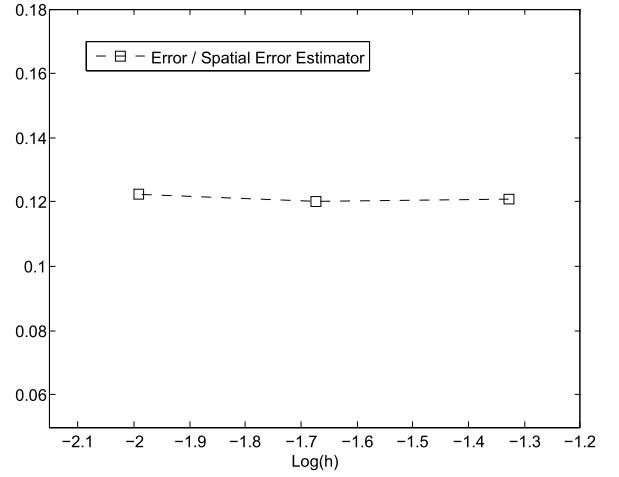
(a) Error convergence: refinement in (h, τ) .



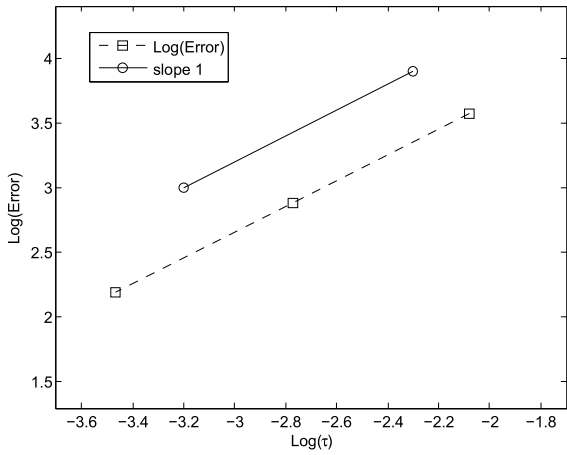
(b) Global effectivity E_G while refining in (h, τ) .



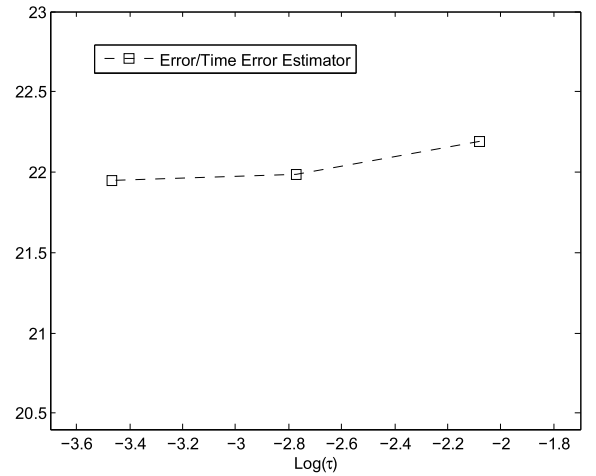
(c) Error convergence in space : refinement in h .



(d) Spatial effectivity E_S while refining in h .



(e) Error convergence in time : refinement in τ .



(f) Temporal effectivity E_T while refining in τ .

Figure 3: Plots of the rate of convergence of the error (Figures (3a),(3c),(3e)), and of the effectivity indices (Figures (3b),(3d),(3f)).

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