Stabilization of a Class of Mixed ODE-PDE port-Hamiltonian Systems with Strong Dissipation Feedback ⋆

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Abstract

This paper deals with the asymptotic stabilization of a class of port-Hamiltonian (pH) 1-D Partial Differential Equations (PDE) with spatial varying parameters, interconnected with a class of linear Ordinary Differential Equations (ODE), with control input on the ODE. The class of considered ODE contains the effect of a proportional term, that can be considered as the proportional action of a controller or a spring in case of mechanical systems. In this particular case of study, it is not possible to directly add damping on the boundary of the PDE. To remedy this problem we propose a control law that makes use of a "strong feedback" term. We first prove that the closed-loop operator generates a contraction strongly continuous semigroup, then we address the asymptotic stability making use of a Lyapunov argument, taking advantage of the pH structure of the original system to be controlled. Furthermore, we apply the proposed control law for the stabilization of a vibrating string with a tip mass and we show the simulation results compared with the application of a simple PD controller.

Key words: Distributed-parameter system, Strong feedback control, Asymptotic stability, Numerical simulations, port-Hamiltonian systems.

1 Introduction

In this paper we are interested in the control design for a class of systems described by a set of hyperbolic Partial Differential Equations (PDE) coupled at the boundary with a set of Ordinary Differential Equations (ODE). We refer to this class of systems as mixed ODE-PDE (m-ODE-PDE). Control design and stability analysis for m-ODE-PDE has raised a significant attention from many researchers because of its wide range of applications. In particular, m-PDE-ODE equations enclose models of rotating and/or translating beams [1, 2, 3], controlled nanotweezer used for DNA manipulation [4] as well as electric transmission lines with load [5].

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In this work, we make use of the distributed parameter port-Hamiltonian (pH) approach introduced in [6] for the modelling and control of physical systems. This formalism has been adapted for the definition of pH Boundary Control Systems [7], where a simple matrix condition suffices to characterize a well-posed (in the Hadamard sense) system [8]. A complete exposition with some further extensions of these first results can be found in [9, 10]. Well-posedness and stabilization problems have been studied in case of static feedback [11], dynamic linear feedback [4] and dynamic nonlinear feedback [12]. In case of dynamic linear feedback, the energy-shaping technique can be used to design the dynamic linear controller, assuring the asymptotic stability of the m-ODE-PDE closed-loop system [13].
The control design and the stabilization problem of m-ODE-PDE have been successfully tackled in different control scenarios using backstepping techniques. In particular, the stabilization problem for sandwiched parabolic m-PDE-ODE systems with control on the PDE boundaries and on the set of ODE has been solved on [14, 15], respectively. Moreover, backstepping con-

trol design has been applied to obtain the exponential stabilization of a class of heterodirectional hyperbolic m-ODE-PDE with actuation on the PDE boundaries [16]. Further, this result has been extended firstly to the same class of systems with space dependent parameters [17], and secondly for a class of heterodirectional m-ODE-PDE-ODE systems with actuation on one set of ODE [18]. In this latter work, exponential stability is achieved trough a control law that, to be implementable, needs the use of an observer. In this work, we propose an asymptotically stabilizing static control strategy for a different class of hyperbolic m-ODE-PDE systems, that in most applications does not need the implementation of a dynamical controller.

It has been proven that linear operator equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \quad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{1}$$

with A generator of a bounded group (*i.e.* $\sup_{t\in\mathbb{R}} ||T(t)|| < \infty$) on a infinite dimensional state space X, and input matrix $B \in \mathcal{L}(\mathbb{R}^n, X)$, are not exponentially stabilizable with classical bounded linear feedback $u = -Fx$ with $F \in \mathcal{L}(X, \mathbb{R}^n)$(see in Lemma 8.4.1 of [19]). However, it has been shown that it is possible to use a "strong dissipation" feedback term $u = K_p \frac{\partial}{\partial t}(Cx)$ instead of the classical dissipation term. This type of feedback has already been applied and studied for specific sets of mixed ODE-PDE. In fact, the *strong dissipation* feedback has been used in [20] to exponentially stabilize a wave equation with dynamic boundary conditions or in [21] for an Euler Bernoulli beam with a tip mass (see also [22, 23] for other examples). Compared to the these previous works that use the *strong dissipation* feedback [24], we extend the class of linear systems that could be interconnected at the boundary, allowing the presence of a position control or, equivalently, the presence of a spring. The combined *strong dissipation* and position control has already been obtained using backstepping techniques in [25] for the specific case of a wave equation with dynamic boundary conditions. In [25], the authors carried out the analysis without position control term, concluding exponential stability of the closed-loop system. Besides, the *strong dissipation* with position control applied to a translating and rotating Timoshenko's beam in contact scenario has already been studied in [26], where exponential stability has been proved.

In this paper we generalize the concept of combined *strong dissipation* and position control for a class m-pH systems that encloses a variety of practical applications. Moreover, we propose a Lyapunov argument to show the asymptotic stability of the closed-loop system. The use of a Lyapunov function instead of the classical frequency domain methods [26] opens the possibility of extending this method to the case of nonlinearities in the set of ODE. The stability proof makes use of the properties of infinite dimensional pH systems. Moreover, the effectiveness of the proposed control law is shown via the application on a clamped vibrating string with a tip mass on the free side, together with a simulation comparison with a simple PD control law. Since the *strong dissipation* feedback is obtained as the time derivative of a signal, we also propose some numerical simulations in case noise is added to the measurement.

The paper is organized as follows. In Section 2 we present the class of considered PDE-ODE together with the proposed control law and the consequent closed-loop system. Sections 3 and 4 contain the C_0-semigroup generation and the asymptotic stability theorems for the closed-loop operator respectively, and they correspond to the main contributions of the proposed work. The application and simulation results for a vibrating string with a tip mass are presented in Section 5. Finally, some concluding remarks and comments to future works are given in Section 6.

2 Preliminaries

In this paper we consider a plant composed by a set of PDE defined on a one-dimensional spatial domain interconnected with a set of ODE. In particular, we assume that the control actions are active on the ODE. This class of models is of practical interest because it includes moving vibrating beams or strings, where the control action acts on a boundary inertia. We begin by defining the set of first order 1D-spatial domain pH PDE

$$\frac{\partial z}{\partial t}(\xi, t) = P_1 \frac{\partial}{\partial \xi} \mathcal{H} z(\xi, t) + (P_0 + G_0)\mathcal{H} z(\xi, t), \tag{2}$$

where $\xi \in [a, b]$, $P_1 \in M_n(\mathbb{R})$ [1] is a non-singular symmetric matrix, $P_0 = -P_0^T \in M_n(\mathbb{R})$, $z \in L^2([a, b], \mathbb{R}^n)$, $G_0 \in M_n(\mathbb{R})$ with $G_0 \leq 0$ and $\mathcal{H} \in L^\infty([a, b]; \mathbb{R}^{n\times n})$ such that $mI \leq \mathcal{H}(\xi) \leq MI$ for $\xi \in [a, b]$ and constants $M > m > 0$, where I is the identity matrix. The boundary flows and efforts are defined following [8]:

$$\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}z)(a, t) \\ (\mathcal{H}z)(b, t) \end{bmatrix}. \tag{3}$$

such that $\frac{d}{dt}(z^T \mathcal{H} z) = f_\partial^T e_\partial$. For the above set of PDE (2) we assume that a part of the boundary conditions are homogeneous, whereas the other part can be intercon-

[1] $M_n(\mathbb{R})$ denotes the space of real $n \times n$ matrices.

nected. Thus, we consider the following input operators:

$$\begin{aligned}\mathcal{B}_1 z(t) &= W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = u_z(t) \\ \mathcal{B}_2 z(t) &= W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = 0.\end{aligned} \tag{4}$$

The output operators are split accordingly:

$$\begin{aligned}\mathcal{C}_1 z(t) &= W_{C,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = y_z(t) \\ \mathcal{C}_2 z(t) &= W_{C,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix} = \tilde{y}_z(t),\end{aligned} \tag{5}$$

such that $\operatorname{rank}(W_{B,1}) = \operatorname{rank}(W_{C,1}) = m$, and $\operatorname{rank}(W_{B,2}) = \operatorname{rank}(W_{C,2}) = n - m$. Note that the output $y_z(t)$ has the same dimension as the input $u_z(t)$. We define the complete input and output operators as the composition of the previously defined operators

$$\begin{aligned}\mathcal{B}z &= \begin{bmatrix}\mathcal{B}_1 z \\ \mathcal{B}_2 z\end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_{B,2}\end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} = W_B\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} \\ \mathcal{C}z &= \begin{bmatrix}\mathcal{C}_1 z \\ \mathcal{C}_2 z\end{bmatrix} = \begin{bmatrix} W_{C,1} \\ W_{C,2}\end{bmatrix}\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} = W_C\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix}.\end{aligned} \tag{6}$$

We define the state space $Z = L^2([a,b],\mathbb{R}^n)$ with inner product $\langle z_1, z_2\rangle_Z = \langle z_1, \mathcal{H}z_2\rangle_{L^2}$ and associated norm $||z||_Z = \sqrt{\langle z, z\rangle_Z}$. We define the operator

$$\mathcal{J}z = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + (P_0 + G_0)(\mathcal{H}z). \tag{7}$$

Throughout the rest of the paper, we use the following set of assumptions

Assumption 1 *For the operator $\mathcal{J}$ and the PDE* (2)-(5) *the following holds:*

(1) The matrix $\left[\begin{smallmatrix} W_B \\ W_C\end{smallmatrix}\right]$ is a full rank $2n \times 2n$ matrix.
(2) For classical solutions of (2)–(5) *there holds $\langle z(t), \mathcal{J}z(t)\rangle_Z = \dot{H}(t) \leq u_z(t)^T y_z(t)$, with $H(t) = \frac{1}{2}||z(t)||_Z^2$.*

From Assumption 1 directly follows that the system

$$\begin{aligned}\dot{z} &= \mathcal{J}z \\ \mathcal{B}_1 z &= u_z,\end{aligned} \tag{8}$$

with domain,

$$D(\mathcal{J}) = \left\{ z \in Z \mid \mathcal{H}z \in H^1([a,b],\mathbb{R}^n), \mathcal{B}_2 z = 0\right\} \tag{9}$$

is a boundary control system [10, Theorem 11.3.2]. This means that for $u_z \in C^2([0,\infty),\mathbb{R}^m)$, $u_z(0) = \mathcal{B}_1\mathcal{H}z(0)$ and $\mathcal{H}z(0) \in D(\mathcal{J})$ the system (8)-(9) has a unique classical solution. It has been shown in [27] that the inequality in item (2) of Assumption 1 holds also when $G_0 = 0$. Therefore, from now on we take $\mathcal{J}z = P_1\frac{\partial}{\partial\xi}(\mathcal{H}z) + P_0(\mathcal{H}z)$. For stability, this corresponds to the worst case scenario, being $G_0 > 0$ less restrictive. We now introduce a technical assumption regarding the solution of equation (2) alone with a specific set of boundary conditions, that will be of crucial importance in the Lyapunov's stability analysis.

Assumption 2 *The system* (2) *subject to the following over-constraining set of boundary conditions*

$$\begin{cases}\mathcal{B}z(t) = 0 \\ \mathcal{C}_1 z(t) = y_z^*\end{cases} \tag{10}$$

with $y_z^* \in \mathbb{R}^m$ constant, admits $z = 0$ as the only solution, so that $y_z^* = 0$.

This assumption can directly be checked in the application case under study, as shown in Section 5 for a vibrating string with a particular set of boundary conditions. This assumption is strongly related to the approximate observability property. This connection is shown to hold for the general class of linear systems

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t) \quad x(0) = x_0 \\ y(t) &= \mathcal{C}x(t),\end{aligned} \tag{11}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on a Hilbert space X, and $\mathcal{C}$ is an admissible output operator from X to the output space Y. In the following, $\Sigma(\mathcal{A},\mathcal{C})$ denotes the linear system (11).

Proposition 3 *For the linear system $\Sigma(\mathcal{A},\mathcal{C})$, the following statements are equivalent*

i) If $y(t)$ is constant, then $x_0 = 0$.
ii) $\Sigma(\mathcal{A},\mathcal{C})$ is approximately observable and its only equilibrium point is 0.

PROOF.
i) $\Rightarrow$ ii) Let x_0 be such that the output $y(t)$ of $\Sigma(\mathcal{A},\mathcal{C})$ is identically zero. By i) we see that $x_0 = 0$, and thus $\Sigma(\mathcal{C},\mathcal{A})$ is approximately observable.
Let x_{eq} be an equilibrium solution, then $0 = \mathcal{A}x_{eq}$, and the corresponding output (corresponding to $x(t) \equiv x_{eq}$) is $y(t) = \mathcal{C}x_{eq}$. This is constant, and so by i) $x_{eq} = 0$.
ii) $\Rightarrow$ i) Let $y(t)$ be a constant output of (11) and let $x(t)$ be the corresponding state trajectory. Define $y_\Delta(t) := y(t+t_1) - y(t) = 0$, $t_1 > 0$. The corresponding state trajectory is $x_\Delta(t) = x(t+t_1) - x(t)$. By approximate observability we have that $x_\Delta(0) = 0$. Thus $x(t_1) - x(0) =$

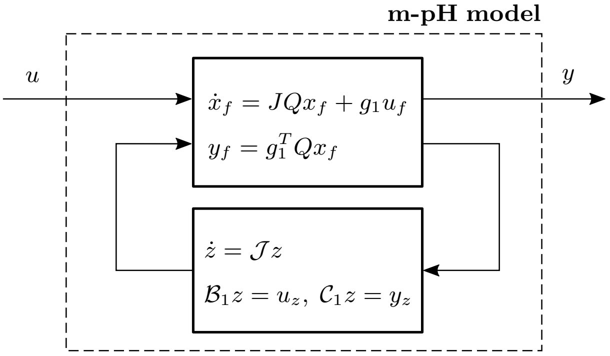

Fig. 1. m-pH model.

0. Since t_1 was arbitrary, we have that $x(t) \equiv x(0)$, and thus $x(0)$ is an equilibrium solution. By assumption, we conclude that $x(0) = x_0 = 0$. $\square$

The considered set of ODE takes the form

$$\begin{cases} \dot{x}_f(t) = JQx_f(t) + g_1 u_f(t) \\ y_f(t) = g_1^T Q x_f(t) = M^{-1} p(t), \end{cases} \tag{12}$$

where $x_f = \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right] \in \mathbb{R}^{2m}$, $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} K & 0 \\ 0 & M^{-1} \end{smallmatrix}\right]$ and $g_1 = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$. The matrices are assumed to be $K = \operatorname{diag}([k_1 \ \dots \ k_m]) \in \mathbb{R}^{m\times m}$, $M = \operatorname{diag}([m_1 \ \dots \ m_m]) \in \mathbb{R}^{m\times m}$, $K, M > 0$. In equation (12), the effect of a spring (or multiple springs) has been included, and it can either represent an actual part of the model or a proportional control action.

We interconnect the finite dimensional system (12) with the infinite dimensional one (2)–(5) with the power preserving interconnection

$$u_z(t) = y_f(t), \quad u_f(t) = -y_z(t) + u(t), \tag{13}$$

to obtain the mixed ODE-PDE pH (m-pH) model depicted in Figure 1. The m-pH model is described by the equations

$$\begin{cases} \dot{x}_p = \begin{bmatrix} \mathcal{J} & 0 \\ -g_1\mathcal{C}_1 & JQ \end{bmatrix} x_p + \begin{bmatrix} 0 \\ g_1 \end{bmatrix} u = \mathcal{A}_p x_p + Bu \\ y = \begin{bmatrix} 0 & g_1^T Q \end{bmatrix} x_p = \mathcal{C}_p x_p = M^{-1} p \end{cases} \tag{14}$$

where $u \in \mathbb{R}^m$ is the new input, $x_p = \left[\begin{smallmatrix} z \\ x_f \end{smallmatrix}\right] \in X_p = L^2([a,b],\mathbb{R}^n) \times \mathbb{R}^{2m}$ is the extended plant's state, and

$$\begin{aligned} \mathcal{D}(\mathcal{A}_p) = \big\{ & x_p \in X_p \mid \mathcal{H}z \in H^1([a,b],\mathbb{R}^n), \\ & \mathcal{B}_1 z = \mathcal{C}_p x_p, \mathcal{B}_2 z = 0 \} \end{aligned} \tag{15}$$

is the domain of the operator $\mathcal{A}_p$. We propose the following control law

$$\begin{aligned} u = & -R_c M^{-1} p + (I - R_c M^{-1} K_p)\mathcal{C}_1 z \\ & -K_p \tfrac{\partial}{\partial t}(\mathcal{C}_1 z), \end{aligned} \tag{16}$$

where the last term is known in the literature of stabilization of mixed PDE-ODE systems as *strong dissipation* feedback, and the matrices are defined as $R_c = \operatorname{diag}([r_1 \ \dots \ r_m]) \in \mathbb{R}^{m\times m}$ and $K_p = \operatorname{diag}([k_{p,1} \ \dots \ k_{p,m}]) \in \mathbb{R}^{m\times m}$, where $r_i, k_{p,i} > 0$ for $i \in \{1, \dots, m\}$. A very similar control law has been obtained in [25] using a baskstepping control design. In case of distributed parameter mechanical systems, the boundary output normally corresponds to velocity and stress measurements. Therefore, the *strong dissipation* term $K_p \frac{\partial}{\partial t}(\mathcal{C}_1)$ can be obtained from the acceleration measurement and from the time derivative of the strain measurement. In the context of control of flexible beams, this term is also referred to as "strain rate feedback" [28]. In the case of boundary controlled flexible beams, this term can practically be obtained from the approximated and filtered time derivative of the strain measurement [22, Section 4]. The closed-loop system with the proposed control law writes

$$\begin{cases} \dot{z} & = \mathcal{J} z \\ \dot{x}_f & = (J - R)Qx_f - g_1 R_c M^{-1} K_p \mathcal{C}_1 z \\ & \quad -g_1 K_p \frac{d}{dt}(\mathcal{C}_1 z), \end{cases} \tag{17}$$

where $R = \left[\begin{smallmatrix} 0 & 0 \\ 0 & R_c \end{smallmatrix}\right]$. In order to analyse the obtained closed-loop system we perform the change of variables $\eta = p + K_p \mathcal{C}_1 z$, such to rewrite the system as

$$\begin{cases} \dot{z} = \mathcal{J} z \\ \dot{v} = (J - R)Qv - g_2 M^{-1} K_p \mathcal{C}_1 z, \end{cases} \tag{18}$$

where, $g_2 = \left[\begin{smallmatrix} I & 0 \end{smallmatrix}\right]^T$ and $v = \left[\begin{smallmatrix} q^T & \eta^T \end{smallmatrix}\right]^T \in \mathbb{R}^{2m}$. This system can be written as a linear operator equation of the form

$$\dot{x} = \mathcal{A}x = \begin{bmatrix} \mathcal{J} & 0 \\ -g_2 M^{-1} K_p \mathcal{C}_1 & (J - R)Q \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} \tag{19}$$

with domain defined as

$$\begin{aligned} & D(\mathcal{A}) = \big\{ x \in L^2([a,b],\mathbb{R}^n) \times \mathbb{R}^{2m} \mid \mathcal{H}z \in H^1, \\ & \mathcal{B}_2 z = 0, \mathcal{B}_1 z = M^{-1}(\eta - K_p \mathcal{C}_1 z) \big\} \end{aligned} \tag{20}$$

and state defined as $x = \left[\begin{smallmatrix} z^T & v^T \end{smallmatrix}\right]^T$.

3 Strongly continuous semigroup generation

In this section we investigate which type of solution function is generated by the closed-loop operator (19)-(20). The closed-loop operator is defined as the non-power preserving interconnection between an infinite and a finite dimensional linear pH systems. Since the interconnection is not power preserving, it is not possible to show the contraction C_0-semigroup generation in $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the energy norm, as in classical interconnected ODE-PDE pH systems [9]. Hence, in the next theorem we show that the closed-loop operator generates a contraction C_0-semigroup in $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with a special (energy-like) weighted norm.

Theorem 4 *Under Assumption 1, there exists a weighted $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ space such that the closed-loop operator* (19) *with domain defined by* (20) *generates a contraction C_0-semigroup on this space, provided $r_i^2 > m_i k_i \quad \forall i\in\{1,\dots,m\}$.*

PROOF. Let $IP(L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m},\mathbb{R})$ be the set of real inner products in $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$. Define the set of inner products parametrized by the matrix Λ

$$\begin{aligned}&\Gamma=\{\gamma_\Lambda\in IP(L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m},\mathbb{R})\mid\\&\gamma_\Lambda(x_1,x_2)=\langle x_1,x_2\rangle_\Gamma=\langle z_1,z_2\rangle_Z+v_1^TM_vv_2\},\end{aligned}\tag{21}$$

where

$$M_v=\begin{bmatrix}\Lambda R_cK_p^{-1} & K_p^{-1}\\ K_p^{-1} & 2\Lambda^{-1}R_c^{-1}K_p^{-1}\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\tag{22}$$

and $\Lambda=\operatorname{diag}([\alpha_1\ \dots\ \alpha_m])$ and $\alpha_i>0$ for $i=\{1,\dots,m\}$. The inner products in Γ are parametrized according to the α_i parameters. Using Schur complements it is easy to show that the matrix M_v is strictly positive definite. In fact, since all the matrices in (22) are diagonal strictly positive definite, A_{22} and $A_{11}-A_{12}A_{22}^{-1}A_{21}$ are strictly positive definite matrices, from which it follows the positive definitiveness of M_v. Hence, the inner product (21) is well-defined for any selection of the weighting parameters α_i. Using the Lumer-Phillips' Theorem (see Theorem 6.1.7 of [10]), we have to show that $\gamma(x,\mathcal{A}x,\Lambda)\le 0$ and that $\operatorname{Ran}(\lambda I-\mathcal{A})=X$ for some $\lambda>0$. We start by the dissipativity of the operator $\mathcal{A}$, taking into account that $\mathcal{C}_1 z=y_z$ and point (2) of Assumption 1

$$\begin{aligned}\gamma_\Lambda(x,\mathcal{A}x)&=\langle z,\mathcal{J}z\rangle_Z-v^TM_vg_2M^{-1}K_py_z\\&+v^TM_v(J-R)Qv\\&=u_z^Ty_z-v^TM_vg_2M^{-1}K_py_z\\&+v^TM_v(J-R)Qv\\&=M^{-1}(\eta-K_py_z)^Ty_z\\&-2\eta^T\Lambda^{-1}(R_c^{-1}K_p^{-1}Kq+K_p^{-1}M^{-1}\eta)\\&-\eta^TM^{-1}y_z-q^TK_p^{-1}R_cM^{-1}\eta\\&-q^TK_p^{-1}Kq+q^T\Lambda R_cK_p^{-1}M^{-1}\eta\\&+\eta^TK_p^{-1}M^{-1}\eta-q^T\Lambda R_cM^{-1}y_z.\end{aligned}\tag{23}$$

Then, after some computation, we obtain

$$\begin{aligned}\gamma_\Lambda(x,\mathcal{A}x)&\le-y_z^TM^{-1}K_py_z-q^TK_p^{-1}Kq\\&-\eta^T(2\Lambda^{-1}K_p^{-1}M^{-1}-K_p^{-1}M^{-1})\eta\\&-q^T\Lambda R_cM^{-1}y_z+\eta^T\left[R_cK_p^{-1}M^{-1}\right.\\&\left.(\Lambda-I)-2\Lambda^{-1}R_c^{-1}K_p^{-1}K\right]q.\end{aligned}\tag{24}$$

The latter inequality can be rewritten in matrix form

$$\gamma(x,\mathcal{A}x,\Lambda)\le-\begin{bmatrix}\eta^T & q^T & y_z^T\end{bmatrix}P\begin{bmatrix}\eta\\ q\\ y_z\end{bmatrix}\tag{25}$$

with $P=\begin{bmatrix}P_{11} & P_{12} & 0\\ P_{12}^T & P_{22} & P_{23}\\ 0 & P_{23}^T & P_{33}\end{bmatrix}$, and

$$\begin{aligned}&P_{11}=(2I-\Lambda)K_p^{-1}M^{-1}\Lambda^{-1},\quad P_{22}=K_p^{-1}K\\&P_{12}=\tfrac{1}{2}R_cK_p^{-1}M^{-1}(I-\Lambda)+\Lambda^{-1}R_c^{-1}K_p^{-1}K\\&P_{23}=+\tfrac{1}{2}\Lambda R_cM^{-1},\quad P_{33}=M^{-1}K_p.\end{aligned}\tag{26}$$

Since $P_{33}>0$ we have $P\ge0$ if and only if

$$\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}-\begin{bmatrix}0\\ P_{23}\end{bmatrix}P_{33}^{-1}\begin{bmatrix}0 & P_{32}\end{bmatrix}\ge0.\tag{27}$$

Thus if and only if

$$\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}-P_{23}P_{33}^{-1}P_{32}\end{bmatrix}\ge0.\tag{28}$$

So if $P_{11}>0$, then $P\ge0$ if and only if

$$P_{22}-P_{23}P_{33}^{-1}P_{32}-P_{21}P_{11}^{-1}P_{12}\ge0.\tag{29}$$

Since all the matrices are diagonal, $P_{11} > 0$ if and only if $\Lambda < 2$ meaning that $0 < \lambda_i < 2$ for $i = \{1, \dots, m\}$. If $\lambda_i = 2$ for an index i, then it is easy to show that (28) cannot hold. So we need $P_{11} > 0$. Since $P_{11}, P_{33} > 0$ and all matrices are diagonal, condition (29) is equivalent to

$$P_{11}P_{22}P_{33} - P_{23}^2 P_{11} - P_{12}^2 P_{33} \geq 0. \tag{30}$$

After some computations and using the matrices definition (26), the left hand side of the last inequality becomes

$$-\frac{1}{4}(-2KM + \Lambda R_c^2)^2 K_p^{-1} \Lambda^{-2} M^{-3} R_c^{-1} \tag{31}$$

so always ≤ 0. The only choice to have (30) fulfilled is therefore

$$\Lambda = 2KMR_c^{-2}. \tag{32}$$

Since $0 < \Lambda < 2$ we need

$$2KMR_c^{-2} < 2 \quad \Leftrightarrow \quad R_c^2 > KM, \tag{33}$$

that since the matrices are diagonal is verified by assumption.

The range condition consists in finding for a certain $\lambda > 0$, $(z, v) \in D(\mathcal{A})$ such that

$$\lambda \begin{bmatrix} z \\ v \end{bmatrix} - \mathcal{A} \begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} f_z \\ f_v \end{bmatrix}, \quad \forall \begin{bmatrix} f_z \\ f_v \end{bmatrix} \in X. \tag{34}$$

Writing the former equation in all its components

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ (\lambda I - (J - R)Q)v + g_2 M^{-1} K_p \mathcal{C}_1 z = f_v \\ \mathcal{B}_1 z = +g_1^T Q v - M^{-1} K_p \mathcal{C}_1 z, \ \mathcal{B}_2 z = 0 \end{cases} \tag{35}$$

and taking into account that $(\lambda I - (J - R)Q)^{-1}$ exists for $\lambda > 0$ since $J = -J^T$ and $R \geq 0$, we solve v in the second equation and substitute it in the third one. The problem becomes finding z such that $\mathcal{H}z \in H^1([a, b], \mathbb{R}^n)$ and

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ \mathcal{B}_1 z + (g_1^T Q(\lambda I - (J - R)Q)^{-1} g_2 + I)M^{-1} K_p \mathcal{C}_1 z = \tilde{f}_v \\ \mathcal{B}_2 z = 0 \end{cases} \tag{36}$$

where $\tilde{f}_v = g_1^T Q(\lambda I - (J - R)Q)^{-1} f_v$. Next, we define

$$Y = \begin{bmatrix} (g_1^T Q(\lambda I - (J - R)Q)^{-1} g_2 + I)M^{-1} K_p & 0 \\ 0 & 0 \end{bmatrix}$$

such to rewrite the problem as

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ (\mathcal{B} + Y\mathcal{C})z = \tilde{f}_v' \end{cases} \tag{37}$$

with $\tilde{f}_v' = \left[\tilde{f}_v^T \ 0 \right]^T$. Using (3) and (6), we rewrite the second equation in (37) as

$$\frac{1}{\sqrt{2}} \begin{bmatrix} I & Y \end{bmatrix} \begin{bmatrix} W_B \\ W_C \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}z)(a) \\ (\mathcal{H}z)(b) \end{bmatrix} = \tilde{f}_v'. \tag{38}$$

Since $[I \quad Y]$ is right invertible and since $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]$ are invertible, we can find a z_p such that $\mathcal{H}z_p \in H^1([a, b], \mathbb{R}^n)$ and (38) is satisfied for $z = z_p$. We define the new variable $z_n = z - z_p$ to obtain

$$\begin{cases} (\lambda I - \mathcal{J})z_n = f_z - (\lambda I - \mathcal{J})z_p \\ \mathfrak{B}_{cl} z_n = (\mathcal{B} + Y\mathcal{C})z_n = 0 \end{cases} \tag{39}$$

From Theorem 3.3.6 of [29], the operator $J_{cl} = \mathcal{J}|_{D(J_{cl})}$ generates a contraction C_0-semigroup on Z if J_{cl} is dissipative, with $D(J_{cl}) = \{z \in Z \mid \mathcal{H}z \in H^1([a, b], \mathbb{R}^n), \ \mathfrak{B}_{cl} z = 0\}$. For every $z_n \in D(J_{cl})$, we take $v = -(\lambda I - (J - R)Q)^{-1} g_2 M^{-1} K_p \mathcal{C}_1 z_n$, so that

$$(\mathcal{B}_1 + M^{-1} K_p \mathcal{C}_1) z_n - g_1^T Q v = 0, \ \mathcal{B}_2 z_n = 0 \tag{40}$$

and hence $(z_n, v) \in D(\mathcal{A})$. Finally, we use Assumption 1 and the fact that $\mathcal{B}_2 z_n = 0$ to write

$$\begin{aligned} \langle J_{cl} z_n, z_n \rangle_Z &= \langle \mathcal{J} z_n, z_n \rangle_Z = \langle \mathcal{B}_1 z_n, \mathcal{C}_1 z_n \rangle_{\mathbb{R}^m} \\ &= \langle -(g_1^T Q(\lambda I - (J - R)Q)^{-1} g_2 \\ &\quad + I)M^{-1} K_p \mathcal{C}_1 z_n, \mathcal{C}_1 z_n \rangle_{\mathbb{R}^m} \\ &\leq -\langle g_1^T Q(\lambda I - (J - R)Q)^{-1} g_2 \\ &\quad M^{-1} K_p \mathcal{C}_1 z_n, \mathcal{C}_1 z_n \rangle_{\mathbb{R}^m} \\ &\quad -\langle M^{-1} K_p \mathcal{C}_1 z_n, \mathcal{C}_1 z_n \rangle_{\mathbb{R}^m}. \end{aligned} \tag{41}$$

For λ large enough and defining μ as the smallest eigenvalue of $M^{-1} K_p$, it is true that

$$\langle J_{cl} z_n, z_n \rangle_Z \leq \frac{1}{2} \mu ||\mathcal{C}_1 z_n||_{\mathbb{R}^m}^2 - \mu ||\mathcal{C}_1 z_n||_{\mathbb{R}^m}^2 < 0. \tag{42}$$

and thus J_{cl} generates a contraction C_0-semigroup. Consequently, the resolvent operator $(\lambda I - \mathcal{J})^{-1}$ exists, and the unique solution of (39) is given by

$$z_n = (\lambda I - \mathcal{J})^{-1}(f_z - (\lambda I - \mathcal{J})z_p). \tag{43}$$

Therefore the choice

$$\begin{aligned} z &= z_n + z_p \\ v &= (\lambda I - (J - R)Q)^{-1}(f_v + g_2 M^{-1} K_p \mathcal{C}_1 z) \end{aligned} \tag{44}$$

defines an element $(z, v) \in D(\mathcal{A})$ for which the range condition is fulfilled, and from the Lumer-Phillips's theorem we conclude that the operator $\mathcal{A}$ generates a contraction C_0-semigroup in the state space X with weighted inner product (21). $\square$

It is well known that if an operator is the infinitesimal generator of a C_0-semigroup in a space equipped with a certain norm, then it generates a C_0-semigroup in all the spaces equipped with equivalent norms. In the next corollary we show that for any selection of the weighting parameters α_i in Λ, the norm defined through (21) is equivalent to the standard norm in $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$. Hence, this directly implies that the closed-loop operator (19)-(20) generates a C_0-semigroup in $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the standard norm.

Corollary 5 *The closed-loop operator* (19)-(20) *generates a C_0-semigroup in $L^2([0,L],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the standard norm*

$$||x|| = \sqrt{\langle z,z\rangle_{L^2} + v^T v} \tag{45}$$

PROOF. It is sufficient to show that the norm associated to the inner product (21)

$$||x||_\Gamma = \sqrt{\langle z,\mathcal{H}z\rangle_{L^2} + v^T M_v v}, \tag{46}$$

is equivalent to the standard norm, *i.e.* that there exist $c < C \in \mathbb{R}^+$ such that

$$c||x|| \le ||x||_\Gamma \le C||x||. \tag{47}$$

The first inequality of (47) can be rewritten as

$$c\sqrt{\langle z,z\rangle_{L^2} + v^T v} \le \sqrt{\langle z,\mathcal{H}z\rangle_{L^2} + v^T M_v v}, \tag{48}$$

that is equivalent to

$$c^2\left(\langle z,z\rangle_{L^2} + v^T v\right) \le \langle z,\mathcal{H}z\rangle_{L^2} + v^T M_v v. \tag{49}$$

The last inequality is fulfilled if

$$\langle z,(c^2 I-\mathcal{H})z\rangle \le 0, \quad v^T(c^2 I - M_v)v \le 0. \tag{50}$$

Since $\mathcal{H}$ and M_v are strictly positive definite, it exists a constant $c\in\mathbb{R}$ such that both inequalities hold. The second inequality in (47) can be rewritten as

$$\sqrt{\langle z,\mathcal{H}z\rangle_{L^2} + v^T M_v v} \le C\sqrt{\langle z,z\rangle_{L^2} + v^T v}, \tag{51}$$

and it holds if

$$\langle z,(\mathcal{H}-C^2 I)z\rangle \le 0, \quad v^T(M_v - C^2 I)v \le 0. \tag{52}$$

Since the entries of both $\mathcal{H}$ and M_v are always finite, it exists a $C\in\mathbb{R}$ such that both these inequalities are fulfilled. We therefore conclude that (46) is equivalent to the standard norm in $L^2([0,L],\mathbb{R}^n)\times\mathbb{R}^{2m}$. As a consequence, since the closed-loop operator (19)-(20) generates a contraction C_0-semigroup in $L^2([0,L],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the norm (46), it also generates a C_0-semigroup in the same space equipped with the standard norm. $\square$

Since by Theorem 4 the operator $\mathcal{A}$ generates a contraction C_0-semigroup $T(t)$ in $L^2([0,L],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the norm (46), we have that $||T(t)||_\Gamma \le 1$. Using (47), the bound of the C_0-semigroup generated by the operator $\mathcal{A}$ in $L^2([0,L],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the standard norm (45) becomes $||T(t)|| \le \frac{C}{c}$.

4 Asymptotic stabilization

In this section we prove the asymptotic stability of the system described by equations (19)–(20), that is equivalent to show the asymptotic stability of system (18). To do so, consider the state space $X = L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ with inner product $\langle x_1,x_2\rangle_X = \langle z_1,z_2\rangle_{L^2} + v_1^T v_2$ and associated norm (45). Before stating the main theorem on asymptotic stability, we need the following lemma that assure the compactness of the trajectories generated by the closed-loop operator.

Lemma 6 *For $x_0\in X$ define the trajectory $\gamma(x_0) = \bigcup_{t\ge 0} T(t)x_0$, where $T(t)$ is the semigroup generated by the operator* (19) *in X. Under Assumption 1, the trajectory $\gamma(x_0)$ is precompact in X for all $x_0\in X$.*

PROOF. Using [19, Theorem 11.2.25] it suffices to show that the operator $\mathcal{A}$ has compact resolvent. To do so, we show that the resolvent operator maps bounded sequences into bounded sequences with a convergent subsequence. We define the sequence

$$\{w_n\} = (\lambda I - \mathcal{A})^{-1}\{x_n\} \tag{53}$$

with $\lambda > 0$ and $\{x_n\}$ a bounded sequence. The sequences are defined such that $\{w_n\} = \left[\,{\{w_{n,1}\}^T}\ {\{w_{n,2}\}^T}\right]^T \in H^1([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ and $\{x_n\} = \left[\,{\{x_{n,1}\}^T}\ {\{x_{n,2}\}^T}\right]^T \in X$. By Assumption 1 and Theorem 4, $\mathcal{A}$ generates a contraction C_0-semigroup on the space $L^2([a,b],\mathbb{R}^n)\times\mathbb{R}^{2m}$ equipped with the weighted norm (46). Therefore, by Corollary 5, $\mathcal{A}$ generates a bounded C_0-semigroup in X, from the Hille-Yoshida Theorem [19, Theorem 2.1.15] it follows that $||(\lambda I-\mathcal{A})^{-1}|| < \frac{C}{c\lambda}$. This implies that $\{w_n\}$ is bounded in X, *i.e.* $||w_{n,1}||_{L^2}, ||w_{n,2}||_{\mathbb{R}^{2m}} < \infty$. The Bolzano–Weierstrass Theorem implies that it also has a converging subsequence in $\mathbb{R}^{2m}$. We compute the H^1 norm of $w_{1,n}$

$$||w_{n,1}||^2_{H^1} = \left|\left|\frac{\partial}{\partial\xi}w_{n,1}\right|\right|^2_{L^2} + ||w_{n,1}||^2_{L^2} \tag{54}$$

Using the operator $\mathcal{J}$ definition and (53) we obtain

$$\begin{aligned}||\tfrac{\partial}{\partial\xi}w_{n,1}||^2_{L^2} &= ||\mathcal{H}^{-1}P_1^{-1}\mathcal{J}w_{n,1} - \mathcal{H}^{-1}\tfrac{\partial\mathcal{H}}{\partial\xi}w_{n,1} \\ &-\mathcal{H}P_1^{-1}P_0w_{n,1}||^2_{L^2} \\ &\leq 2||\mathcal{H}^{-1}\left(\mathcal{H}^{-1}(P_1^{-1}\lambda - P_1P_0\mathcal{H} - \tfrac{\partial\mathcal{H}}{\partial\xi}\right) \\ &w_{n,1}||^2_{L^2} + 2||\mathcal{H}^{-1}P_1^{-1}x_{n,1}||^2_{L^2} < \infty.\end{aligned} \tag{55}$$

Thus, $\{w_{n,1}\}$ is a bounded sequence in H^1 and from the Sobolev embedding Theorem, $\{w_{n,1}\}$ has a converging subsequence in L^2. Therefore, $\mathcal{A}$ has a compact resolvent and the statement follows. □

Now we state the main contribution of this paper, *i.e.* the asymptotic stability of system (19). We prove that with the proper choice of control parameters the closed-loop system is asymptotically stable.

Theorem 7 *Under Assumptions 1 and 2, the system* (19) *with domain defined by* (20) *is asymptotically stable, i.e.* $\lim_{t\to\infty} x(t) \to 0$, *if the control gains* $k_{p,i}, r_i, k_i, i = 1, .., m$ *are chosen such that*

$$r_i^2 > 2m_ik_i, \quad k_{p,i} > 0 \tag{56}$$

PROOF. Define as set of candidate Lyapunov's functions Υ, the set of half the norm of the set defined in (21):

$$\Upsilon = \{V : X \to \mathbb{R} \mid V(x) = \frac{1}{2}\gamma_\Lambda(x, x)\} \tag{57}$$

The time derivative of an element $V(x) \in \Upsilon$, computed in $x_0 \in X$, is defined as

$$\dot{V}(x_0) = \limsup_{t\to 0}\frac{V(x(t,x_0)) - V(x_0)}{t}, \tag{58}$$

and it can be proven that $\dot{V}(x_0) = dV(x_0)\mathcal{A}x$, where $dV(x_0)$ represents the Fréchet derivative of $V(x)$ in x_0. Using the inner-product symmetry and inequality (25) we obtain

$$\begin{aligned}\dot{V}(x_0) &= \tfrac{1}{2}\gamma_\Lambda(\mathcal{A}x, x) + \tfrac{1}{2}\gamma_\Lambda(x, \mathcal{A}x) = \gamma_\Lambda(x, \mathcal{A}x) \\ &\leq -\begin{bmatrix}\eta^T & q^T & y_z^T\end{bmatrix} P \begin{bmatrix}\eta \\ q \\ y_z\end{bmatrix},\end{aligned} \tag{59}$$

where the matrix P is defined with components as in (26). Note that using relation (32) and the fact that all the matrices are diagonal, the term P_{12} can be rewritten as

$$\begin{aligned}P_{12} &= \tfrac{1}{2}R_cK_p^{-1}M^{-1}(I-\Lambda) + \Lambda^{-1}R_c^{-1}K_p^{-1}K \\ &= K_p^{-1}M^{-1}R_c[\tfrac{1}{2}(I-\Lambda) + \Lambda^{-1}R_c^{-2}MK] \\ &= K_p^{-1}M^{-1}R_c[\tfrac{1}{2}(I-\Lambda) + \tfrac{1}{2}I] \\ &= \tfrac{1}{2}K_p^{-1}M^{-1}R_c(2I-\Lambda).\end{aligned} \tag{60}$$

Then, use relation (32) to write

$$\begin{aligned}\dot{V}(x_0) &\leq -[\eta^T(2I-\Lambda)K_p^{-1}M^{-1}\Lambda^{-1}\eta \\ &+\eta^T(2I-\Lambda)K_p^{-1}M^{-1}R_cq + q^TK_p^{-1}Kq \\ &+q^T\Lambda R_cM^{-1}y_z + y_z^TM^{-1}K_py_z] \\ &= -[(\Lambda^{-\frac{1}{2}}\eta + \tfrac{1}{2}R_c\Lambda^{\frac{1}{2}}q)^T((2I-\Lambda)K_p^{-1} \\ &M^{-1})(\Lambda^{-\frac{1}{2}}\eta + \tfrac{1}{2}R_c\Lambda^{\frac{1}{2}}q) + \tfrac{1}{4}q^T\Lambda^2R_c^2K_p^{-1} \\ &M^{-1}q + q^T\Lambda R_cM^{-1}y_z + y_z^TM^{-1}K_py_z] \\ &= -[(\Lambda^{-\frac{1}{2}}\eta + \tfrac{1}{2}R_c\Lambda^{\frac{1}{2}}q)^T((2I-\Lambda)K_p^{-1} \\ &M^{-1})(\Lambda^{-\frac{1}{2}}\eta + \tfrac{1}{2}R_c\Lambda^{\frac{1}{2}}q) \\ &+(\tfrac{1}{2}\Lambda R_cK_p^{-\frac{1}{2}}q + K_p^{\frac{1}{2}}y_z)^TM^{-1}(\tfrac{1}{2}\Lambda R_c \\ &K_p^{-\frac{1}{2}}q + K_p^{\frac{1}{2}}y_z).\end{aligned} \tag{61}$$

In order to use the La Salle's invariance principle, we show that the largest invariant subset E of $\Omega = \{x_0 \in X \mid \dot{V}(x_0) = 0\}$ consists of only the origin of the state space. To do so, using relation (32) in the last inequality of (61), we characterize the set for which the Lyapunov function's time derivative is equal to zero:

$$\Omega = \left\{x_0 \in X \mid \eta = -KMR_c^{-1}q, y_z = -K_p^{-1}\eta\right\}. \tag{62}$$

Then, substitute the former relations in the closed-loop dynamic (19)-(20) to obtain

$$\dot{z} = \mathcal{J}z, \quad \dot{\eta} = 0, \quad \dot{q} = 0 \tag{63}$$

with operator $\mathcal{J}$ domain

$$D(\mathcal{J}) = \left\{z \in Z | z \in H^1([a,b],\mathbb{R}^n), \mathcal{B}z = 0\right\}, \tag{64}$$

$\mathcal{C}_2z(t) = \tilde{y}(t)$, and the other part of the output

$$\mathcal{C}_1z(t) = y_z(t) = -KMR_c^{-1}K_p^{-1}q(t). \tag{65}$$

System (63) implies that η and q must be constant along time, i.e. $\eta(t) = \eta^*$ $q(t) = q^*$. Hence, $x_0 \in E$ should verify

$$\begin{aligned}&\dot{z}(t) = \mathcal{J}z(t) \\ &\mathcal{B}z(t) = 0 \quad \mathcal{C}_1z(t) = -KMR_c^{-1}K_p^{-1}q^*\end{aligned} \tag{66}$$

Using Assumption 2, we know that the only solution of (66) is $z = 0$, which in turn implies $y_z = 0$ and consequently $q^* = \eta^* = 0$ using the relations in (62). Thus the largest invariant set $E \subset \Omega$ corresponds to $E = \{0\}$. Then, since the solutions are pre-compact in the weighted space X, we can conclude by LaSalle's invariance principle that the solution converges asymptotically to the origin. $\square$

In the next proposition we show that also the control input converges to zero, provided that the initial conditions of the infinite dimensional part of the system are sufficiently smooth.

Proposition 8 *Under the assumptions of Theorem 7, the control input $u(t)$ in (15) converges to zero as $t \to \infty$, when the initial condition x_0 lies in the domain of $\mathcal{A}^2$.*

PROOF. We first recall that $p(t) = \eta(t) - K_p \mathcal{C}_1 z(t)$. The first term of the control input (15) is well-posed. The second term is well-posed for $z(t) \in H^1$. From [10, Theorem 5.2.2] we know that $x(t) \in D(\mathcal{A})$ whenever $x_0 \in D(\mathcal{A})$. So under the condition of the proposition, we have that $x(t) \in D(\mathcal{A})$ and in particular $z(t) \in H^1$. By Theorem 7, we know that the C_0-semigroup generated by the closed-loop operator (18) is stable, and thus $p(t) \to 0$ as $t \to \infty$. To study the behaviour of $\mathcal{C}_1 z(t)$ we write $\mathcal{C}_1 z(t) = [\mathcal{C}_1, 0]\, x(t) := \mathcal{C}^e x(t)$. Since $\mathcal{C}^e \mathcal{A}^{-1}$ is a bounded operator and since $\mathcal{A}T(t)x_0 = T(t)\mathcal{A}x_0$ for all $x_0 \in D(\mathcal{A})$, we find that $\mathcal{C}_1 z(t) = \mathcal{C}^e \mathcal{A}^{-1} \mathcal{A} x(t) \to 0$. So it remain to consider the third term of (15). Using [10, Theorem 5.2.2] once more, we see that $\frac{d}{dt}(\mathcal{C}_1 z(t))$ equals

$$\begin{aligned} \tfrac{d}{dt}(\mathcal{C}_1 z(t)) &= \tfrac{d}{dt}(\mathcal{C}^e x(t)) \\ &= \tfrac{d}{dt}\mathcal{C}^e \mathcal{A}^{-1} \mathcal{A} x(t) = \mathcal{C}^e \mathcal{A}^{-1} \mathcal{A}^2 x(t) \end{aligned} \tag{67}$$

provided $x_0 \in D(\mathcal{A}^2)$. For these initial conditions, have that $\mathcal{A}^2 T(t) x_0 = T(t)\mathcal{A}^2 x_0$, and as above this shows that the last term of (15) converges to zero as time goes to infinity, and thus this concludes the proof. $\square$

5 Example

To illustrate the applicability and the stability in closed-loop of the proposed control law (16), we propose the example of a clamped-free string with a mass connected at the free side, as shown in Figure 2. This system can be modelled by the following set of equations:

$$\begin{aligned} &\tfrac{\partial}{\partial t}\left(\tfrac{\partial w}{\partial t}(\xi,t) + \dot{q}(t)\right) = \tfrac{1}{\rho(\xi)}\tfrac{\partial}{\partial \xi}\left(T(\xi)\tfrac{\partial w}{\partial \xi}(\xi,t)\right) \\ &m_1 \tfrac{\partial^2 w}{\partial t^2}(0,t) = T(0)\tfrac{\partial w}{\partial \xi}(0,t) + f(t) \\ &w(\xi,0) = w_0(\xi) \quad w(L,t) = 0 \end{aligned} \tag{68}$$

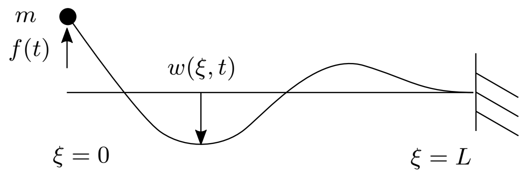

Fig. 2. Vibrating string with tip mass.

for $\xi \in [0, L]$.

We design a proportional control law such that $f(t) = -kw(0,t) + u(t)$, and define the state variables $z_1(\xi,t) = \frac{\partial w}{\partial \xi}$, $z_2(\xi,t) = \rho\left(\frac{\partial w}{\partial t} + \dot{q}\right)$ and $p = m\frac{\partial w}{\partial t}(0,t)$, $q = w(0,t)$. Then, the system (68) admits a pH representation in the form of (14)

$$\begin{cases} \dot{x}_p(t) = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}\mathcal{H} & 0 \\ -g_1 \mathcal{C}_1 \mathcal{H} & JQ \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ g_1 \end{bmatrix} u(t) \\ y_p(t) = \begin{bmatrix} 0 & g_1^T Q \end{bmatrix} x_p(t) = \mathcal{C}_p x_p(t) = \frac{1}{m} p(t) \end{cases} \tag{69}$$

with $x_p = [z \;\; x_f]^T \in L^2([0,L],\mathbb{R}^2) \times \mathbb{R}^2$, $z(t) = [z_1(\xi,t)\; z_2(\xi,t)]^T$ and $x_f = [q\; p]^T$. The system's matrices are defined as

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathcal{H} = \begin{bmatrix} T(\xi) & 0 \\ 0 & \rho^{-1}(\xi) \end{bmatrix} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$Q = \begin{bmatrix} k & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \quad g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{70}$$

The PDE's input output operators are defined as

$$\begin{aligned} \begin{bmatrix} \mathcal{B}_1 z(t) \\ \mathcal{B}_2 z(t) \end{bmatrix} &= \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} \rho(0)^{-1} z_2(0,t) \\ \rho(L)^{-1} z_2(L,t) \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}_1 z(t) \\ \mathcal{C}_2 z(t) \end{bmatrix} &= \begin{bmatrix} W_{C,1} \\ W_{C,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} -T(0) z_1(0,t) \\ T(L) z_1(L,t) \end{bmatrix}, \end{aligned} \tag{71}$$

where we consider

$$\begin{aligned} u_z(t) &= \mathcal{B}_1 z(t) = \rho(0)^{-1} z_2(0,t) \\ y_z(t) &= \mathcal{C}_1 z(t) = -T(0) z_1(0,t). \end{aligned} \tag{72}$$

According to (68) and to the state variables definition, the input operators are set such that $u_z(t) = \frac{1}{m}p(t)$ and $\mathcal{B}_2 z(t) = \rho(1)^{-1} z_2(1,t) = 0$.
In order to fulfil Assumption 1, we write

$$\begin{bmatrix} W_B \\ W_C \end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_{B,2} \\ W_{C,1} \\ W_{C,2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \tag{73}$$

and we can see that it is full rank. Furthermore, we compute

$$\begin{aligned}\dot{H}(t) &= \tfrac{1}{2}\tfrac{d}{dt}\langle z(t), \mathcal{H}z(t)\rangle_{L^2} \\ &= \tfrac{1}{2}[(\mathcal{H}z(t))^T P_1 \mathcal{H}z(t)]_0^1 \\ &= T(1)z_1(1,t)\rho(L)^{-1}z_2(L,t) \\ &\qquad -T(0)z_1(0,t)\rho^{-1}(0)z_2(0,t),\end{aligned} \tag{74}$$

and using the previously defined input/output (72) together with $\rho(L)^{-1}z_2(L,t) = 0$, we can write

$$\dot{H}(t) = u_z(t)y_z(t). \tag{75}$$

Remark 9 *The dynamical equations of the overhead crane in [25], together with a proportional position control term, fit the framework of equation* (69) *with a different definition of the boundary operators. Therefore, this example can be seen as an addendum of the aforementioned paper, giving a proof for the asymptotic stability of the closed-loop system in case the position control is added to the strong dissipation term.*

In the next proposition we show that Assumption 2 holds for the introduced system, such then to be able to apply the result on asymptotic stability obtained in Theorem 7.

Proposition 10 *The system*

$$\dot{z} = P_1\frac{\partial}{\partial\xi}(\mathcal{H}z) \tag{76}$$

with matrices defined in (70)*, boundary conditions*

$$\mathcal{B}z = \begin{bmatrix}\mathcal{B}_1 z\\ \mathcal{B}_2 z\end{bmatrix} = \begin{bmatrix}\rho(0)^{-1}z_2(0,t)\\ \rho(L)^{-1}z_2(L,t)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{77}$$

and output operator defined as

$$y_z(t) = \mathcal{C}_1 z = Tz_1(0,t) \tag{78}$$

is approximately observable and its only equilibrium point is $z = 0$.

PROOF. Since operator (76) generates a unitary group, it is a skew-adjoint operator, hence its eigenvalues belong to the imaginary axis. Moreover, since operator (76) generates a unitary group and its resolvent is compact, we know by Theorem A.4.20 [30], that its eigenvectors forms an orthonormal basis. Since an orthonormal basis is a special case of a Rietz-Basis, operator (76) is a Rietz-spectral operator. Consequently, using theorem 4.2.3 of [30], to check that the system is approximately observable we have to show that there exists no eigenvector in the kernel of $\mathcal{C}_1$. To show this, assume by contradiction that there exists an eigenvector v such that $C_1 v = 0$. We group the former equation with the first equality of (77) such to obtain $(Hv)(0) = 0$. We integrate both side of the eigenvalue problem's equation

$$iwv(\xi) = P_1\frac{\partial}{\partial\xi}(\mathcal{H}v)(\xi) \tag{79}$$

to obtain

$$\begin{aligned}iw\textstyle\int_0^s v(\xi)d\xi &= P_1\textstyle\int_0^s \frac{\partial}{\partial\xi}(\mathcal{H}v)(\xi)d\xi \\ &= P_1[(\mathcal{H}v)(s) - (\mathcal{H}v)(0)] = P_1(\mathcal{H}v)(s).\end{aligned} \tag{80}$$

With P_1 being full rank, the former equation is equivalent to

$$(\mathcal{H}v)(s) = iwP_1^{-1}\int_0^s \mathcal{H}(\xi)^{-1}\mathcal{H}(\xi)v(\xi)d\xi, \tag{81}$$

from which we can get

$$||(\mathcal{H}v)(s)|| \leq |w|\cdot||P_1^{-1}||\int_0^s ||\mathcal{H}(\xi)^{-1}||\cdot||\mathcal{H}v(\xi)||d\xi.$$

Now, using the Gronwall's Lemma we obtain that

$$\int_0^s ||(\mathcal{H}v)(s)||d\xi \leq 0 \tag{82}$$

that implies $(\mathcal{H}v)(s) \equiv 0$, which since $\mathcal{H}(\xi) \geq mI$ with $m > 0$ implies $v(s) \equiv 0$, that is a contradiction to the fact that v is an eigenvector.
The equilibrium positions of (76) are the solutions of

$$P_1\frac{\partial}{\partial\xi}(\mathcal{H}z) = 0. \tag{83}$$

The previous equation implies that $\rho(\xi)^{-1}z_2(\xi,t)$ and $T(\xi)z_1(\xi,t)$ must be constants, and because of the boundary conditions (77), these constants are zero

$$\rho(\xi)^{-1}z_2(\xi,t) = T(\xi)z_1(\xi,t) = 0. \tag{84}$$

Since $\rho(\xi)$ and $T(\xi)$ are strictly positive by assumption, the only possibility is $z_1(\xi,t) = z_2(\xi,t) = 0$, that concludes the proof. □

We apply the control law (16) to the system (69)

$$u(t) = -\frac{r}{m}p(t) + (1 - \frac{r}{m}k_p)\mathcal{C}_1 z(t) - k_p\frac{\partial}{\partial t}(\mathcal{C}_1 z(t)). \tag{85}$$

Note that system (69) was already endowed with the proportional control action. After defining the change of variables $\eta = p + k_p \frac{\partial}{\partial t}(\mathcal{C}_1 z)$, the closed-loop system can be written as an evolution equation with the same form of equation (19):

$$\dot{x} = \mathcal{A}x = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) \\ -\frac{r}{m}\eta - kq \\ \frac{1}{m}(\eta - k_p \mathcal{C}_1 z) \end{bmatrix} \tag{86}$$

with $x = [z\ \eta\ q]^T$, and domain of the operator $\mathcal{A}$

$$D(\mathcal{A}) = \{x \in X \mid z \in H^1, \mathcal{B}_1 z = m^{-1}(\eta \\ -k_p \mathcal{C}_1 z), \mathcal{B}_2 z = 0\} \tag{87}$$

Thanks to Theorem 7, if the control parameters satisfy the conditions in equation (56), the system described by (86)-(87) is asymptotically stable. To demonstrate the performances of the proposed control law (85), we compare the simulation results with the ones obtained using a classical PD controller

$$r(t) = -\frac{r}{m}p(t) - kq(t), \tag{88}$$

where the parameters are chosen as in (85). For the simulations we approximate the vibrating string with a structure preserving finite difference discretization scheme [31], where the space domain has been divided in 100 discretizing elements. The total simulation time is 10 seconds, with a sample time equal to 0.001 seconds. The system's parameters are selected such that $T = 1\ (N)$, $\rho = 1\ (kg/m)$ and $m = 1\ kg$. The controller's gains $k = 6.25$, $r = 5$, $k_p = 1$ are selected such to satisfy the condition (56) with $\alpha = 0.5$. The *strong dissipation* term in (85) is obtained as the difference quotient of $\mathcal{C}_1 z(\xi, t) = T(0) z_1(0, t) = T(0)\frac{\partial w}{\partial \xi}(0, t)$ between two successive discrete instants of time. Simulations have been done using the Matlab ® Simulink® environment and the "ode23tb" time integration algorithm.
Figures 3 and 4 show respectively the string deformation evolution along time with the control actions described by equations (88) and (85). In the shown simulations the system is initialized with the initial condition $x(\xi, 0) = \frac{1}{2}(1-\xi) + \frac{2}{5}\sin(\pi\xi)$.

It is clear from the simulations that the proposed control law stabilizes faster the string equation with the tip mass. Figure 5 shows the comparison between the system's energy time evolutions for the two different control actions. One can see that the energy of the closed-loop system with the "Strong dissipation" feedback (85) converges faster to zero than the one with the PD controller (88). It can be noticed that the closed-loop system with the PD control (88) has a decreasing energy function, while it is not the case for the system in closed-loop with the "Strong dissipation" feedback (85). This is the reason why it was not possible to choose the closed-loop

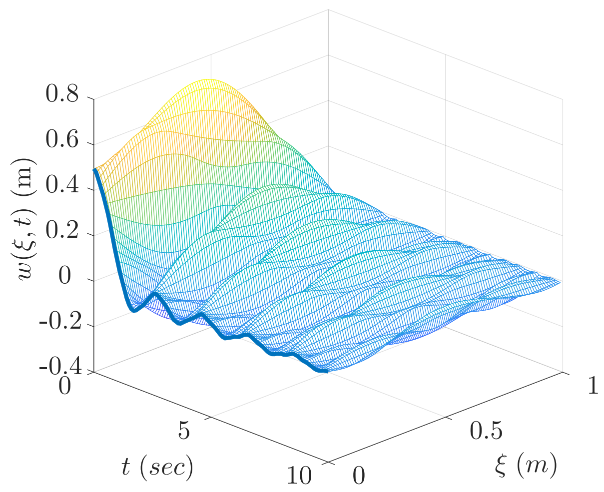

Fig. 3. String's deformation along time with PD control action.

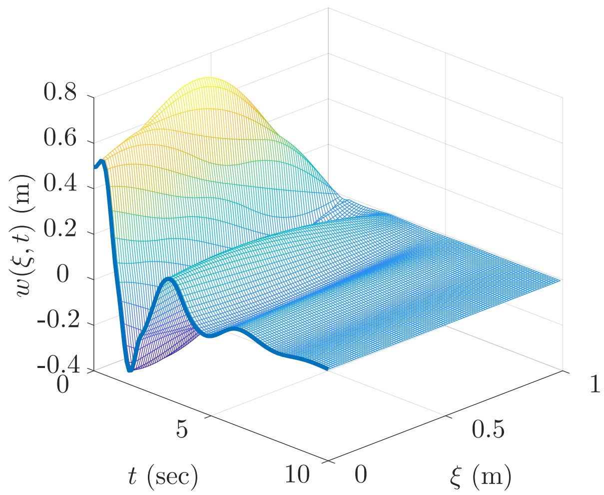

Fig. 4. String's deformation along time with "Strong dissipation" control action.

energy as a Lyapunov function in the stability proof of Theorem 7. Always in Fig. 5 we show also that the selected Lyapunov function (57) is decreasing along the system's trajectories if the control parameters are chosen such to satisfy the conditions in equation (56).

We now investigate the effect in closed-loop of the noise presence in the strain measurement. The noise is obtained using the "Band-limited white noise" block with *Noise power*= 0.001, *Sample time*= 0.001 and *Seed*= [23341]. As previously mentioned, the *strong dissipation* feedback is obtained by the time differentiation of the strain measurement $\mathcal{C}_1 z(t) = -T(0)z_1(0, t) = -T(0)\frac{\partial w}{\partial \xi}(0, t)$. In real applications, the strain measurement, coming from a strain gauge sensor, is subjected

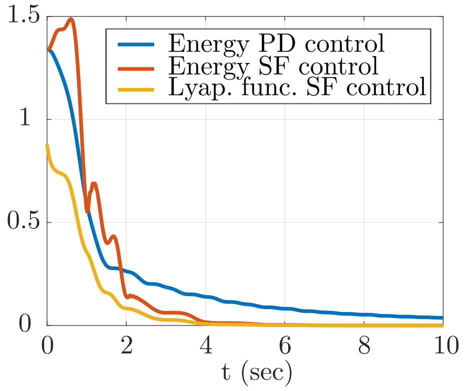

Fig. 5. Energy and Lyapunov function evolution along time.

to noise. The time derivation of a noised signal amplifies the disturbance, making the control law not feasible because of the actuator's saturation. To attenuate the noise effect on the control law computation, we propose the use of a first-order low-pass filter after the strain measurement acquisition as shown in the control scheme of Figure 6. The dynamic equation of the first-order filter writes

$$\dot{y}_f = -r_f y_f + u_f, \tag{89}$$

The velocity of propagation of the considered wave equation is $v = \sqrt{\frac{T}{\rho}} = 1\ (m/s)$, while the fundamental harmonic is given by $f_0 = \frac{1}{2L}v = 0.5\ (Hz)$. Therefore, we choose the filter's parameter r_f such to have a cut-off frequency of $10\ (Hz)$, *i.e.* $r_f = 0.1$. In this way, we are able to capture the essential dynamics of the considered wave equation and to filter the simulated white noise of the strain gauge sensor. In Figure 7 we show the string deformation along time in the presence of noise and filtered strain measurement. In the same figure it is also plotted the mass position $w(0,t)$ in presence of noisy measurement with both the presence and not of the filter after the acquisition. We highlight that the tip mass time trajectory in presence of noise but without filter, is very similar to the tip mass time trajectory without noise and filter. This can be explained by the fact that the tip mass dynamics acts as a low pass filter for the high frequencies introduced by the noise. Nevertheless, in Figure 8 we show that the applied control law in presence of noise would be infeasible using a real actuator, while the control law obtained with a filtered measurement is better implementable.

Remark 11 *The presence of a filter* (89) *after the strain measurement add a dynamic in the closed-loop system that is not considered in the control design of Section 2 and in the stability results of Section 3 and 4. The presence of this additional closed-loop dynamic could be the starting point for further stability investigations.*

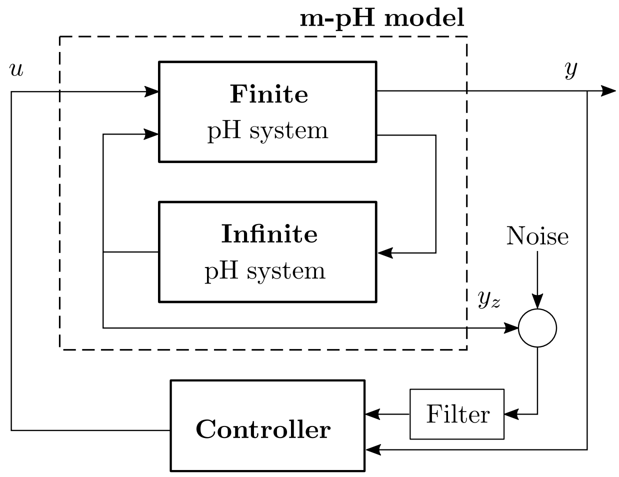

Fig. 6. String's deformation along time with "Strong dissipation" control action in presence of noise.

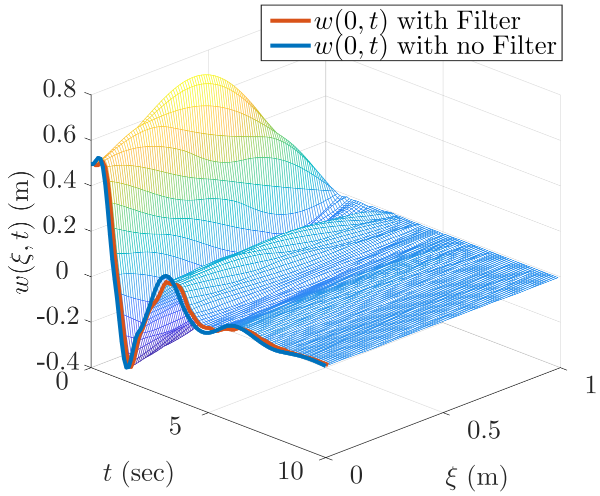

Fig. 7. String's deformation along time with "Strong dissipation" control action in presence of noise.

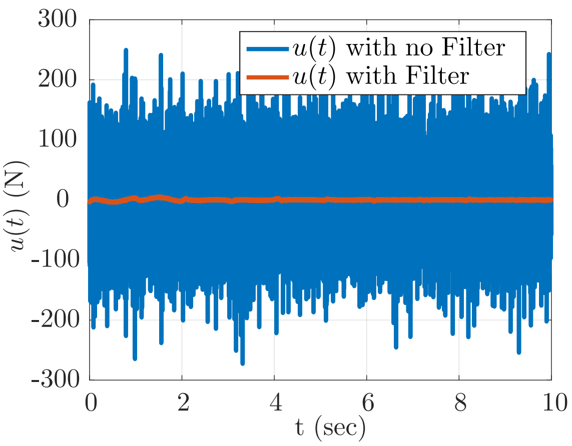

Fig. 8. Control force $u(t)$ with and without filtered strain measurement.

6 Conclusions

In this paper we have developed a control strategy that asymptotically stabilizes a class of mixed PDE-ODE systems with actuation in the ODE part. The closed-loop operator has been obtained after an appropriate change of coordinates, and it has been shown to generate a contraction C_0-semigroup in an appropriate space equipped with a weighted inner product, provided the correct selection of the control parameters. Afterwards, the weighted inner product has been shown to be equivalent to the standard inner product, therefore the closed-loop operator has been shown to generates a bounded C_0-semigroup in the same space equipped with the standard inner product. Further, a Lyapunov based stability proof has been used to show the asymptotic stability of the closed-loop system under a proper selection of the control parameters. This analysis takes advantage of the pH structure of the system to be controlled. Finally, the control law has been applied to a clamped vibrating string with a tip mass in the free side and control action on the mass' dynamic. Simulation results have been given such to validate the theoretical development.

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