Multi-qubit doilies: Enumeration for all ranks and classification for ranks four and five

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Abstract

For $N \geq 2$, an N-qubit doily is a doily living in the N-qubit symplectic polar space. These doilies are related to operator-based proofs of quantum contextuality. Following and extending the strategy of Saniga et al. (Mathematics 9 (2021) 2272) that focused exclusively on three-qubit doilies, we first bring forth several formulas giving the number of both linear and quadratic doilies for any $N > 2$. Then we present an effective algorithm for the generation of all N-qubit doilies. Using this algorithm for $N = 4$ and $N = 5$, we provide a classification of N-qubit doilies in terms of types of observables they feature and number of negative lines they are endowed with. We also list several distinguished findings about N-qubit doilies that are absent in the three-qubit case, point out a couple of specific features exhibited by linear doilies and outline some prospective extensions of our approach.

1 Introduction

The doily is a remarkable piece of finite geometry that occurs in a number of disguises. Here, we mention the most prominent ones.

1. *The doily as a duad-syntheme geometry.* Let us recall a famous Sylvester's construction of the doily [1]. Given a six-element set $M_6 \equiv \{1, 2, 3, 4, 5, 6\}$, a *duad* is an unordered pair $(ij) \in M_6$, $i \neq j$, and a *syntheme* is a set of three pairwise disjoint duads, i. e. a set $\{(ij), (kl), (mn)\}$ where $i, j, k, l, m, n \in M_6$ are all distinct. The point-line incidence structure whose points are duads and whose lines are synthemes, with incidence being inclusion, is isomorphic to the doily, as also illustrated in Figure 1.

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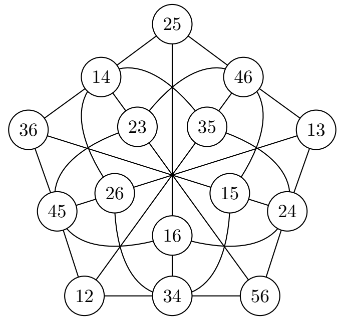


Figure 1: A duad-syntheme model of the doily.

2. *The doily as the Cremona–Richmond configuration.* It is a particular 15_3-configuration, i. e. a self-dual configuration of 15 points and 15 lines, with three points on a line and, dually, three lines through a point such that it contains no triangles [2, 3]. Up to isomorphism, there are altogether 245,342 15_3-configurations, of which only the doily enjoys the property of being triangle-free.

3. *The doily as a generalized quadrangle.* A generalized quadrangle $\mathrm{GQ}(s,t)$ of order (s,t) is an incidence structure of points and lines (blocks) where every point is on $t+1$ lines ($t>0$), and every line contains $s+1$ points ($s>0$) such that if p is a point and L is a line, p not on L, then there is a unique point q on L such that p and q are collinear. The doily is isomorphic to the unique generalized quadrangle with $s=t=2$ [4].

4. *The doily as a symplectic polar space.* Given a d-dimensional projective space $\mathrm{PG}(d,2)$ over the two-elements field $\mathbb{F}_2=\{0,1\}$ of modulo-2 arithmetic, a *polar space* $\mathcal{P}$ in this projective space consists of the projective subspaces that are *totally isotropic/singular* with respect to a given non-singular bilinear form [5, 6]; $\mathrm{PG}(d,2)$ is called the *ambient projective space* of $\mathcal{P}$. A projective subspace of maximal dimension in $\mathcal{P}$ is called a *generator*; all generators have the same (projective) dimension $r-1$. One calls r the *rank* of the polar space. The *symplectic polar space* $\mathcal{W}(2N-1,2)$, $N\geq 1$, consists of all the points of $\mathrm{PG}(2N-1,2)$, $\{(x_1,x_2,\ldots,x_{2N}) : x_j\in\{0,1\}, j\in\{1,2,\ldots,2N\}\}\backslash\{(0,0,\ldots,0)\}$, together with the totally isotropic subspaces with respect to the standard symplectic form

$$\sigma(x,y)=x_1y_{N+1}-x_{N+1}y_1+x_2y_{N+2}-x_{N+2}y_2+\cdots+x_Ny_{2N}-x_{2N}y_N. \quad (1)$$

Throughout the paper, the space name $\mathcal{W}(2N-1,2)$ is often shortened as W_N. This space features

$$|W_N|_p=4^N-1$$

points and

$$|W_N|_g=(2+1)(2^2+1)\cdots(2^N+1)$$

generators. The doily is isomorphic to the symplectic polar space of rank $N=2$, $\mathcal{W}(3,2)$.

5. *Multi-qubit doilies.* This paper is about doilies related to Kochen–Specker operator-based proofs of quantum contextuality, to be called N-qubit doilies or multi-qubit doilies. We

follow the terminology and notation of Section 2 of [7], to which the reader can refer for more finite-geometric background. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices, I the identity matrix, '$\otimes$' the tensor product of matrices and $\mathcal{I}_N \equiv I_{(1)} \otimes I_{(2)} \otimes \ldots \otimes I_{(N)}$, and let $\mathcal{S}_N = \{G_1 \otimes G_2 \otimes \cdots \otimes G_N : \; G_j \in \{I, X, Y, Z\}, \; j \in \{1, 2, \ldots, N\}\} \backslash \{\mathcal{I}_N\}$. The $4^N - 1$ N-qubit observables of $\mathcal{S}_N$ can be bijectively identified with the $4^N - 1$ points of $\mathcal{W}(2N-1, 2)$ in such a way that any two commuting observables are represented by collinear points and the product of the three observables lying on a line of $\mathcal{W}(2N-1, 2)$ is $+\mathcal{I}_N$ or $-\mathcal{I}_N$ (see, for example, [8, Section 5.3.2]). If the symplectic form in the ambient space $\mathrm{PG}(2N-1, 2)$, defining $\mathcal{W}(2N-1, 2)$, is given by Eq. (1), then the corresponding bijection reads

$$G_j \leftrightarrow (x_j, x_{j+N}), \; j \in \{1, 2, \ldots, N\}, \tag{2}$$

with the assumption that

$$I \leftrightarrow (0, 0), \; X \leftrightarrow (0, 1), \; Y \leftrightarrow (1, 1), \; \text{and } Z \leftrightarrow (1, 0). \tag{3}$$

To briefly illustrate this property, let us consider the three-qubit $\mathcal{W}(5, 2)$ and one of its lines, say $(0, 1, 1; 1, 1, 0)$, $(1, 0, 0; 0, 0, 1)$ and $(1, 1, 1; 1, 1, 1)$. Using the correspondences (2) and (3) we find that the corresponding observables are $X \otimes Y \otimes Z$, $Z \otimes I \otimes X$ and $Y \otimes Y \otimes Y$, respectively; these observables indeed pairwise commute and their product is $+I \otimes I \otimes I$.

In what follows, W_N will always be understood as having its points labeled by the N-qubit observables as described above, and any doily lying in it, together with the inherited labeling, will be called an N-qubit doily ($N \geq 2$). Slightly rephrased, an *N-qubit doily* is a doily whose points are bijectively identified with 15 specific observables from $\mathcal{S}_N$, such that any two commuting observables share the same line, and, given any line, the product of (any) two observables lying on it is, up to a sign, equal to the remaining observable on it. A line of an N-qubit doily will be called *positive* (resp. *negative*) if the product of its three observables is $+\mathcal{I}_N$ (resp. $-\mathcal{I}_N$). To avoid any possible misunderstanding, it is worth mentioning that the product of observables is the (ordinary) matrix product, denoted by a dot (.), induced by the following multiplication table of Pauli matrices.

.	X	Y	Z
X	I	iZ	$-iY$
Y	$-iZ$	I	iX
Z	iY	$-iX$	I

From here on, the geometrical points are considered to be finite words on the four-letter alphabet $\{I, X, Y, Z\}$ that encode the observables $G_1 \otimes G_2 \otimes \cdots \otimes G_N$, while omitting the symbol $\otimes$ for the tensor product and forgetting in the sequel about the matrix nature of I, X, Y and Z.

Contributions and paper outline. Our contributions start in Section 2, with a presentation of several facts about N-qubit doilies that motivates the design of an effective algorithm to enumerate N-qubit doilies for any rank N (Section 4). By geometric considerations, we first establish in Section 3 closed formulas for the numbers of N-qubit doilies. As these numbers increase rapidly with N, the enumeration algorithm can in practice only be executed for small numbers of qubits. We use it in Section 5 to classify N-qubit doilies for $N = 4$ and $N = 5$, according to their types of observables and their configurations of negative lines. We thus produce precise tables for the number of doilies in each category/class, reproduced in the appendices of this paper. Section 5 also analyzes these results and points out various findings about N-qubit doilies that are absent in the known three-qubit case. Section 6 concludes and outlines some prospective extensions of our approach.

2 Some basic facts about multi-qubit doilies

2.1 Patterns formed by negative lines

It is a straightforward task to work out possible types of patterns of negative lines an N-qubit doily can be endowed with. This classification follows readily from the facts that each grid in the doily must contain an odd number of negative lines and that two different grids have two intersecting lines in common. And as a grid has an even number of lines the types of configurations come in complementary pairs, as depicted in Figure 2.

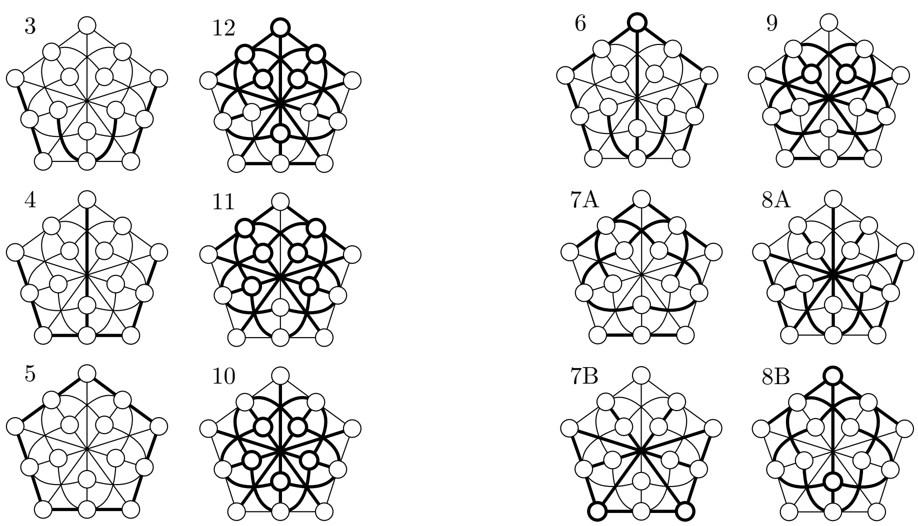
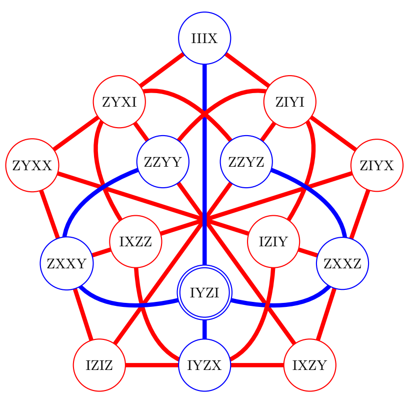


Figure 2: Generic representatives of the twelve different types of configurations of negative lines (bold) that can be found in a multi-qubit doily.

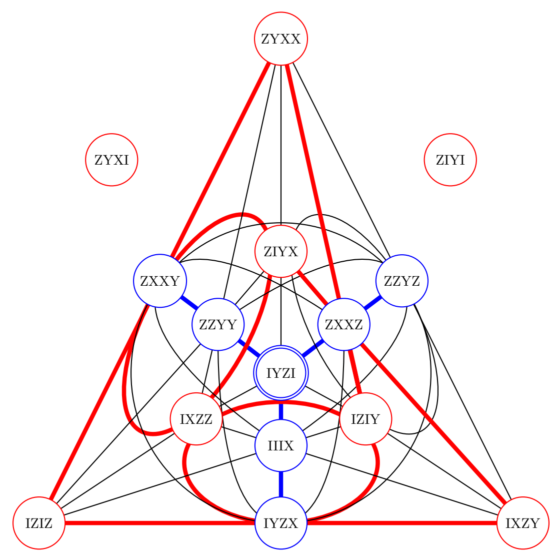
Let us give a brief description of the individual types of configurations. In Type 3 the three negative lines are pairwise disjoint and lie in a grid; that is, their dual is a tricentric triad. Type 4 features three pairwise disjoint lines not belonging to a grid and their unique transversal. In Type 5 the five negative lines form a pentagon. Type 6 contains the three lines from Type 3 plus three concurrent lines, whose point of concurrence is not lying on any of the three former lines. Type 7A contains six lines forming a hexagon and a unique line disjoint from any of the six. Type 7B is a particular union of two Types 4 and an extra line or, equivalently, is composed of the five lines of a grid and two disjoint lines. A two-qubit doily features just a Type 3 pattern, while in a three-qubit doily we can find all the patterns from Type 3 to Type 7A inclusive [7].

2.2 Linear and quadratic doilies

Following [7], we will also distinguish between two kinds of doilies, referred to as linear and quadratic. A *linear* N-qubit doily spans a $\mathrm{PG}(3,2)$ of the ambient $\mathrm{PG}(2N-1,2)$. This means that the three lines of a perp-set of such a doily are coplanar, i. e. lie in a $\mathrm{PG}(2,2)$ of the $\mathrm{PG}(3,2)$, a tricentric triad corresponds to a line of the $\mathrm{PG}(3,2)$ and the plane defined by a unicentric triad of the doily passes through its center. Figure 3 serves as a graphical illustration of these features for $N=4$. In the doily we selected a perp-set (blue) and colored the remaining lines red. The model of $\mathrm{PG}(3,2)$ is based on a 3-D tetrahedral model of Polster [9]; our version features all the points but not all the lines of the model in order to avoid too crowded appearance of the figure. The two red points at the side lie on the line passing via $IYZI$ that would be perpendicular to the plane of the drawings. Each black line of the $\mathrm{PG}(3,2)$ is non-isotropic and corresponds to a tricentric triad in the doily.



(a)



(b)

Figure 3: A linear four-qubit doily with one of its perp-sets highlighted in blue color (a) and the corresponding $\mathrm{PG}(3,2)$ of $\mathrm{PG}(7,2)$ it spans (b). One can readily see that the three lines of the perp-set lie in a plane of the $\mathrm{PG}(3,2)$ and the three points on a non-isotropic line of the space (black) correspond to a tricentric triad of the doily.

A quadratic N-qubit doily spans a $\mathrm{PG}(4,2)$ of the ambient $\mathrm{PG}(2N-1,2)$, being, in fact, isomorphic to the geometry formed by 15 points and 15 lines lying on a parabolic quadric $Q(4,2)$ in

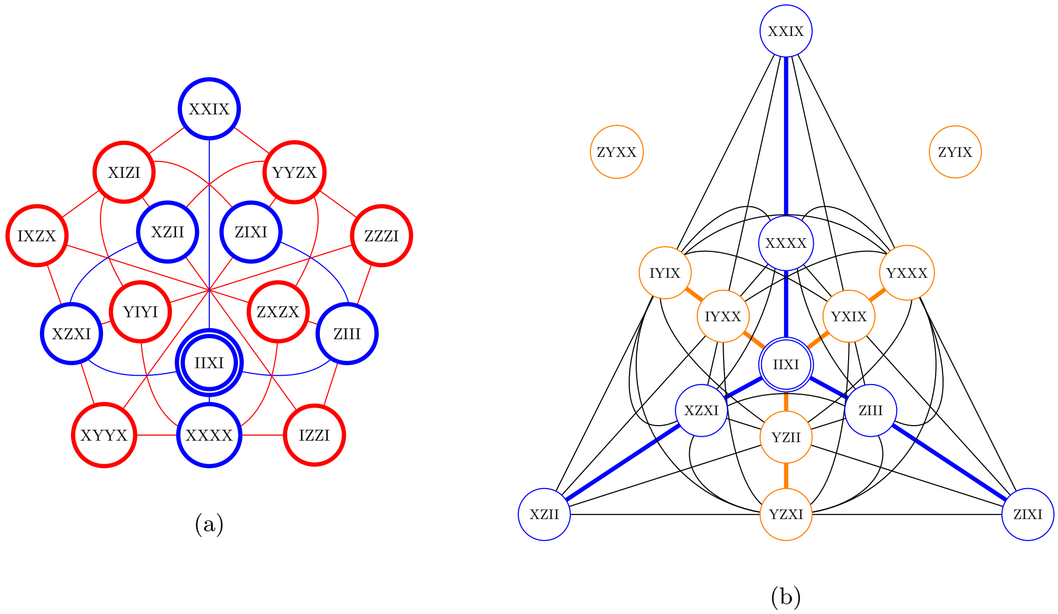


Figure 4: An illustration of the fact that a perp-set of a quadratic (four-qubit) doily (a) spans a PG(3, 2) (b) of the PG(4, 2) spanned by the doily.

this PG(4, 2). This quadric, as any other parabolic quadric in PG(4, 2), has a remarkable property that all its tangent hyperplanes pass through the same point J, called the nucleus (see, e. g., [5]). Any tricentric triad of such a doily defines a plane in the PG(4, 2) that contains J; a unicentric triad also defines a plane, this plane passing through the remaining third point lying on the line defined by J and the (unique) center of the triad. Moreover, all the 15 PG(3, 2)s passing through J intersect our quadric in three concurrent lines that form a perp-set of the doily. Figure 4 offers a pictorial illustration of some of these properties. We again take a four-qubit doily, where we highlighted a perp-set (blue). Now the three lines of the perp-set are not coplanar as in the case of linear doily, but span a PG(3, 2). We colored the remaining eight points (and the totally-isotropic lines) of the PG(3, 2) in yellow in order to stress the property that the only points shared by the doily and this PG(3, 2) are the (blue) points of the perp-set. There are two "distinguished" points of the PG(3, 2), namely $ZYXX$ and $ZYIX$, which lie on the remaining seventh line passing via $IIXI$; the point $ZYXX$ is nothing but the nucleus of the parabolic quadric our particular doily is located on. Given a perp-set, we know that there are four tricentric and four unicentric triads contained in it. In our particular perp, the four *tri*centric triads are $\{XXIX, XZII, ZIXI\}$, $\{XXIX, XZXI, ZIII\}$, $\{XZXI, XXXX, ZIXI\}$ and $\{XXXX, ZIII, XZII\}$; one can readily check that the product of the three observables in any of them is $ZYXX$ (the nucleus). The four *uni*centric triads of our perp-set are $\{XZXI, XXXX, ZIII\}$, $\{XZXI, ZIXI, XXIX\}$, $\{ZIII, XZII, XXIX\}$ and $\{XXX, XZII, ZIXI\}$; the product of the observables in any of them is $ZYIX$, i. e. the second distinguished point. By this construction we get a (different) PG(3, 2) for any of the 15 perp-sets of the doily; and because in any of these perp-sets the four tricentric triads always define the nucleus, $ZYXX$, we get altogether 15 PG(3, 2)s that share the point $ZYXX$, these 15 spaces lying in that particular PG$(4,2)$ of the ambient PG$(7,2)$ that contains the quadric of our selected doily.

In a recent paper [7], four of the authors have thoroughly analyzed and classified three-qubit doilies. To this end, they first explicitly computed all 63 perp-sets, 36 hyperbolic quadrics and 28 elliptic quadrics living in $\mathcal{W}(5,2)$. Then, employing the fact that a linear doily is isomorphic to the intersection of two perp-sets with non-collinear nuclei, they computed and classified all $63 \times 32/3! = 336$ linear doilies of the $\mathcal{W}(5,2)$. In the next step, making use of the property that a quadratic doily is isomorphic to the intersection of an elliptic quadric and a hyperbolic quadric, they generated and classified all $36 \times 28 = 1\,008$ quadratic doilies of the $\mathcal{W}(5,2)$. The procedure described above is, however, not a viable one for $N > 3$, as we would first need to compute all $\mathcal{W}(5,2)$s living in a particular $\mathcal{W}(2N-1,2)$, $N > 3$, and then in each of them compute 336 linear and 1 008 quadratic doilies following the strategy of [7]. Instead, we shall follow (in Section 4) a different, and reasonably faster, approach that makes use of some properties of an ovoid of a doily. In particular, we shall start with a particular N-qubit ovoid, i. e. a set of five mutually anticommuting N-qubit observables whose product is $\pm\mathcal{I}_N$, and introduce a unique algebro-geometrical recipe with the help of which one can find all the N-qubit doilies having this particular ovoid in common. Before embarking on this path, however, we shall introduce several general formulas for the number of both linear and quadratic doilies of $\mathcal{W}(2N-1,2)$, valid for any $N \geq 2$, so that we already have certain important numbers at hand to validate some of our subsequent, mostly computer-assisted, results.

2.3 Contextuality degree

All multi-qubit doilies are observable-based proofs of the Kochen–Specker theorem, that establishes that no Non-Contextual Hidden Variables (NCHV) model can reproduce the outcomes of quantum mechanics. This contextuality property is related to a linear problem, as follows. Let A be the incidence matrix of the points on the lines of a finite geometry, such as the doily. Its coefficients are in the two-elements field $\mathbb{F}_2 = \{0,1\}$, its l rows correspond to the geometric lines and its p columns to the geometric points (for the doily, $l = p = 15$). The positive (resp. negative) nature of a line is encoded by a 0 (resp. 1) for the corresponding coefficient of the *valuation vector* E in $\mathbb{F}_2^l$. Then a quantum geometry is contextual iff there is no vector x such that $Ax = E$. The *contextuality degree* is the minimal Hamming distance between a vector Ax and the vector E [10]. The contextuality degree is the minimal number of line valuations that one should change to make the quantum geometry satisfiable by an NCHV model.

Proposition 1. *All multi-qubit doilies have a contextuality degree of 3.*

Proof. All multi-qubit doilies have the same incidence matrix A. Accordingly, the only parameter that is changing between all the doilies is the vector E, which only depends on the configuration of their negative lines. We have seen that there are only 12 such configurations. For each of these 12 configurations, we have computed the Hamming distance between Ax and E, for all vectors x in $\mathbb{F}_2^{15}$. It turns out that the minimal Hamming distance is always 3. □

In practice, we did not write by hand the 12 possible E vectors, but we computed these vectors from the 5-qubit doilies, because, as described later, we have checked by enumeration that these doilies present all the configurations.

3 Numbers of multi-qubit doilies

This section proposes and justifies closed formulas for the numbers of linear and quadratic doilies in $\mathcal{W}(2N-1,2)$. Before all we introduce some well-known formulas. First, we introduce the Gaussian (binomial) coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-k+i}-1}{q^i-1} = \frac{(q^n-1)\dots(q^{n-k+1}-1)}{(q^k-1)\dots(q-1)} \tag{4}$$

where $0 \le k \le n$ and q is a power of a prime, which gives the number of subspaces of dimension[1] $k-1$ in a projective space $\mathrm{PG}(n-1,q)$ of dimension $n-1$ over $\mathbb{F}_q$. More generally, the number of $(k-1)$-dimensional spaces of $\mathrm{PG}(n-1,q)$ that pass through a fixed $(l-1)$-dimensional space is

$$\begin{bmatrix} n-l \\ k-l \end{bmatrix}_q . \tag{5}$$

Next, for a symplectic polar space $\mathcal{W}(2N-1,q)$ embedded in a projective space $\mathrm{PG}(2N-1,q)$, the number of its k-dimensional spaces is given by (see, e. g., [11, Lemma 2.10])

$$\begin{bmatrix} N \\ k+1 \end{bmatrix}_q \prod_{i=1}^{k+1} \left(q^{N+1-i}+1\right) \tag{6}$$

and the number of k-dimensional spaces through a fixed m-dimensional space [11, Corollary 2.11] equals

$$\begin{bmatrix} N-m-1 \\ k-m \end{bmatrix}_q \prod_{i=1}^{k-m} \left(q^{N-m-i}+1\right) . \tag{7}$$

Further, let $\perp$ be a symplectic polarity of $\mathrm{PG}(n,q)$ and let denote by $S^{\perp}$ the polar space of a subspace S. If S is of dimension k, then $S^{\perp}$ has dimension $n-k-1$. A projective subspace S of $\mathrm{PG}(n,q)$ is called *isotropic* if $S \cap S^{\perp} \neq \emptyset$ and *non-isotropic* if $S \cap S^{\perp} = \emptyset$. An isotropic S is called *totally isotropic* if $S \subseteq S^{\perp}$. It is easy to see that if S is a totally isotropic subspace, then every subspace contained in S is also totally isotropic. Moreover,

$$S \subseteq T^{\perp} \Rightarrow T \subseteq S^{\perp}. \tag{8}$$

In order to prove the two theorems below, we will need a couple of lemmas.

Lemma 2. *If a $PG(3,2)$ of the ambient $PG(5,2)$ equipped with a symplectic polarity $\perp$ contains a totally-isotropic $PG(2,2)$, then it contains exactly three such $PG(2,2)$s, passing through a common (totally-isotropic) $PG(1,2)$.*

Proof. First, there are no totally-isotropic $\mathrm{PG}(3,2)$s in the $\mathrm{PG}(5,2)$. Given a totally-isotropic $\mathrm{PG}(1,2)$ of $\mathrm{PG}(5,2)$, S, there are (see Eq. (7) for $q=2$, $N=3$, $k=2$ and $m=1$) three totally-isotropic $\mathrm{PG}(2,2)$s passing through it. Denoting these as $T_i^{\perp}$ $(i=1,2,3)$, the lemma then follows from the fact that $S^{\perp} \cong \mathrm{PG}(3,2)$, $T_i^{\perp} = T_i$, and property (8). □

[1] All dimensions in this section are projective dimensions.

Remark 3. For $N > 3$, $\mathrm{PG}(2N-1,2)$ features also totally-isotropic $\mathrm{PG}(3,2)$s; any other of its $\mathrm{PG}(3,2)$s endowed with totally-isotropic $\mathrm{PG}(2,2)$s has the property as described in Lemma 2.

Lemma 4. *If a* $PG(4,2)$ *of the ambient* $PG(7,2)$ *equipped with a symplectic polarity* $\perp$ *contains a totally-isotropic* $PG(3,2)$*, then it contains exactly three such* $PG(3,2)$*s, passing through a common (totally-isotropic)* $PG(2,2)$*.*

Proof. The proof parallels that of the preceding lemma. First, there are no totally-isotropic $\mathrm{PG}(4,2)$s in the $\mathrm{PG}(7,2)$. Given a totally-isotropic $\mathrm{PG}(2,2)$ of $\mathrm{PG}(7,2)$, S, there are (see Eq. (7) for $q=2$, $N=4$, $k=3$ and $m=2$) three totally-isotropic $\mathrm{PG}(3,2)$s passing through it. Denoting these as $T_i^{\perp}$ ($i=1,2,3$), the lemma then follows from the fact that $S^{\perp} \cong \mathrm{PG}(4,2)$, $T_i^{\perp} = T_i$, and property (8). □

Remark 5. For $N > 4$, $\mathrm{PG}(2N-1,2)$ features also totally-isotropic $\mathrm{PG}(4,2)$s; any other of its $\mathrm{PG}(4,2)$s endowed with totally-isotropic $\mathrm{PG}(3,2)$s has the property as described in Lemma 4.

Next, through a (totally-isotropic) point of $\mathrm{PG}(5,2)$, S, there pass 15 totally-isotropic $\mathrm{PG}(1,2)$s, $T_j^{\perp}$ ($j=1,2,3,\ldots,15$) and the same number of $\mathrm{PG}(2,2)$s. Given the facts that $S^{\perp} \cong \mathrm{PG}(4,2)$ and $T_j \cong \mathrm{PG}(3,2)$, a $\mathrm{PG}(4,2)$ of $\mathrm{PG}(5,2)$ will contain 15 $\mathrm{PG}(3,2)$s of type defined by Lemma 2 concurring at a point, namely the pole of this particular $\mathrm{PG}(4,2)$. As $\mathrm{PG}(4,2)$ contains altogether 31 $\mathrm{PG}(3,2)$s, each of the remaining 16 $\mathrm{PG}(3,2)$s does not contain totally-isotropic $\mathrm{PG}(2,2)$s and so hosts a unique linear doily. As each such doily can be viewed as a projection of a quadratic doily from the pole, a $\mathrm{PG}(4,2)$ is found to be spanned by 16 quadratic doilies.

Remark 6. If a $\mathrm{PG}(4,2)$ of the ambient $\mathrm{PG}(2N-1,2)$, $N > 3$, is devoid of totally-isotropic $\mathrm{PG}(3,2)$s, then it is of the type described above, i. e. it entails 16 quadratic doilies.

3.1 Number of linear doilies

Theorem 7. *For any* $N \geq 2$ *the number of linear doilies in* $\mathcal{W}(2N-1,2)$ *is*

$$D_l(N) = \begin{bmatrix} 2N \\ 4 \end{bmatrix}_2 - \begin{bmatrix} N \\ 4 \end{bmatrix}_2 \prod_{i=1}^{4} \left(2^{N+1-i}+1\right) - 7 \begin{bmatrix} N \\ 3 \end{bmatrix}_2 2^{2N-6} \prod_{i=1}^{3} \left(2^{N+1-i}+1\right)/3. \tag{9}$$

Proof. A linear doily of $\mathcal{W}(2N-1,2)$ spans a particular $\mathrm{PG}(3,2)$ of the ambient $\mathrm{PG}(2N-1,2)$ that does not contain any totally-isotropic $\mathrm{PG}(2,2)$. And since any such $\mathrm{PG}(3,2)$ is spanned by a single linear doily, the number of linear doilies of $\mathcal{W}(2N-1,2)$ is thus equal to the number of $\mathrm{PG}(3,2)$s that are devoid of totally-isotropic planes. To find this number, from Eq. (4) we first note that there are altogether

$$\begin{bmatrix} 2N \\ 4 \end{bmatrix}_2 \tag{10}$$

$\mathrm{PG}(3,2)$s in $\mathrm{PG}(2N-1,2)$, out of which

$$\begin{bmatrix} N \\ 4 \end{bmatrix}_2 \prod_{i=1}^{4} \left(2^{N+1-i}+1\right) \tag{11}$$

(Eq. (6) with $k=3$ and $q=2$) are totally isotropic.

To ascertain the cardinality of the remaining PG(3, 2)s that feature totally-isotropic PG(2, 2)s, we proceed as follows. We first observe that by Eq. (6) with $k = 2$ and $q = 2$ there are

$$\begin{bmatrix} N \\ 3 \end{bmatrix}_2 \prod_{i=1}^{3} \left(2^{N+1-i} + 1\right) \tag{12}$$

totally-isotropic PG(2, 2)s in PG(2N − 1, 2). Next, with $k = 3$ and $m = 2$ in (7), it follows that there are

$$\begin{bmatrix} N-3 \\ 1 \end{bmatrix}_2 \left(2^{N-3} + 1\right) = 2^{2(N-3)} - 1 \tag{13}$$

totally-isotropic PG(3, 2)s passing through a totally-isotropic PG(2, 2). And since the total number of PG(3, 2)s passing via a PG(2, 2) of PG(2N − 1, 2) is

$$\begin{bmatrix} 2N-3 \\ 4-3 \end{bmatrix}_2 = 2^{2N-3} - 1 \tag{14}$$

(as stemming from Eq. (5) for $n = 2N$, $k = 4$, $l = 3$ and $q = 2$), through a totally-isotropic PG(2, 2) there pass

$$2^{2N-3} - 1 - \left(2^{2(N-3)} - 1\right) = 7 \ \times \ 2^{2N-6} \tag{15}$$

isotropic PG(3, 2)s apart from those that are totally isotropic. Hence, the number of those PG(3, 2)s of PG(2N − 1, 2) that are endowed with totally-isotropic PG(2, 2)s − with the exclusion of totally isotropic ones − amounts to (12) × (15)/3, where we also took into account (see Remark 3) that any such PG(3, 2) features just three totally-isotropic PG(2, 2)s. All in all, there are

$$\begin{bmatrix} 2N \\ 4 \end{bmatrix}_2 - \begin{bmatrix} N \\ 4 \end{bmatrix}_2 \prod_{i=1}^{4} \left(2^{N+1-i} + 1\right) - 7 \begin{bmatrix} N \\ 3 \end{bmatrix}_2 2^{2N-6} \prod_{i=1}^{3} \left(2^{N+1-i} + 1\right) /3$$

PG(3, 2)s in the ambient PG(2N − 1, 2) that are devoid of totally-isotropic PG(2, 2)s, and so the same number of linear doilies in $\mathcal{W}(2N-1, 2)$. □

Given the fact that the three lines of a perp-set of a linear doily span a PG(2, 2), and namely that PG(2, 2) that features just three totally-isotropic PG(1, 2)s, we arrive at the interesting expression

$$D_l(N) = \frac{4}{15} 4^{N-3} \Theta_2(N), \tag{16}$$

for the number of linear doilies in $\mathcal{W}(2N-1, 2)$, where

$$\Theta_2(N) = \frac{1}{16} 2^{2N} \prod_{i=1}^{2} \frac{2^{N-2+i} - 1}{2^i - 1} \prod_{i=1}^{2} (2^{N+1-i} + 1) \tag{17}$$

is the number of those PG(2, 2)s of the ambient PG(2N − 1, 2) each of which features just three totally-isotropic PG(1, 2)s.

3.2 Number of quadratic doilies

Theorem 8. *For any $N \geq 3$ the number of quadratic doilies in $\mathcal{W}(2N-1,2)$ is*

$$D_q(N) = 16\left(\begin{bmatrix}2N\\5\end{bmatrix}_2 - \begin{bmatrix}N\\5\end{bmatrix}_2 \prod_{i=1}^{5}\left(2^{N+1-i}+1\right) - 15\begin{bmatrix}N\\4\end{bmatrix}_2 2^{2N-8}\prod_{i=1}^{4}\left(2^{N+1-i}+1\right)/3\right). \quad (18)$$

Proof. A quadratic doily of $\mathcal{W}(2N-1,2)$ spans a particular PG$(4,2)$ of the ambient PG$(2N-1,2)$ that does not contain any totally-isotropic PG$(3,2)$. And since any such PG$(4,2)$ is spanned by (see Remark 6) 16 such doilies that are all unique to this space, the number of quadratic doilies of $\mathcal{W}(2N-1,2)$ is thus equal to 16 times the number of PG$(4,2)$s that are devoid of totally-isotropic PG$(3,2)$s. To find the latter number, we again start with Eq. (4) that tells us that there are altogether

$$\begin{bmatrix}2N\\5\end{bmatrix}_2 \quad (19)$$

PG$(4,2)$s in PG$(2N-1,2)$, out of which

$$\begin{bmatrix}N\\5\end{bmatrix}_2 \prod_{i=1}^{5}\left(2^{N+1-i}+1\right) \quad (20)$$

(Eq. (6) with $k=4$ and $q=2$) are totally isotropic.

To ascertain the cardinality of the remaining isotropic PG$(4,2)$s, we proceed as follows. We first observe that by Eq. (6) with $k=3$ and $q=2$ there are

$$\begin{bmatrix}N\\4\end{bmatrix}_2 \prod_{i=1}^{4}\left(2^{N+1-i}+1\right) \quad (21)$$

totally-isotropic PG$(3,2)$s in PG$(2N-1,2)$. Next, with $k=4$, $m=3$ and $q=2$ in (7) it follows that there are

$$\begin{bmatrix}N-4\\1\end{bmatrix}_2 \left(2^{N-4}+1\right) = 2^{2(N-4)} - 1 \quad (22)$$

totally-isotropic PG$(4,2)$s passing through a totally-isotropic PG$(3,2)$. And since the total number of PG$(4,2)$s passing via a PG$(3,2)$ of PG$(2N-1,2)$ is

$$2^{2N-4} - 1 \quad (23)$$

(as stemming from Eq. (5) for $n=2N$, $k=5$, $l=4$ and $q=2$), through a totally-isotropic PG$(3,2)$ there pass

$$2^{2N-4} - 1 - \left(2^{2(N-4)} - 1\right) = 15 \times 2^{2N-8} \quad (24)$$

PG$(4,2)$s that feature totally-isotropic PG$(3,2)$s apart from those that are totally isotropic. Hence, the number of those PG$(4,2)$s of PG$(2N-1,2)$ that contain totally-isotropic PG$(3,2)$s – with the exclusion of totally isotropic ones – amounts to $(21) \times (24)/3$, where we also took into account (see Remark 5) that any such PG$(4,2)$ features just three totally-isotropic PG$(3,2)$s. All in all, there are

$$\begin{bmatrix}2N\\5\end{bmatrix}_2 - \begin{bmatrix}N\\5\end{bmatrix}_2 \prod_{i=1}^{5}\left(2^{N+1-i}+1\right) - 15\begin{bmatrix}N\\4\end{bmatrix}_2 2^{2N-8}\prod_{i=1}^{4}\left(2^{N+1-i}+1\right)/3 \quad (25)$$

PG(4, 2)s in the ambient PG(2N − 1, 2) that are not endowed with any totally-isotropic PG(3, 2)s, the number of quadratic doilies of $\mathcal{W}(2N-1,2)$ being just 16 times this number. □

Employing further the fact that the three lines of a perp-set of a quadratic doily span a PG(3, 2), in particular that PG(3, 2) that contains just three totally-isotropic PG(2, 2)s, we find the compact formula

$$D_q(N) = \frac{48}{15} 4^{N-3} \Theta_3(N) \tag{26}$$

for the number of quadratic doilies in $\mathcal{W}(2N-1,2)$, where

$$\Theta_3(N) = \frac{7}{3} 2^{2N-6} \prod_{i=1}^{3} \frac{2^{N-3+i}-1}{2^i-1} \prod_{i=1}^{3} \left(2^{N+1-i}+1\right) \tag{27}$$

is the number of those PG(3, 2)s of the ambient PG(2N − 1, 2) each of which features just three totally-isotropic PG(2, 2)s.

N	$D_l(N)$	$D_q(N)$	$D(N)$
2	1	-	1
3	336	1 008	1 344
4	91 392	1 370 880	1 462 272
5	23 744 512	1 495 904 256	1 519 648 768
6	6 100 942 848	1 555 740 426 240	1 561 841 369 088
7	1 563 272 675 328	1 599 227 946 860 544	1 600 791 219 535 872
8	400 289 425 260 544	1 639 185 196 441 927 680	1 639 585 485 867 188 224
9	102 479 956 839 235 584	1 678 929 132 897 196 572 672	1 679 031 612 854 035 808 256

Table 1: First numbers $D(N)$ (resp. $D_l(N)$, $D_q(N)$) of (resp. linear, quadratic) N-qubit doilies.

Comparing expressions (9) and (18), one gets

$$D_q(N) = (4^{N-2} - 1) D_l(N). \tag{28}$$

Consequently, the total number of doilies is

$$D(N) = 4^{N-2} D_l(N). \tag{29}$$

For $2 \le N \le 9$ the numbers of N-qubit doilies are collected in Table 1. Since the quadratic doilies of $\mathcal{W}(2N-1,2)$ span a PG(4, 2), it has no geometrical meaning to consider quadratic doilies for $N = 2$. It can nevertheless be noticed that Eq. (18) also holds for $N = 2$, and consistently gives $D_q(2) = 0$.

4 Generation of all N-qubit doilies

An N-qubit doily can be represented by an isomorphism f, sometimes called a (*doily*) *labeling*, mapping the points of W_2 to distinct points of W_N, preserving commutations and anticommutations, and such that $f(a.b) = \pm f(a).f(b)$ for any two commuting points/observables a and b (the dot (.) denotes the matrix product). This section describes an algorithm for the enumeration of all N-qubit doilies, for any $N \geq 2$, by construction of one of their labelings.

Let us start with some definitions. An *N-qubit ovoid* is a 5-set of mutually anticommuting N-qubit observables whose product is the identity $\mathcal{I}_N$. A *triad* is a 3-set of mutually anticommuting N-qubit observables. A *center* of a triad is a point commuting with the three points of the triad. A *unicentric triad* is a triad that has only one center. Let ε denote the empty word. The *lexicographic order* $<$ on words is such that $\varepsilon < u$ for all non-empty word u, and $a.u < b.v$ if and only if either $a < b$, or $a = b$ and $u < v$, for any letters a and b and words u and v.

In order to avoid to consider several times objects that are similar but differently ordered, we define as follows a total order among letters and words, and then extend it to all tuples and sets of objects of the same nature, such as lines, sets of lines, etc. Pauli observables, encoded as words on the alphabet $\{I, X, Y, Z\}$, are totally ordered by the lexicographic order $<$ induced by the order on letters, also denoted $<$, such that $I < X < Z < Y$. These orders are chosen so that their binary counterpart through the encoding $I \to 00$, $X \to 01$, $Z \to 10$, $Y \to 11$ is the lexicographic order on bit vectors (aka. bytes or binary words) induced by the order $0 < 1$ on bits. This order $<$ extends further to tuples $(a_1, a_2, \ldots, a_n)$ of words, by considering them as words $a_1 a_2 \cdots a_n$ and re-using the former lexicographic order on words. It also extends to sets of words, by associating canonically to each set the tuple $(a_1, a_2, \ldots, a_n)$ of its elements written in increasing order ($a_i < a_j$ when $i < j$), and so on at any level of the hierarchy of objects of the same nature, such as a point-line geometries, seen as sets of lines, that are sets of points.

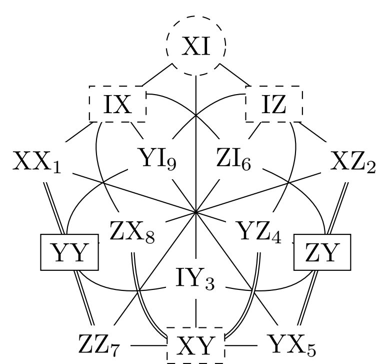


Figure 5: The 2-qubit doily W_2, the ovoid O_2 (framed), the triad T_2 (framed and dashed), its center c_2 (circled and dashed) and the completion order (subscripted). The negative lines are doubled.

The algorithm relies on the following predefined elements, depicted in Fig. 5: the 2-qubit doily W_2, the ovoid $O_2 \equiv \{IX, IZ, XY, ZY, YY\}$ in W_2, the unicentric triad $T_2 \equiv \{IX, IZ, XY\}$ in O_2, the center $c_2 \equiv XI$ of T_2, and the sequence of lines

$$S \equiv (XI, IX, XX), (XI, IZ, XZ), (XI, XY, IY), (ZY, XX, YZ), (ZY, XZ, YX), (ZY, IY, ZI), (YY, XX, ZZ), (YY, XZ, ZX), (YY, IY, YI).$$

In the figure the third element of the tuples in this *completion order* is numbered from 1 to 9.

The algorithm itself is presented in Algorithm 1, where $f(a) \leftarrow b$ denotes the assignment of b as the image of a by f.

Algorithm 1 Doily generation algorithm.

1: **for each** ovoid $O = \{o_1, o_2, o_3, o_4, o_5\}$ in W_N, with $o_1 < o_2 < o_3 < o_4 < o_5$ **do**
2: $\quad f(IX) \leftarrow o_1 \mid\mid f(IZ) \leftarrow o_2 \mid\mid f(XY) \leftarrow o_3 \mid\mid f(ZY) \leftarrow o_4 \mid\mid f(YY) \leftarrow o_5$
3: $\quad$ **for each** center c of $\{o_1, o_2, o_3\}$ in W_N that anticommutes with o_4 and o_5 **do**
4: $\qquad f(c_2) \leftarrow c$
5: $\qquad$ **for each** line (p, q, r) in the order of the sequence S **do** $f(r) \leftarrow |f(p).f(q)|$ **end for**
6: $\qquad$ **if** O is not the smallest ovoid of f **then** discard f **end if**
7: $\qquad \ldots$ ▷ location for a potential treatment of f
8: $\quad$ **end for**
9: **end for**

On Line 2 a doily labeling f is partially defined by the choice of images for the 5 points of the ovoid O_2 of W_2. These images are the points of some ovoid $O = \{o_1, o_2, o_3, o_4, o_5\}$ of N-qubit observables. The points are assigned in increasing order so that to avoid duplicates. As these five assignments are independent, they can be performed in parallel.

Then (on Line 3) the algorithm looks for a point c that commutes with the first three points of O and that anticommutes with its last two points o_4 and o_5. On Line 4 this point becomes the image by f of the center c_2 of the triad T_2 of O_2.

The completion step on Line 5 computes one by one the images of all the other points of W_2 by f, in the order described by the sequence of lines S. At each iteration of this loop, for the line (p, q, r), the values $f(p)$ and $f(q)$ are known. By definition of a doily line, the image by f of the third point r is the product of the images $f(p)$ and $f(q)$ of the first two points, up to a possible minus sign, removed by the operation $|\_|$ that denotes absolute value.

Knowing that each doily features 6 ovoids, the same doily is generated 6 times before Line 6, whose statement keeps only one of them, namely the doily d generated from the ovoid that is the smallest (according to the lexicographic order) among the 6 ovoids in d.

On Line 7 various treatments of the generated doilies f can be added, such as a storage, or the computation of classification criteria defined in Section 5.

4.1 Justification of the generation algorithm

First of all, the fact that doily labelings encode multi-qubit doilies is a direct consequence of the definition of a multi-qubit doily. Then, the properties of correctness and completeness for the doily enumeration algorithm mainly come from the following definition and proposition, whose proof is illustrated by Figure 6.

Definition 9 (Doily Root)**.** Any pair (O, c) such that O is an ovoid of W_N and c is a point of W_N that commutes with exactly three points of O and anticommutes with the other two points is called a *(multi-qubit) doily root.*

Proposition 10. *Any doily root* (O, c) *of* W_N *determines exactly one* N*-qubit doily.*

Proof. Let $O = \{o_1, o_2, o_3, o_4, o_5\}$ and c be such that (O, c) is a multi-qubit doily root. The fact that c commutes with o_1, o_2 and o_3 implies that there exist three multiplicative factors a_1, a_2 and $a_3 \in \{-1, 1\}$ such that $\{c, o_i, a_i\ c.o_i\}$ are isotropic lines of W_N, for $i = 1, 2, 3$. This is depicted in Figure 6 by the 3 points $\pm c.o_i$. But because c anticommutes with o_4 and o_5 we also have

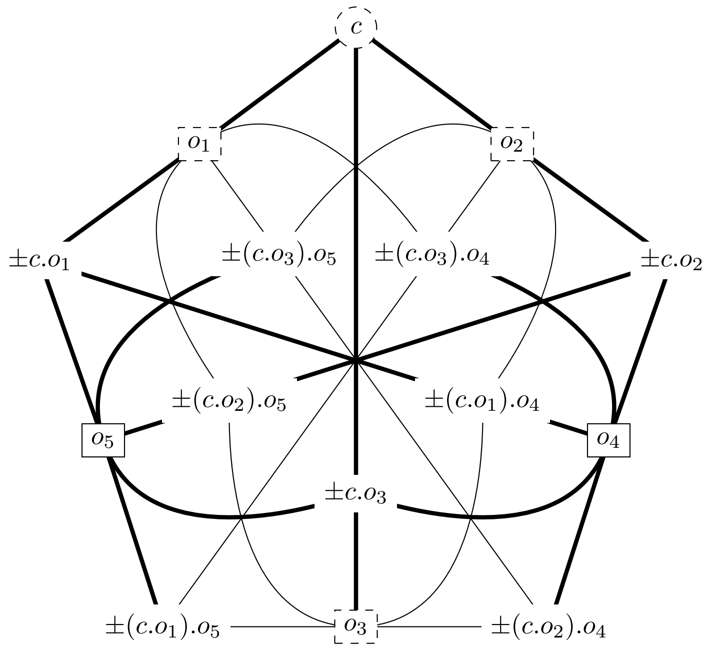


Figure 6: Expression of each observable according to the completion order followed by the generation algorithm, from the doily root $(\{o_1, o_2, o_3, o_4, o_5\}, c)$. The thick lines are the lines used to compute these expressions.

that the observables $a_i\ c.o_i$ commute with o_4 and o_5. Therefore there exist multiplicative factors $a_{ij} \in \{-1, 1\}$ such that $\{a_i\ c.o_i, o_j, a_{ij}\ (c.o_i).o_j\}$ are isotropic lines, for $i = 1, 2, 3$ and $j = 4, 5$. This is depicted in Figure 6 by the 6 points of the form $\pm(c.o_i).o_j$. These 9 points are computed by the completion step of the generation algorithm, in the order $a_1\ c.o_1$, $a_2\ c.o_2$, $a_3\ c.o_3$, $a_{14}\ (c.o_1).o_4$, $a_{24}\ (c.o_2).o_4$, $a_{34}\ (c.o_3).o_4$, $a_{15}\ (c.o_1).o_5$, $a_{25}\ (c.o_2).o_5$, $a_{35}\ (c.o_3).o_5$. The 9 geometric lines thus identified are depicted by thick lines in Figure 6. Finally, it is easy to check that the six 3-sets represented by the thin lines in Figure 6 indeed are geometric lines. A noticeable property is that the product of the three points on each of these lines contains twice the center c and once each point of the ovoid O. By applying the known commutation and anticommutation relations between these points, it comes that the product of both centers annihilates. So, modulo a possible minus sign, it remains the product of all observables of the ovoid, known to equal identity. Therefore, the product of the three observables on each line equals $\pm\mathcal{I}_N$. Consequently, these 15 points and 15 lines form a doily, shown in Figure 6, so the algorithm is correct.

Each multi-qubit doily features at least one ovoid and the first loop explores all ovoids in W_N. So, each doily is found six times before Line 6, since each multi-qubit doily features exactly six ovoids. As the statement on this line always keeps one of them (the one that has been produced from the smallest of its ovoids), the algorithm is also complete. $\square$

4.2 Algorithmic complexity and implementation details

The enumeration algorithm explores all 4-tuples of observables likely to form an ovoid (the fifth point in the ovoid is computed as the product of the previous four), and then explores all observables to find c (on Line 3 of Algorithm 1). Therefore, the complexity of the algorithm is estimated to be $O\left(4^{5N}\right)$, when the time unit is the duration to check whether two observables commute.

For efficiency reasons, we have implemented the algorithm in the C language, which allows for many optimizations. The total code is composed of about 2 300 lines and 50 functions, some of which implementing the classification process presented in Section 5. Some factors make the algorithm implementation more efficient than the former one presented in [7]: The new algorithm has a lower complexity; compared to the previously used language Magma [12], the low-level language C allows to perform fast operations on bit vector representations of the observables, using as few CPU instructions as necessary, and to split the workload into multiple threads.

The calculations were run on Linux Ubuntu, on a PC equipped with an Intel (R) Core(TM) i7-8665U 1.90 GHz and 15 GB RAM. The code was compiled with gcc 9.3.0 with optimization Ofast and is multi-threaded with OpenMP.

5 Multi-doily classification process and results

This section presents our classification criteria of N-qubit doilies and the classification results for $N = 4$ and $N = 5$.

5.1 Classification criteria

The classification parameters adopted are the same as in [7]. The classification of an N-qubit doily is based on: 1) its signature, i. e. the number of its observables containing a given number of I : $N-1$, $N-2$, $N-3$, ... respectively named types A, B, C, ...; 2) the configuration of its negative lines, as described in Section 2; and 3) its linear or quadratic character.

To find the line configuration of a doily, the first discriminatory factor is the number of negative lines, since for each number of negative lines except 7 and 8, there is only one configuration possible. Then the property used to distinguish configurations 7A from 7B and 8A from 8B is to count the number of observables contained in at least one negative line, since this number is different between A and B.

We use the following property to check whether a doily is linear or quadratic. Given an N-qubit doily, we pick up in it a tricentric triad (here we take the image of $\{XY, ZY, YI\}$). If the product of the corresponding three observables is $\pm i\mathcal{I}_N$, then the doily is linear, otherwise it is quadratic. This is because any tricentric triad is a line in the ambient $\mathrm{PG}(3,2)$ if a doily spans a $\mathrm{PG}(3,2)$.

5.2 Database of numerical results

Using the program described in Section 4, we were able to classify all doilies for $N = 3$ (2 016 ovoids), $N = 4$ (548 352 ovoids) and $N = 5$ (142 467 072 ovoids). This classification is a treatment added on Line 7 of the algorithm presented in Algorithm 1, that determines the complete type of each generated doily, counts the number of doilies for each type, and registers it in a result table.

The sums of the numbers of linear and quadratic doilies found in each of the above-mentioned cases correspond exactly to those stemming from eqs. (9) and (18), respectively, summarized in

Table 1. The results of our classification are collected in Appendix A (three qubits), Appendix B (four qubits) and Appendix C (five qubits). The data for three qubits are in complete agreement with those of [7]; we found 11 different types of doilies of which five are linear and six quadratic. The 95 distinct types of four-qubit doilies split into 24 linear and 71 quadratic ones, whereas amongst 447 types of five-qubit doilies one finds 89 linear and 358 quadratic.

The structure of the classification table in each appendix is the same: the first column gives the type, the next N columns feature the numbers of observables of the corresponding types in a doily of the given type, the ν column shows the doily's character, and the remaining columns contain information about how many doilies of the given type are endowed with a particular number of negative lines (the blank space stands for zero here). The types are ordered in decreasing order of the number of observables containing no Is, in case of equality in decreasing order of the number of observables containing one I, and so on up to the number A of observables containing $N-1$ Is. For instance, for 4 qubits, the type 1 contains the maximal number 12 of D-type observables, and the last type 95 contains only A- and B-type observables. For a given signature, the type of quadratic doilies precedes that of linear ones.

The result tables are stored in `https://quantcert.github.io/`.

The C code for classification runs in 0.3 s for 4 qubits with 1.4 MB of memory and 12 min with 1.8 MB of memory for 5 qubits. The memory usage is low because the doilies are not stored, all the measurements are performed on the fly.

5.3 Remarks about five-qubit doilies

Let us have a closer look at the five-qubit case. The $3^2 \times \binom{5}{2} = 90$ observables of type B and $3^4 \times \binom{5}{4} = 405$ observables of type D lie on an elliptic quadric $\mathcal{Q}^-_{(YYYYY)}(9,2)$ of $\mathcal{W}(9,2)$. This special quadric $\mathcal{Q}^-_{(YYYYY)}(9,2)$, like any non-degenerate quadric, is a *geometric hyperplane* of $\mathcal{W}(9,2)$. As a doily is also a *subgeometry* of $\mathcal{W}(9,2)$, it either lies fully in $\mathcal{Q}^-_{(YYYYY)}(9,2)$ (in which case $B \cup D = 15$, such a doily will be called special), or shares with $\mathcal{Q}^-_{(YYYYY)}(9,2)$ a set of points that form a geometric hyperplane, in particular an ovoid ($B \cup D = 5$), a perp-set ($B \cup D = 7$) and/or a grid ($B \cup D = 9$) and being referred to as ovoidal, perpial and/or gridal, respectively.

From Appendix C one can infer a number of interesting properties. We first notice that signatures with $B \cup C$ being even or odd are endowed with even or odd numbers of negative lines, respectively.

We also observe that there are 12 different signatures with $A = C = E = 0$, i. e., signatures featuring solely special doilies.

Further, there are 17 particular signatures such that each features observables of every type and no two types have the same cardinality. Out of them, six are ovoidal (e. g., 2-1-3-4-5), seven perpial (e. g., 1-3-2-4-5) and four gridal (e. g., 2-4-3-5-1).

If all doilies of a particular signature have just five or just six negative lines, then each doily is ovoidal; if a signature features just seven negative lines, then all of its doilies are perpial.

Among 33 distinct signatures with four negative lines only, one finds 12 ovoidal, 11 perpial and 9 gridal ones; doilies of the remaining signature, viz. 0-8-0-7-0, are special.

Next, there are 15 different signatures whose doilies are endowed with 12 (i. e., the maximum number of) negative lines. Out of them, five are ovoidal, five perpial and four gridal; the doilies of the remaining signature, namely 0-0-0-15-0, are special. Similarly, there are 35 distinct signatures whose doilies contain 11 (i. e., the maximum odd number of) negative lines; out of them, 10 are ovoidal, 11 perpial and 12 gridal, with the remaining two signatures, viz. 0-1-0-14-0 and 0-3-0-12-0, featuring solely special doilies.

5.4 Specific behavior of linear doilies

Finally, this section and the next one briefly mention some properties of linear doilies. Like the three- and four-qubit cases, a linear five-qubit doily can be either ovoidal or gridal and always contains an odd number of negative lines. Also, 75 types of linear doilies share their signatures with their quadratic siblings. However, there are 14 different signatures that are genuinely linear, of which eight cases are ovoidal.

From our results on three-, four- and five-qubit cases it follows that a linear doily (a) always features an odd number of negative lines, and (b) does not share a perp-set with the distinguished quadric.

We conjecture that Property (a) holds for any number of qubits $N \geq 2$, but we have not yet found of proof of it; we surmise that it has something to do with the fact that a linear doily is "squeezed" into a PG$(3, 2)$, compared to a quadratic doily that enjoys more degrees of freedom being stretched out in a PG$(4, 2)$. Property (b) can readily be proved to hold for any $N \geq 3$, as follows.

Proof. Let us consider a linear doily with one of its perp-set; on Figure 7a, this perp-set is illustrated in bold font. Any perp-set of any doily features four tricentric triads; in our perp-set these triads are: $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{1, 5, 6\}$ and $\{2, 6, 4\}$. Now, we know that any tricentric triad of a linear doily corresponds to a non-isotropic line in the ambient projective space, the four lines plus the three (totally isotropic) lines of the perp-set forming a Fano plane in this space, which is illustrated in Figure 7b. The assumption that our perp-set also lies on the distinguished quadric would mean that the whole plane would lie in the distinguished quadric and so would be totally isotropic, a contradiction. □

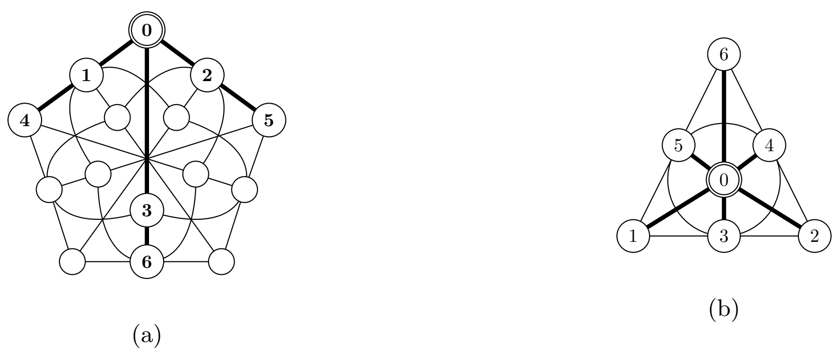


Figure 7: Graphical arguments for the property that a linear doily cannot share a perp-set with the distinguished quadric.

5.5 A distinguished hexad of (linear) doilies

One knows that given an ovoid, there is a unique linear doily containing this ovoid. Now, take any quadratic doily. As each of its six ovoids defines a unique linear doily, we have a unique hexad of doilies tied to each quadratic doily. This holds for any $N \geq 3$. Figure 8 illustrates this property for $N = 3$. It features a quadratic doily in the middle, its six ovoids depicted explicitly as pentads of points located on bold gray lines and the corresponding six linear doilies; for better readability, the points of the corresponding ovoids are illustrated by double-circles.

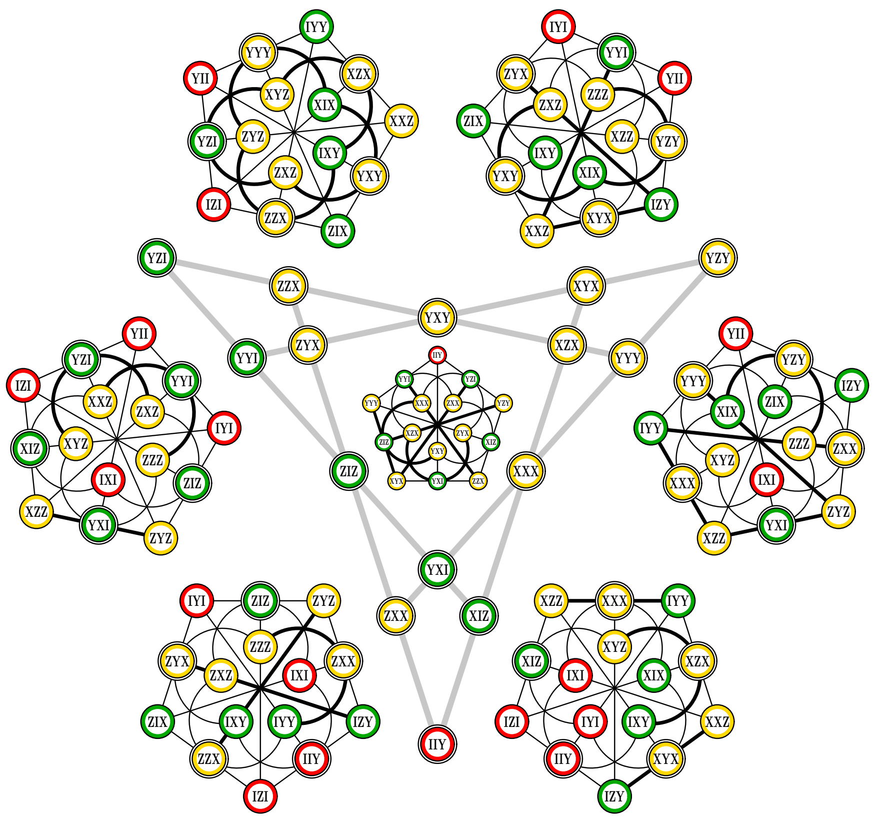


Figure 8: A particular hexad of linear doilies in the three-qubit symplectic polar space.

6 Conclusion

There are a number of intriguing extensions and generalizations of the ideas and findings presented in this paper. We shall mention a few of them.

An interesting situation that will be worth addressing occurs in the case of $N = 4$. Given a $\mathrm{PG}(3,2)$ of the ambient $\mathrm{PG}(7,2)$ of $\mathcal{W}(7,2)$, its polar space is another $\mathrm{PG}(3,2)$. Hence, $\mathrm{PG}(3,2)$s in $\mathrm{PG}(7,2)$ come in polar pairs. Taking into account the fact that a non-isotropic $\mathrm{PG}(3,2)$ features a unique linear doily of $\mathcal{W}(7,2)$, the above property means that also linear doilies of $\mathcal{W}(7,2)$ occur in pairs. That is, picking up any linear four-qubit doily, there exists a unique linear doily such that each of its 15 observables commutes with each observable of the selected doily. This observation raises several interesting questions. For example, it would be interesting to ascertain which signatures are/can be paired, or which cardinalities of negative lines can occur in such self-polar pairs; we have already checked by hand a few examples where both doilies in a pair have the same signature and feature the same number of negative lines. There are (see Appendix B) altogether 24 different signatures featured by linear four-qubit doilies. We can then create a graph on 24 vertices such that its two edges are connected if there exists a pair of linear doilies exhibiting the corresponding signatures; we can even add a weight to an edge showing how many pairs of doilies feature this particular pair of signatures. This graph, as it follows from the examples checked, will also have edges joining a vertex to itself when the two paired signatures are identical. So, being an interesting graph of its own, it will also reveal some finer traits of the relation between individual linear doilies in $\mathcal{W}(7,2)$!

A particular case deserving closer attention is $N = 6$. Here, let us formally view any six-qubit observable as a 'syntheme' partitioned into three two-qubit observables ('duads'). Given a partition, we find a set of linear doilies such that any doily in the set features 15 particular observables such that when restricted to the same duad we get a two-qubit doily; that is, any such doily can formally be regarded as being composed of three two-qubit doilies. Moreover, each partition features a prominent doily having all the three duads identical. The next worth-exploring case in this respect is $N = 9$, as $\mathcal{W}(17,2)$ hosts not only composites comprising three doilies having the same number of qubits (namely three), but also those whose compounds feature different numbers of qubits (namely four, three and two).

Another prospective, but much more challenging, task will be to count and classify all rank-three spaces, $\mathcal{W}(5,2)$s, living in a particular W_N, for $N \geq 4$. The case $N = 4$ was already briefly examined in [7]. To address higher rank cases, we plan to employ the strategy that is the direct and natural generalization of the ovoid-based algorithm for doilies described in this paper. Geometrically, an N-qubit ovoid is a set of five points lying on a certain elliptic quadric of a PG$(3,2)$ in the ambient PG$(2N-1,2)$. Hence, its analogue will be a set of 27 N-qubit observables lying on an elliptic quadric of PG$(5,2)$ in the ambient PG$(2N-1,2)$, and a triad of the ovoid will have its counterpart in a quadratic doily located on the quadric. A root of an N-qubit $\mathcal{W}(5,2)$ will thus comprise an elliptic quadric and an off-quadric point such that its associated observable commutes with each of the 15 observables of a doily located in the quadric and anticommutes with the remaining 12 observables. It is obvious that this task will require a more elaborate generation algorithm and a more complex computer code to be successfully accomplished.

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References

[1] J.J. Sylvester. XLIV. Elementary researches in the analysis of combinatorial aggregation. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 24(159):285–296, 1844. https://doi.org/10.1080/14786444408644856.

[2] L. Cremona. *Teoremi stereometrici dal quali si deducono le proprietà dell' esagrammo di Pascal (Stereometric theorems from which the properties of Pascal's hexagram are deduced)*. Atti dell'Academia dei Lincei (Proceedings of the Accademia dei Lincei), coi tipi del salviucci edition, 1877.

[3] H. W. Richmond. The figure formed from six points in space of four dimensions. *Mathematische Annalen*, 53:161–176, 1900.

[4] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles*. European Mathematical Society, 2009.

[5] J. W. P. Hirschfeld and J. A. Thas. *General Galois Geometries*. Clarendon Press, 1991.

[6] P. J. Cameron. *Projective and Polar Spaces*. University of London, Queen Mary and Westfield College, 1992.

[7] M. Saniga, H. de Boutray, F. Holweck, and A. Giorgetti. Taxonomy of Polar Subspaces of Multi-Qubit Symplectic Polar Spaces of Small Rank. *Mathematics*, 9(18):2272, 2021. http://dx.doi.org/10.3390/math9182272.

[8] F. Holweck. Geometric constructions over $\mathbb{C}$ and $\mathbb{F}_2$ for Quantum Information. In *Quantum Physics and Geometry*, volume 25 of *Lecture Notes of the Unione Matematica Italiana*, pages 87–124. Springer Nature, 2019. https://arxiv.org/abs/1810.04258.

[9] B. Polster. *A Geometrical Picture Book*. Springer New York, 1998.

[10] H. de Boutray, F. Holweck, A. Giorgetti, P.-A. Masson, and M. Saniga. Contextuality degree of quadrics in multi-qubit symplectic polar spaces. 2022. https://arxiv.org/abs/2105.13798.

[11] M. De Boeck, M. Rodgers, L. Storme, and A. Švob. Cameron-Liebler sets of generators in finite classical polar spaces. *Journal of Combinatorial Theory Series A*, 167(C):340–388, 2019. http://dx.doi.org/10.1016/j.jcta.2019.05.005.

[12] W. Bosma, J. Cannon, and C. Playoust. The Magma Algebra System I: The User Language. *Journal of Symbolic Computation*, 24(3):235–265, 1997. http://dx.doi.org/10.1006/jsco.1996.0125.

A Taxonomy of 3-qubit doilies

	Observables				Configuration of negative lines											
Type	A	B	C	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	1	5	9	q				108								
2	0	7	8	q					81							
3	2	5	8	l			162									
4	3	5	7	q		324										
5	0	9	6	l	9				27							
6	2	7	6	q	216		162									
7	4	5	6	l	54											
8	2	9	4	l	81											
9	4	7	4	q	81											
10	0	15	0	q	36											
11	6	9	0	l	3											

B Taxonomy of 4-qubit doilies

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	3	0	12	q	216				648				648			
2	0	4	0	11	q				3888			3888					
3	0	5	0	10	q	972		1944		4860	1944			1944			
4	1	0	5	9	q	648								648			
5	3	0	3	9	l	144											
6	0	6	0	9	q		1296		5184								
7	0	1	6	8	q	972				3888						972	
8	1	1	5	8	q				7776								
9	2	1	4	8	q	1944		1944									
10	2	1	4	8	l	972					972						
11	0	7	0	8	q			1944		972							
12	0	2	6	7	q				15 552			11 664	19 440				
13	1	2	5	7	q	7776		13 608			15 552			1944			
14	1	2	5	7	l	3888					7776						
15	2	2	4	7	q		11 664						3888				
16	3	2	3	7	q	1944		1944									
17	0	8	0	7	q		3888										
18	0	1	8	6	q	648		3888		1944	15 552			11 664			
19	1	1	7	6	q		19 440		18 144				11 664				
20	0	3	6	6	q	7452		21 384		30 132	46 656			8424			
21	0	3	6	6	l	2592		1944		4860	11 664			648		324	
22	2	1	6	6	q	3888		9720			1944						

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
23	1	3	5	6	q		46 656		32 400				11 664				
24	2	3	4	6	q	11 016		11 664			4860						
25	2	3	4	6	l			3888									
26	3	3	3	6	q		3888										
27	4	3	2	6	l	324											
28	0	9	0	6	q	2700		1944		324							
29	0	0	10	5	q				1296				3888				
30	1	0	9	5	q			5832						1944			
31	1	0	9	5	l	648					1944						
32	0	2	8	5	q		7776		23 328			15 552	19 440				
33	1	2	7	5	q	15 552		46 656			19 440			3888			
34	0	4	6	5	q		11 664		19 440				7776				
35	2	2	6	5	q		31 104						3888				
36	1	4	5	5	q	7776		11 664			3888						
37	1	4	5	5	l			7776									
38	2	4	4	5	q		7776										
39	0	1	10	4	q					1944	3888						
40	0	1	10	4	l			1944						1944			
41	1	1	9	4	q		7776		7776				3888				
42	0	3	8	4	q	6480		17 496		23 328	15 552			3888			
43	2	1	8	4	q	1944					1944						
44	2	1	8	4	l			1944									
45	1	3	7	4	q		27 216		7776				3888				
46	0	5	6	4	q	23 328		23 328		14 580	13 608						
47	0	5	6	4	l	972		7776		2916							
48	2	3	6	4	q	8424		15 552									
49	1	5	5	4	q		31 104		3888								
50	3	3	5	4	q		7776										
51	2	5	4	4	q	3888		3888									
52	2	5	4	4	l	1458											
53	4	5	2	4	q	486											
54	0	11	0	4	q	972											
55	0	2	10	3	q		3888		6480			3888	3888				
56	1	2	9	3	q	4536		3888									
57	1	2	9	3	l			7776									
58	0	4	8	3	q		15 552		24 624			3888					
59	2	2	8	3	q		15 552										
60	1	4	7	3	q	28 512		27 216			3888						
61	3	2	7	3	l	648											
62	0	6	6	3	q		19 440		3888								
63	2	4	6	3	q		19 440										
64	1	6	5	3	q	7776		3888									

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
65	1	6	5	3	l	3888											
66	3	4	5	3	q	3888											
67	0	3	10	2	q					3888							
68	0	3	10	2	l	648				1944							
69	1	3	9	2	q		7776		1296								
70	0	5	8	2	q	9720		11 664		3888							
71	2	3	8	2	q	5832		3888									
72	2	3	8	2	l	1944											
73	1	5	7	2	q		7776										
74	0	7	6	2	q	8748		5832		972							
75	0	7	6	2	l	1944											
76	2	5	6	2	q	5832											
77	2	7	4	2	q	1944											
78	1	4	9	1	q	3888		3888									
79	1	4	9	1	l	1944											
80	0	6	8	1	q		3888										
81	1	6	7	1	q	3888											
82	3	4	7	1	q	1944											
83	0	5	10	0	q	2592											
84	1	5	9	0	q				432								
85	0	7	8	0	q	6480				324							
86	2	5	8	0	l			648									
87	3	5	7	0	q		1296										
88	0	9	6	0	q	4266											
89	0	9	6	0	l	36				108							
90	2	7	6	0	q	864		648									
91	4	5	6	0	l	216											
92	2	9	4	0	l	324											
93	4	7	4	0	q	324											
94	0	15	0	0	q	144											
95	6	9	0	0	l	6											

C Taxonomy of 5-qubit doilies

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	0	1	5	9	q						58 320			19 440			
2	1	0	0	5	9	q				12 960								
3	0	1	1	4	9	q		58 320		233 280				233 280		116 640		19 440
4	0	2	1	3	9	q	68 040		174 960		116 640	466 560			262 440		116 640	
5	1	2	0	3	9	q				12 960								
6	0	3	1	2	9	q		116 640		421 200			116 640	174 960		116 640		
7	0	4	1	1	9	q			29 160		58 320	29 160						
8	0	5	1	0	9	q				9720								
9	0	0	0	7	8	q				19 440			58 320					
10	0	1	0	6	8	q			58 320		238 140	247 860			145 800			
11	0	0	2	5	8	q				291 600			233 280	233 280		58 320		
12	1	0	1	5	8	q			58 320			58 320						
13	0	2	0	5	8	q		116 640		291 600			174 960	233 280		116 640		
14	0	1	2	4	8	q	58 320		349 920		495 720	787 320			816 480		58 320	
15	1	1	1	4	8	q		58 320		233 280				58 320				
16	0	3	0	4	8	q	82 620		58 320		277 020	422 820			145 800		43 740	
17	2	1	0	4	8	l			19 440									
18	0	2	2	3	8	q		174 960		816 480			291 600	583 200		116 640		
19	1	2	1	3	8	l			38 880									
20	0	4	0	3	8	q				58 320			58 320					
21	0	3	2	2	8	q	29 160		349 920			116 640			58 320		29 160	
22	0	5	0	2	8	q	29 160				53 460				29 160		14 580	
23	0	5	2	0	8	q			14 580									
24	0	7	0	0	8	q					7290							
25	0	0	1	7	7	q					349 920	174 960			174 960			
26	0	1	1	6	7	q		174 960		991 440			524 880	758 160		291 600		58 320
27	0	0	3	5	7	q	68 040		670 680		641 520	1 399 680			816 480		408 240	
28	1	0	2	5	7	q		291 600		583 200				291 600				
29	0	2	1	5	7	q	116 640		758 160		874 800	1 458 000			1 166 400		58 320	
30	0	1	3	4	7	q		1 574 640		3 888 000			1 458 000	3 382 560		1 341 360		174 960
31	1	1	2	4	7	q	233 280		1 049 760			583 200			233 280			
32	1	1	2	4	7	l			174 960						58 320			
33	0	3	1	4	7	q		233 280		583 200			291 600	524 880				
34	2	1	1	4	7	q		213 840						58 320				
35	0	2	3	3	7	q	554 040		1 195 560		845 640	2 682 720			729 000		58 320	
36	0	2	3	3	7	l			320 760						145 800			
37	1	2	2	3	7	q		388 800		233 280				233 280				
38	0	4	1	3	7	q			58 320		116 640	58 320						
39	2	2	1	3	7	q	58 320		116 640			58 320						
40	3	2	0	3	7	q		38 880										
41	0	3	3	2	7	q		758 160		583 200				233 280		58 320		
42	0	5	3	0	7	q		29 160										
43	0	1	0	8	6	q	43 740		58 320		335 340	131 220			233 280		14 580	
44	0	0	2	7	6	q		524 880		1 516 320			1 224 720	2 507 760		758 160		77 760
45	1	0	1	7	6	q	136 080		466 560			466 560			136 080			
46	0	2	0	7	6	q		174 960		174 960			233 280	291 600		58 320		
47	2	0	0	7	6	q		58 320		19 440								
48	0	1	2	6	6	q	1 156 680		2 012 040		4 753 080	7 231 680			3 625 560		1 341 360	
49	1	1	1	6	6	q		874 800		349 920				641 520				
50	0	3	0	6	6	q	97 200		174 960		184 680	58 320			272 160		29 160	
51	0	3	0	6	6	l	8100				24 300	14 580					4860	

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
52	2	1	0	6	6	q	64 800		87 480			29 160						
53	0	0	4	5	6	q		1 691 280		3 965 760			1 574 640	4 490 640		1 283 040		369 360
54	1	0	3	5	6	q	729 000		2 012 040			1 953 720			437 400			
55	1	0	3	5	6	l			349 920						77 760			
56	0	2	2	5	6	q		2 566 080		5 890 320			2 099 520	4 782 240		758 160		
57	2	0	2	5	6	q		291 600		58 320								
58	1	2	1	5	6	q	272 160		563 760			233 280			58 320			
59	3	0	1	5	6	l	6480											
60	0	1	4	4	6	q	972 000		3 265 920		2 624 400	6 619 320			2 935 440		437 400	
61	0	1	4	4	6	l	87 480		583 200		145 800	437 400			466 560		29 160	
62	1	1	3	4	6	q		3 849 120		1 769 040				2 216 160				
63	0	3	2	4	6	q	1 182 600		1 837 080		1 078 920	2 653 560			806 760		58 320	
64	2	1	2	4	6	q	204 120		495 720			174 960						
65	2	1	2	4	6	l	29 160					29 160						
66	0	5	0	4	6	q	58 320				87 480	29 160			58 320			
67	2	3	0	4	6	q	21 060		19 440									
68	0	2	4	3	6	q		2 449 440		2 896 560			466 560	2 216 160		174 960		
69	1	2	3	3	6	q	291 600		1 253 880			699 840			204 120			
70	1	2	3	3	6	l	64 800					116 640						
71	0	4	2	3	6	q		583 200		544 320				233 280				
72	2	2	2	3	6	q		758 160						174 960				
73	3	2	1	3	6	q	29 160		29 160									
74	0	3	4	2	6	q	97 200		116 640		145 800	320 760			38 880			
75	0	3	4	2	6	l			29 160						9720			
76	1	3	3	2	6	q		58 320		116 640				58 320				
77	0	5	2	2	6	q	204 120		233 280		29 160	116 640						
78	2	3	2	2	6	q	29 160		58 320			29 160						
79	2	3	2	2	6	l			29 160									
80	3	3	1	2	6	q		29 160										
81	4	3	0	2	6	l	1620											
82	0	6	2	1	6	q		58 320										
83	0	5	4	0	6	q	4860											
84	0	7	2	0	6	q	19 440		14 580									
85	0	9	0	0	6	q	810				2430							
86	0	0	1	9	5	q	9720		116 640		524 880	379 080			388 800		58 320	
87	1	0	0	9	5	q		58 320		194 400				58 320				
88	0	1	1	8	5	q		583 200		2 682 720			1 574 640	1 866 240		583 200		174 960
89	1	1	0	8	5	q	116 640		233 280			116 640						
90	0	0	3	7	5	q	1 010 880		3 557 520		3 440 880	7 290 000			6 006 960		1 749 600	
91	1	0	2	7	5	q		2 157 840		2 799 360				1 807 920				
92	0	2	1	7	5	q	408 240		1 982 880		1 399 680	2 741 040			1 545 480		204 120	
93	0	2	1	7	5	l	29 160		233 280		116 640	87 480			233 280			
94	0	1	3	6	5	q		6 415 200		13 996 800			6 765 120	14 463 360		3 265 920		349 920
95	1	1	2	6	5	q	2 274 480		4 257 360			3 265 920			816 480			
96	0	3	1	6	5	q		933 120		2 157 840			816 480	758 160		116 640		
97	2	1	1	6	5	q		699 840						233 280				
98	0	0	5	5	5	q	1 166 400		4 694 760		4 257 360	9 185 400			6 356 880		1 399 680	
99	0	0	5	5	5	l	87 480		355 752		204 120	554 040			437 400		87 480	
100	1	0	4	5	5	q		5 423 760		2 566 080				3 324 240				
101	0	2	3	5	5	q	2 624 400		6 298 560		3 265 920	7 989 840			2 449 440		116 640	
102	2	0	3	5	5	q	87 480		145 800									
103	1	2	2	5	5	q		1 399 680		816 480				349 920				
104	0	4	1	5	5	q	291 600		408 240		320 760	612 360						
105	2	2	1	5	5	q	174 960		233 280			58 320						

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
106	0	1	5	4	5	q		7 290 000		9 477 000			2 974 320	8 164 800		1 516 320		58 320
107	1	1	4	4	5	q	1 807 920		3 965 760			2 507 760			524 880			
108	1	1	4	4	5	l	116 640					233 280						
109	0	3	3	4	5	q		2 916 000		2 274 480			116 640	1 108 080				
110	2	1	3	4	5	q		1 137 240		58 320				233 280				
111	1	3	2	4	5	q	262 440		320 760			58 320						
112	0	5	1	4	5	q		58 320		174 960								
113	0	2	5	3	5	q	1 399 680		3 819 960		1 020 600	3 674 160			758 160		58 320	
114	0	2	5	3	5	l	116 640				116 640	349 920						
115	1	2	4	3	5	q		2 332 800		933 120				699 840				
116	0	4	3	3	5	q	524 880		816 480		58 320	349 920						
117	2	2	3	3	5	q	233 280		291 600			58 320						
118	0	3	5	2	5	q		758 160		583 200				174 960				
119	1	3	4	2	5	q	58 320		116 640			58 320						
120	1	3	4	2	5	l			58 320									
121	0	5	3	2	5	q		116 640										
122	2	3	3	2	5	q		58 320										
123	0	4	5	1	5	q	58 320		58 320									
124	0	0	2	9	4	q		1 224 720		2 332 800			1 574 640	3 615 840		1 166 400		349 920
125	1	0	1	9	4	q	291 600		495 720			670 680			233 280			
126	1	0	1	9	4	l			174 960						58 320			
127	0	1	2	8	4	q	1 472 580		3 645 000		4 432 320	9 127 080			4 403 160		1 210 140	
128	0	1	2	8	4	l	102 060		466 560		306 180	714 420			466 560		43 740	
129	1	1	1	8	4	q		1 224 720		1 166 400				524 880				
130	2	1	0	8	4	q			116 640									
131	2	1	0	8	4	l	14 580					14 580						
132	0	0	4	7	4	q		5 598 720		11 586 240			5 598 720	14 288 400		4 607 280		933 120
133	1	0	3	7	4	q	2 410 560		6 356 880			5 190 480			1 516 320			
134	0	2	2	7	4	q		4 315 680		6 706 800			2 449 440	4 665 600		291 600		
135	2	0	2	7	4	q		174 960		58 320								
136	1	2	1	7	4	q	233 280		233 280			116 640						
137	1	2	1	7	4	l	77 760					58 320						
138	0	1	4	6	4	q	6 371 460		15 163 200		10 249 740	26 448 120			9 856 080		1 501 740	
139	1	1	3	6	4	q		7 290 000		4 121 280				3 324 240				
140	0	3	2	6	4	q	1 258 740		2 595 240		1 822 500	2 813 940			787 320		14 580	
141	0	3	2	6	4	l	174 960				102 060	466 560					14 580	
142	2	1	2	6	4	q	690 120		1 078 920			466 560						
143	1	3	1	6	4	q		174 960		58 320								
144	2	3	0	6	4	q	43 740					14 580						
145	0	0	6	5	4	q		4 782 240		7 678 800			3 557 520	9 506 160		2 041 200		349 920
146	1	0	5	5	4	q	1 458 000		3 440 880			2 157 840			524 880			
147	1	0	5	5	4	l	145 800					262 440						
148	0	2	4	5	4	q		11 838 960		14 171 760			2 099 520	7 523 280		583 200		
149	1	2	3	5	4	q	2 157 840		4 082 400			2 041 200			233 280			
150	0	4	2	5	4	q		1 574 640		583 200				291 600				
151	2	2	2	5	4	q		816 480						174 960				
152	3	2	1	5	4	q	58 320		58 320									
153	0	1	6	4	4	q	2 945 160		7 771 140		4 228 200	10 585 080			3 207 600		204 120	
154	0	1	6	4	4	l	194 400		116 640		204 120	699 840			58 320		29 160	
155	1	1	5	4	4	q		6 356 880		2 361 960				1 866 240				
156	0	3	4	4	4	q	2 930 580		4 432 320		1 851 660	4 126 140			437 400		14 580	
157	2	1	4	4	4	q	335 340		408 240			145 800						
158	2	1	4	4	4	l			43 740									
159	1	3	3	4	4	q		1 399 680		583 200				233 280				

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
160	0	5	2	4	4	q	364 500		349 920		160 380	174 960						
161	0	5	2	4	4	l	7290				21 870							
162	2	3	2	4	4	q	233 280		160 380			58 320						
163	2	5	0	4	4	l	2430											
164	0	2	6	3	4	q		3 849 120		2 332 800			349 920	1 341 360		58 320		
165	1	2	5	3	4	q	1 108 080		1 749 600			729 000			87 480			
166	1	2	5	3	4	l			116 640									
167	0	4	4	3	4	q		1 924 560		466 560				174 960				
168	2	2	4	3	4	q		524 880						58 320				
169	1	4	3	3	4	q	19 440		58 320									
170	0	6	2	3	4	q		116 640										
171	0	3	6	2	4	q	272 160		466 560		87 480	174 960						
172	0	3	6	2	4	l			58 320									
173	1	3	5	2	4	q		379 080		116 640								
174	0	5	4	2	4	q	306 180		145 800		160 380	131 220						
175	2	3	4	2	4	q			58 320									
176	2	3	4	2	4	l	14 580											
177	3	3	3	2	4	q		58 320										
178	0	7	2	2	4	q	58 320		58 320									
179	2	5	2	2	4	q	29 160		29 160									
180	4	5	0	2	4	q	2430											
181	0	7	4	0	4	q	7290											
182	0	9	2	0	4	q	7290											
183	0	0	3	9	3	q	907 200		3 032 640		1 720 440	5 452 920			3 936 600		1 720 440	
184	0	0	3	9	3	l	113 400				194 400	612 360					194 400	
185	1	0	2	9	3	q		2 216 160		1 185 840				1 982 880				
186	0	1	3	8	3	q		6 881 760		11 547 360			4 782 240	12 013 920		1 924 560		174 960
187	1	1	2	8	3	q	933 120		2 332 800			1 516 320			233 280			
188	1	1	2	8	3	l	116 640					233 280						
189	2	1	1	8	3	q		524 880						174 960				
190	0	0	5	7	3	q	2 021 760		7 115 040		4 315 680	14 929 920			7 153 920		1 982 880	
191	1	0	4	7	3	q		4 782 240		3 304 800				2 799 360				
192	0	2	3	7	3	q	3 674 160		7 727 400		4 082 400	9 593 640			2 536 920		145 800	
193	0	2	3	7	3	l	408 240		174 960		233 280	1 108 080			58 320			
194	2	0	3	7	3	q	116 640		233 280									
195	1	2	2	7	3	q		2 157 840		758 160				641 520				
196	2	2	1	7	3	q	58 320		58 320									
197	0	1	5	6	3	q		13 471 920		20 256 480			6 590 160	15 396 480		1 458 000		174 960
198	1	1	4	6	3	q	3 946 320		8 019 000			5 015 520			699 840			
199	0	3	3	6	3	q		4 918 320		4 335 120			174 960	1 749 600		58 320		
200	2	1	3	6	3	q		1 166 400						233 280				
201	1	3	2	6	3	q	437 400		524 880			116 640						
202	1	3	2	6	3	l			116 640									
203	0	0	7	5	3	q	1 205 280		3 586 680		2 507 760	6 006 960			2 225 880		174 960	
204	0	0	7	5	3	l	58 320		174 960		116 640	291 600			136 080			
205	1	0	6	5	3	q		1 399 680		1 283 040				583 200				
206	0	2	5	5	3	q	5 365 440		10 847 520		3 849 120	9 622 800			1 458 000			
207	1	2	4	5	3	q		5 307 120		1 632 960				1 224 720				
208	0	4	3	5	3	q	1 078 920		1 778 760		116 640	524 880						
209	0	4	3	5	3	l			349 920									
210	2	2	3	5	3	q	408 240		816 480			174 960						
211	1	4	2	5	3	q		291 600										
212	3	2	2	5	3	q		116 640										
213	0	1	7	4	3	q		6 123 600		6 862 320			1 807 920	3 615 840		116 640		

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
214	1	1	6	4	3	q	1 253 880		2 216 160			933 120			116 640			
215	1	1	6	4	3	l			291 600									
216	0	3	5	4	3	q		5 715 360		3 265 920			174 960	1 108 080				
217	2	1	5	4	3	q		758 160						58 320				
218	1	3	4	4	3	q	884 520		1 399 680			349 920						
219	3	1	4	4	3	l	9720											
220	0	5	3	4	3	q		1 166 400		174 960								
221	2	3	3	4	3	q		466 560										
222	1	5	2	4	3	l	29 160											
223	3	3	2	4	3	q	29 160											
224	0	2	7	3	3	q	1 973 160		2 507 760		1 137 240	1 574 640			174 960			
225	0	2	7	3	3	l	29 160		233 280		87 480							
226	1	2	6	3	3	q		1 749 600		116 640				233 280				
227	0	4	5	3	3	q	835 920		1 283 040		116 640	174 960						
228	2	2	5	3	3	q	145 800		204 120									
229	1	4	4	3	3	q		699 840		116 640								
230	0	6	3	3	3	q	116 640		58 320									
231	0	3	7	2	3	q		524 880		213 840			58 320					
232	1	3	6	2	3	q	291 600		291 600			58 320						
233	1	3	6	2	3	l	29 160											
234	0	5	5	2	3	q		524 880		58 320								
235	2	3	5	2	3	q		145 800										
236	1	5	4	2	3	q	58 320		29 160									
237	3	3	4	2	3	q	29 160											
238	0	4	7	1	3	q	58 320											
239	0	6	5	1	3	q	58 320											
240	0	0	4	9	2	q		1 283 040		3 168 720			1 166 400	3 615 840		1 049 760		116 640
241	1	0	3	9	2	q	602 640		1 487 160			933 120			456 840			
242	1	0	3	9	2	l	97 200					233 280						
243	0	1	4	8	2	q	2 464 020		7 202 520		4 126 140	10 847 520			3 557 520		262 440	
244	0	1	4	8	2	l	233 280		262 440		320 760	933 120			204 120		29 160	
245	1	1	3	8	2	q		4 082 400		2 332 800				1 516 320				
246	2	1	2	8	2	q	233 280		291 600			174 960						
247	2	1	2	8	2	l			58 320									
248	0	0	6	7	2	q		1 807 920		3 868 560			1 866 240	3 790 800		408 240		
249	1	0	5	7	2	q	699 840		2 216 160			991 440			291 600			
250	0	2	4	7	2	q		7 406 640		7 523 280			1 458 000	3 090 960		116 640		
251	2	0	4	7	2	q		116 640										
252	1	2	3	7	2	q	1 312 200		2 216 160			787 320			87 480			
253	1	2	3	7	2	l			174 960									
254	2	2	2	7	2	q		524 880						58 320				
255	0	1	6	6	2	q	4 830 840		8 922 960		6 765 120	11 401 560			2 303 640		116 640	
256	1	1	5	6	2	q		6 648 480		1 749 600				1 166 400				
257	0	3	4	6	2	q	3 105 540		3 645 000		1 370 520	2 478 600			262 440			
258	0	3	4	6	2	l	34 020		583 200		102 060							
259	2	1	4	6	2	q	481 140		437 400			174 960						
260	1	3	3	6	2	q		1 224 720		77 760								
261	2	3	2	6	2	q	58 320		58 320									
262	2	3	2	6	2	l	14 580											
263	0	0	8	5	2	q		874 800		1 283 040			641 520	699 840				
264	1	0	7	5	2	q	379 080		699 840			233 280						
265	1	0	7	5	2	l			116 640									
266	0	2	6	5	2	q		9 097 920		6 531 840			816 480	1 924 560				
267	1	2	5	5	2	q	2 216 160		2 857 680			524 880						

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
268	0	4	4	5	2	q		2 624 400		583 200				58 320				
269	2	2	4	5	2	q		641 520										
270	1	4	3	5	2	q	233 280		116 640									
271	1	4	3	5	2	l	29 160											
272	0	1	8	4	2	q	1 662 120		2 449 440		1 137 240	1 516 320			233 280			
273	0	1	8	4	2	l	29 160		291 600		87 480							
274	1	1	7	4	2	q		2 187 000		233 280				58 320				
275	0	3	6	4	2	q	3 888 000		3 863 700		1 166 400	1 370 520			29 160			
276	2	1	6	4	2	q	116 640		116 640									
277	2	1	6	4	2	l	14 580											
278	1	3	5	4	2	q		2 099 520		116 640								
279	0	5	4	4	2	q	575 910		466 560		123 930							
280	0	5	4	4	2	l	87 480											
281	2	3	4	4	2	q	204 120		58 320									
282	2	5	2	4	2	q	29 160											
283	0	2	8	3	2	q		2 099 520		1 049 760			116 640					
284	1	2	7	3	2	q	874 800		758 160			116 640						
285	1	2	7	3	2	l	87 480											
286	0	4	6	3	2	q		1 108 080		194 400								
287	2	2	6	3	2	q		349 920										
288	1	4	5	3	2	q	291 600		116 640									
289	0	3	8	2	2	q	349 920		262 440		29 160							
290	0	3	8	2	2	l	58 320											
291	1	3	7	2	2	q		174 960										
292	0	5	6	2	2	q	379 080		145 800		29 160							
293	2	3	6	2	2	q	72 900											
294	0	7	4	2	2	q	58 320											
295	2	5	4	2	2	q	29 160											
296	0	0	5	9	1	q	116 640		145 800		291 600	991 440			145 800		58 320	
297	0	0	5	9	1	l			116 640						116 640			
298	1	0	4	9	1	q		699 840		291 600				291 600				
299	2	0	3	9	1	q	38 880		116 640									
300	0	1	5	8	1	q		2 216 160		2 857 680			1 283 040	2 274 480				
301	1	1	4	8	1	q	933 120		1 691 280			758 160			116 640			
302	1	1	4	8	1	l			233 280									
303	2	1	3	8	1	q		583 200						58 320				
304	0	0	7	7	1	q	77 760		466 560		291 600	349 920			174 960			
305	1	0	6	7	1	q		466 560		272 160				116 640				
306	0	2	5	7	1	q	2 653 560		4 694 760		1 895 400	2 303 640			291 600			
307	0	2	5	7	1	l	58 320		466 560		174 960							
308	1	2	4	7	1	q		2 566 080		291 600				291 600				
309	2	2	3	7	1	q	116 640		233 280									
310	3	2	2	7	1	q		58 320										
311	0	1	7	6	1	q		2 449 440		2 916 000			699 840	583 200				
312	1	1	6	6	1	q	1 224 720		1 341 360			291 600						
313	0	3	5	6	1	q		4 607 280		1 691 280			233 280	233 280				
314	2	1	5	6	1	q		466 560										
315	1	3	4	6	1	q	933 120		991 440			116 640						
316	1	3	4	6	1	l	87 480											
317	2	3	3	6	1	q		233 280										
318	0	0	9	5	1	q	58 320		29 160		174 960	58 320			29 160			
319	0	0	9	5	1	l	9720				29 160							
320	0	2	7	5	1	q	3 265 920		3 732 480		1 399 680	933 120						
321	1	2	6	5	1	q		2 332 800		233 280								

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
322	0	4	5	5	1	q	1 574 640		1 341 360		58 320	116 640						
323	0	4	5	5	1	l	174 960											
324	2	2	5	5	1	q	291 600		116 640									
325	1	4	4	5	1	q		583 200										
326	2	4	3	5	1	q	58 320											
327	0	1	9	4	1	q		991 440		622 080			58 320	58 320				
328	1	1	8	4	1	q	349 920		291 600									
329	1	1	8	4	1	l	58 320											
330	0	3	7	4	1	q		2 507 760		408 240								
331	1	3	6	4	1	q	874 800		262 440									
332	0	5	5	4	1	q		349 920										
333	1	5	4	4	1	q	145 800											
334	3	3	4	4	1	q	29 160											
335	0	2	9	3	1	q	913 680		787 320		204 120	58 320						
336	0	2	9	3	1	l	58 320											
337	1	2	8	3	1	q		233 280										
338	0	4	7	3	1	q	816 480		408 240		58 320							
339	2	2	7	3	1	q	116 640											
340	0	6	5	3	1	q	58 320											
341	0	3	9	2	1	q		87 480										
342	1	3	8	2	1	q	116 640											
343	1	5	6	2	1	q	58 320											
344	0	0	0	15	0	q		311 040		155 520			136 080	641 520		186 624		103 680
345	0	1	0	14	0	q	534 600		1 603 800		831 060	2 741 040			1 759 320		554 040	
346	0	2	0	13	0	q		3 207 600		4 432 320			1 399 680	3 965 760		349 920		
347	0	3	0	12	0	q	2 227 500		3 936 600		1 383 480	4 169 880			926 640		29 160	
348	0	4	0	11	0	q		3 790 800		2 099 520			136 080	874 800				
349	0	5	0	10	0	q	1 526 040		1 648 512		330 480	592 920			9720			
350	0	0	6	9	0	q		58 320		77 760			58 320	58 320				
351	1	0	5	9	0	q	12 960								3240			
352	1	0	5	9	0	l			58 320									
353	2	0	4	9	0	q		58 320										
354	3	0	3	9	0	l	3960											
355	0	6	0	9	0	q		939 600		142 560								
356	0	1	6	8	0	q	165 240		174 960		311 040	262 440			58 320		4860	
357	0	1	6	8	0	l	14 580				43 740							
358	1	1	5	8	0	q		466 560		97 200								
359	2	1	4	8	0	q	38 880		9720									
360	2	1	4	8	0	l	19 440					4860						
361	0	7	0	8	0	q	340 200		126 360		19 440							
362	0	0	8	7	0	q		58 320		97 200			58 320	58 320				
363	1	0	7	7	0	q	58 320		58 320									
364	0	2	6	7	0	q		816 480		777 600			174 960	97 200				
365	1	2	5	7	0	q	505 440		476 280			136 080			9720			
366	1	2	5	7	0	l	77 760					38 880						
367	2	2	4	7	0	q		174 960						19 440				
368	3	2	3	7	0	q	38 880		9720									
369	0	8	0	7	0	q		19 440										
370	0	1	8	6	0	q	304 560		281 880		213 840	106 920			58 320			
371	1	1	7	6	0	q		213 840		90 720				58 320				
372	0	3	6	6	0	q	1 373 760		952 560		340 200	291 600			42 120			
373	0	3	6	6	0	l	81 000		9720		24 300	58 320			3240		1620	
374	2	1	6	6	0	q	77 760		77 760			9720						
375	1	3	5	6	0	q		349 920		162 000				58 320				

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
376	2	3	4	6	0	q	84 240		58 320			24 300						
377	2	3	4	6	0	l			19 440									
378	3	3	3	6	0	q		19 440										
379	4	3	2	6	0	l	1620											
380	0	9	0	6	0	q	55 080		9720		1620							
381	0	0	10	5	0	q				6480				19 440				
382	1	0	9	5	0	q			29 160						9720			
383	1	0	9	5	0	l	12 960					9720						
384	0	2	8	5	0	q		855 360		233 280			77 760	97 200				
385	1	2	7	5	0	q	427 680		408 240			97 200			19 440			
386	0	4	6	5	0	q		349 920		97 200				38 880				
387	2	2	6	5	0	q		155 520						19 440				
388	1	4	5	5	0	q	184 680		58 320			19 440						
389	1	4	5	5	0	l			38 880									
390	2	4	4	5	0	q		38 880										
391	0	1	10	4	0	q	252 720		87 480		9720	19 440						
392	0	1	10	4	0	l			9720						9720			
393	1	1	9	4	0	q		38 880		38 880				19 440				
394	0	3	8	4	0	q	1 577 880		349 920		160 380	77 760			19 440			
395	2	1	8	4	0	q	9720					9720						
396	2	1	8	4	0	l			9720									
397	1	3	7	4	0	q		136 080		38 880				19 440				
398	0	5	6	4	0	q	750 870		116 640		72 900	68 040						
399	0	5	6	4	0	l	4860		38 880		14 580							
400	2	3	6	4	0	q	110 160		77 760									
401	1	5	5	4	0	q		155 520		19 440								
402	3	3	5	4	0	q		38 880										
403	2	5	4	4	0	q	19 440		19 440									
404	2	5	4	4	0	l	7290											
405	4	5	2	4	0	q	2430											
406	0	11	0	4	0	q	4860											
407	0	2	10	3	0	q		136 080		32 400			19 440	19 440				
408	1	2	9	3	0	q	51 840		19 440									
409	1	2	9	3	0	l			38 880									
410	0	4	8	3	0	q		77 760		123 120			19 440					
411	2	2	8	3	0	q		77 760										
412	1	4	7	3	0	q	142 560		136 080			19 440						
413	3	2	7	3	0	l	3240											
414	0	6	6	3	0	q		97 200		19 440								
415	2	4	6	3	0	q		97 200										
416	1	6	5	3	0	q	38 880		19 440									
417	1	6	5	3	0	l	19 440											
418	3	4	5	3	0	q	19 440											
419	0	3	10	2	0	q	383 940				19 440							
420	0	3	10	2	0	l	3240				9720							
421	1	3	9	2	0	q		38 880		6480								
422	0	5	8	2	0	q	332 910		58 320		19 440							
423	2	3	8	2	0	q	29 160		19 440									
424	2	3	8	2	0	l	9720											
425	1	5	7	2	0	q		38 880										
426	0	7	6	2	0	q	43 740		29 160		4860							
427	0	7	6	2	0	l	9720											
428	2	5	6	2	0	q	29 160											
429	2	7	4	2	0	q	9720											

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
430	1	4	9	1	0	q	19 440		19 440									
431	1	4	9	1	0	l	9720											
432	0	6	8	1	0	q		19 440										
433	1	6	7	1	0	q	19 440											
434	3	4	7	1	0	q	9720											
435	0	5	10	0	0	q	12 960											
436	1	5	9	0	0	q				1080								
437	0	7	8	0	0	q	32 400				810							
438	2	5	8	0	0	l			1620									
439	3	5	7	0	0	q		3240										
440	0	9	6	0	0	q	21 330											
441	0	9	6	0	0	l	90				270							
442	2	7	6	0	0	q	2160		1620									
443	4	5	6	0	0	l	540											
444	2	9	4	0	0	l	810											
445	4	7	4	0	0	q	810											
446	0	15	0	0	0	q	360											
447	6	9	0	0	0	l	10											