Energy Decay Estimates of the Axially Moving Kirchhoff-type Beam

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Abstract: In this paper, the energy decay rates for the axially moving Kirchhoff-type beams with the nonlinear boundary feedback control are established. The nonlinear boundary control, which satisfies a various constraint condition, is a negative feedback of the transverse velocity at the right boundary of beam. Several energy decay rates are provided due to various growth restriction on the nonlinear boundary feedback near the origin and at infinity. To verify the proposed control approach, numerical simulations are shown by the finite element method.

Keywords: Axially moving; Kirchhoff-type beam; Boundary control; Energy decay.

1. INTRODUCTION

Vibration control of the axially moving beam systems has attracted much attention due to many engineering applications, such as, aerial cable tram ways, oil pipelines, magnetic tapes, paper sheet processes, fiber winding power, band saws and transmission belts, for example, see He and He (2017); Shah and Hong (2018); Liu (2016); Hong (2022). It is well known that boundary control is one of the main efficient and practical ways for suppressing vibration of beam systems due to several advantages such as fewer sensors and easy implementation. It may be worthwhile of noting that, boundary controllability and stabilization for non-moving and moving beams such as axially moving steel strip Yang et al. (2004), fixed Euler-Bernoulli beam Wu and Wang (2014), axially moving viscoelastic strip Kelleche and Tatar (2017) and nonlinear axially moving beam Cheng-Wu-Guo (2021), etc. have been investigated by many researchers in recent years.

In this paper, we investigate the energy decay estimates of the following Kirchhoff-type beam

$$\begin{cases} w_{tt} + \lambda w_{xxxx} = M(\|w_x(t)\|^2) w_{xx} - 2v w_{xt}, \\ M(\|w_x(t)\|^2) w_x(L,t) - \lambda w_{xxx}(L,t) - v w_t(L,t) = U(t), \\ w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, \\ w(x,0) = f_1(x),\ w_t(x,0) = f_2(x), \end{cases} \tag{1}$$

for all $x \in (0, L)$ and $t > 0$, where $w(x,t)$ stands for the transversal deflection of beam at the position x and at time t, L denotes the length of beam, v is the moving speed of beam, M is a continuous non decreasing differentiable function ($M \in C^1(0,\infty)$) with $M(s) \geq a > 0$ ($\forall s \geq 0$), U represents control input, $\lambda > 0$, f_1 and f_2 are the bending stiffness coefficient, the initial displacement, and the initial velocity of the system, respectively.

When $M(\cdot) \equiv$ *constant* in (1), i.e., without considering the tension change caused by the vibration of the beam in the deflection process, the system is simplified as a moving Euler Bernoulli beam. The exponential delay rate of the moving Euler-Bernoulli beam with the linear boundary control was considered in Choi et al. (2004). In Wickert (1992), a coupled system of transverse and longitudinal vibrations can be reduced to the axially moving Kirchhoff beams under the quasi-static stretch assumption, where $\int_0^L w_x^2(x,t)dx$ is called the Kirchhoff correction in Arosio (1993). When $M(\|w_x(t)\|^2) = a + b\int_0^1 w_x^2(x,t)dx$ ($a, b > 0$) in (1), the absolute stability for the axially moving Kirchhoff-beam (1) with $v > 0$ is established by the integral-type multiplier method in Cheng-Wu-Guo (2021). For the non-moving Kirchhoff-type beam (1) with $v = 0$ proposed by Woinowsky-Krieger in Woinowsky (1950), some adaptive boundary output feedback controls were proposed in Guo and Guo (2007) and Kobayashi et al. (2009).

When the bending stiffness is not taken into account, i.e., $\lambda = 0$, system (1) can be reduced to the moving Kirchhoff string. For the moving Kirchhoff string under boundary feedback control, we refer for instance to Li et al. (2008); Shahruz (1998); Wu et al. (2014); Cheng-Wu-Guo (2022). The boundary stabilization of other nonlinear moving beam where the axial strain $1/L\int_0^L w_x^2(x,t)dx$ is

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approximated as $w_x^2(x,t)$ at position x has been examined in Kelleche and Tatar (2017).

The main purposes of this paper is to study energy decay rates of the axially moving Kirchhoff-type system (1) under nonlinear boundary control. Since various growth constraints of nonlinear feedback at infinity and near the origin are provided, several energy decay rates of axially moving Kirchhoff-type system (1) are obtained. By invoking the method provided by Lasiecka (1993), we conclude that if the nonlinear feedback grows linearly at infinity, the decay rate will be guaranteed by the solution of a nonlinear ordinary differential equation (ODE). If we impose some additional restrictions on the growth of nonlinear damping near the origin, we find that when the damping term increases linearly near the origin, the energy function decays exponentially, and when the damping increases as a power function near the origin, the energy function decays as a polynomial.

The rest of the presented paper is organized as follows. In Sect. 2, decay estimates of energy for the resulting closed-loop system is addressed. The proof of main results are given in Sect. 3. In Sect. 4, some simulation results are provided to illustrate the theoretical results. A brief conclusion follows in Section 5.

2. MAIN RESULT

In this work, a boundary feedback input control

$$U(t) = -F(w_t(L,t)) \tag{2}$$

is adopted, where $w_t(L,t)$ denotes the transverse velocity at the right boundary of beam and $F(\cdot)$ is a continuous monotone increasing function with $F(0) = 0$ and satisfies

$$k_1 \le \frac{F(s)}{s} \le k_2, \quad \forall\ |s| \ge 1, \tag{3}$$

for the given constants $0 < k_1 \le k_2$. Roughly speaking, to produce a dissipative effect on the beam, the negative feedback of the transverse velocity at the right eyelet of boundary $x = L$ is achieved by the boundary control law (2), which is consistent with the fact that negative velocity feedback helps to increase damping in most inertial systems, such as large flexible systems and pipelines conveying fluid.

For convenience of the reader, we present a closed-loop system shown by the following integro-partial differential equations

$$\begin{cases} w_{tt} + \lambda w_{xxxx} = M(\|w_x(t)\|^2)w_{xx} - 2vw_{xt}, \\ M(\|w_x(t)\|^2)w_x(L,t) - \lambda w_{xxx}(L,t) - vw_t(L,t) \\ \qquad\qquad = -F(w_t(L,t)), \\ w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, \\ w(x,0) = f_1(x), w_t(x,0) = f_2(x), \end{cases} \tag{4}$$

by substituting (2) into (1), for all $x \in (0,L)$ and $t \ge 0$.

Let

$$E(t) := \frac{1}{2}\int_0^L w_t^2 dx + \frac{1}{2}\widetilde{M}(\|w_x\|^2) + \frac{\lambda}{2}\int_0^L w_{xx}^2 dx \tag{5}$$

stand for the beam energy, where $\widetilde{M}(s) = \int_0^s M(\theta)d\theta$. In what follows, we mainly focus on the stability analysis of closed-loop systems (4), since the proof of well-posedness of the problem (4) is similar to Cheng-Wu-Guo (2021), where the Faedo-Galerkin method is used to complete the two estimates of solutions.

Now, we present the idea given by Lasiecka (1993), which will play a crucial role in establishing the stability for the axially moving Kirchhoff-type beam (1). In this context, it is important to point out other important works in the previous literature that considered explicit decay rate estimates, such as Alabau-Boussouira (2005, 2010); Alabau-Boussouira-Ammari (2011), Martinez (1999).

Assume that a map $W(s)$ is concave and strictly increasing for $s \ge 0$, with $W(0) = 0$, satisfying

$$W(sF(s)) \ge s^2 + [F(s)]^2,\ \forall\ |s| \le N, \tag{6}$$

for some constant $N > 0$. According to the property of $F(s)$, such a function $W(s)$ can always be constructed; for detail, see Lasiecka (1993). Set

$$\hat{W}(s) = W(\frac{s}{T}),\ \forall\ s \ge 0, \tag{7}$$

where T is a constant to be determined later. For $\delta > 0$, $\delta\mathcal{I} + \hat{W}$ is invertible and strictly increasing, where $\mathcal{I}$ is the identity mapping. Let us define a map

$$P(s) = (\delta\mathcal{I} + \hat{W})^{-1}(\hat{\delta}s) \tag{8}$$

for a constant $\hat{\delta} > 0$, which is a strictly increasing, positive and continuous function with $P(0) = 0$. Define $Q(s) = s - (\mathcal{I} + P)^{-1}(s)$ for $s \ge 0$. Then $Q(s)$ is also a positive, continuous and strictly increasing function. Recalling the ODE system given by

$$\begin{cases} \frac{d}{dt}S(t) + Q(S(t)) = 0, \quad t > 0 \\ S(0) = s_0, \end{cases} \tag{9}$$

if $P(t) > 0$ defined in (8) for any $t > 0$, we have $\lim_{t\to\infty} S(t) = 0$, as discussed in Lasiecka (1993) (see also Cavalcanti (2007)). From the above preliminary work, we state our stability result.

Theorem 1. Let $E(t)$ be the energy function of the closed-loop system (4) given by (5). Assume that the assumption (3) on F is satisfied. Then there exist a positive constant $T_0 > 0$ such that

$$E(t) \le S(\frac{t}{T_0} - 1) \tag{10}$$

for all $t > T_0$ with $\lim_{t\to\infty} S(t) = 0$, where $S(t)$ is the solution of ODE system (9) with $s_0 = E(0)$.

It is worth noting that the asymptotic stability of energy function to the closed-loop system (4) can only be obtained from the above theorem. If the nonlinear feedback function F fulfills additional specific growth conditions at the origin, a clear energy decay rate is deduced by applying Theorem 1, as discussed in Cavalcanti (2014). Therefore, we have the following further result.

Corollary 2. Under the assumptions of Theorem 1, let $p \in [1, \infty)$ and we assume that there exists two positive constants k_1, k_2, such that

$$k_1|s|^p \le |F(s)| \le k_2|s|^{1/p}, \ \forall\ |s| \le 1. \tag{11}$$

Then the energy $E(t)$ along the solution of closed-loop system (4) satisfies

$$E(t) \le Ct^{\frac{2}{1-p}}, \ if\ p > 1, \tag{12}$$

and

$$E(t) \le Ce^{-\mu t}, \ if\ p = 1, \tag{13}$$

where μ, C are positive constants.

3. PROOF OF MAIN RESULTS

In order to prove the main result, some auxiliary lemmas need to be completed.

Lemma 3. Let w be the solution of closed-loop system (4), then for any $T > S > 0$

$$E(T) = E(S) - \int_S^T F(w_t(L,t))w_t(L,t)dt. \tag{14}$$

Proof. According to the boundary condition in (4), the variational structure corresponding to (4) is provided by

$$\int_0^L w_{tt}ydx + 2v\int_0^L w_{xt}ydx + M(\|w_x(t)\|^2)\int_0^L y_x w_x dx$$
$$+\lambda\int_0^L w_{xx}y_{xx}dx - [vw_t(L,t) - F(w_t(L,t))]y(L,t) = 0 \tag{15}$$

for any $y \in H^2 := \{y \in H^2(0,L); y(0) = y_x(0) = 0\}$. From (15), taking $y = w_t$ and applying the derivative of energy function $E(t)$ defined in (5), then we obtain

$$\int_0^L w_{tt}w_t dx + M(\|w_x(t)\|^2)\int_0^L w_{xt}w_x dx + \lambda\int_0^L w_{xx}w_{xxt}dx$$
$$= -F(w_t(L,t))w_t(L,t), \tag{16}$$

where we use the equation $\int_0^L w_{xt}w_t dx = \frac{1}{2}w_t^2(L,t)$ due to $w_t(0,t) = 0$. Hence, we have

$$\dot{E}(t) = -F(w_t(L,t))w_t(L,t). \tag{17}$$

Since $F(w_t(L,t))w_t(L,t) \ge 0$, we obtain that the energy function $E(t)$ is non-increasing and $E(t) \le E(0)$ for all $t \ge 0$. Therefore, integrating on both sides of (17) from T to S, our desired result follows.

Lemma 4. If w is the solution of closed-loop system (4), then the following results hold:

$$\int_0^L xw_{tt}w_x dx = \int_0^L \left(\frac{1}{2}w_t^2 + [xw_x w_t]_t\right)dx - \frac{L}{2}w_t^2(L,t), \tag{18}$$

$$\int_0^L xw_x w_{xxxx}dx = Lw_x(L,t)w_{xxx}(L,t) + \frac{3}{2}\int_0^L w_{xx}^2 dx, \tag{19}$$

$$\int_T^S\int_0^L [xw_x w_t + 2vxw_x^2]_t dxdt \le 2\max\{\frac{L^2+4vL}{a}, 1\}E(S) \tag{20}$$

for any $T > S > 0$.

Proof. From the basic derivative rules and integration by parts, one has, for all $t > 0$,

$$\int_0^L xw_{tt}w_x dx = \int_0^L [xw_x w_t]_t dx - \int_0^L xw_{xt}w_t dx \tag{21}$$

and

$$\int_0^L xw_{xt}w_t dx = \frac{1}{2}\int_0^L [xw_t^2]_x dx - \frac{1}{2}\int_0^L w_t^2 dx$$
$$= \frac{L}{2}w_t^2(L,t) - \frac{1}{2}\int_0^L w_t^2 dx. \tag{22}$$

Substituting (22) into (21), we immediately get the equation (18). Integrating by parts and the boundary value condition (4) implies

$$\int_0^L xw_x w_{xxxx}dx$$
$$= Lw_x(L,t)w_{xxx}(L,t) - \int_0^L w_{xxx}(xw_{xx} + w_x)dx$$
$$= Lw_x(L,t)w_{xxx}(L,t) + \int_0^L w_{xx}(xw_{xxx} + 2w_{xx})dx$$
$$= \frac{3}{2}\int_0^L w_{xx}^2 dx + Lw_x(L,t)w_{xxx}(L,t). \tag{23}$$

Recalling the definition of energy $E(t)$ and applying the mean value inequality yield

$$\int_0^L (xw_t w_x + 2vxw_x^2)dx$$
$$\le (\frac{L^2}{2} + 2vL)\int_0^L w_x^2 dx + \frac{1}{2}\int_0^L w_t^2 dx,$$
$$\le \max\{\frac{L^2+4vL}{a}, 1\}E(t). \tag{24}$$

It is easy to check that for any $T > S$,

$$\left|\int_S^T\int_0^L [xw_x(x,t)w_t(x,t) + 2vxw_x^2(x,t)]_t dxdt\right|$$
$$\le \left|\int_0^L xw_x(x,T)w_t(x,T) + 2vxw_x^2(x,T)dx\right|$$

$$+\left|\int_0^L xw_x(x,S)w_t(x,S)+2vxw_x^2(x,S)dx\right|. \quad (25)$$

Thanks to (24) and $E(T) \le E(S)$ for any $T > S$, we can derive that

$$\left|\int_S^T\int_0^L [xw_x(x,t)w_t(x,t)+2vxw_x^2(x,t)]_t dxdt\right|$$
$$\le 2\max\{\frac{L^2+4vL}{a},1\}E(S). \quad (26)$$

Thus, the estimate (20) holds.

Proof of Theorem 1.

Taking the inner product with xw_x on both sides of the first equation in (4) gives

$$\langle xw_x, w_{tt}\rangle + \langle xw_x, 2vw_{xt}\rangle + \lambda\langle xw_x, w_{xxxx}\rangle$$
$$= M(\|w_x(t)\|^2)\langle xw_x, w_{xx}\rangle. \quad (27)$$

In light of the basic derivative rules and integration by parts, one obtains

$$\langle xw_x, 2vw_{xt}\rangle = 2v\int_0^L xw_x w_{xt}dx = v\int_0^L [xw_x^2]_t dx \quad (28)$$

and

$$M(\|w_x(t)\|^2)\langle xw_x, w_{xx}\rangle = \frac{M(\|w_x(t)\|^2)}{2}Lw_x^2(L,t)$$
$$-\frac{M(\|w_x(t)\|^2)}{2}\int_0^L w_x^2 dx. \quad (29)$$

Due to the fact that $M(s)$ is a continuous non decreasing function with $M(s) \ge a$, then it easy to find $sM(s) \ge \widehat{M}(s) = \int_0^s M(\tau)d\tau$ for any $s \ge 0$. Recalling the definition of energy function $E(t)$, and inserting (18), (19), (28) and (29) into (27) leads to

$$E(t) \le \frac{1}{2}\int_0^L w_t^2 dx + \frac{1}{2}M(\|w_x(t)\|^2)\int_0^L w_x^2 dx + \frac{3\lambda}{2}\int_0^L w_{xx}^2 dx$$
$$= -\int_0^L [xw_x w_t + 2vxw_x^2]_t dx + \frac{L}{2}w_t^2(L,t)$$
$$+\frac{L}{2}M(\|w_x(t)\|^2)w_x^2(L,t)$$
$$-L\lambda w_x(L,t)w_{xxx}(L,t).$$

Taking the boundary value condition (4) into account, the inequality above becomes

$$E(t) \le -\int_0^L [xw_x w_t + 2vxw_x^2]_t dx + \frac{L}{2}w_t^2(L,t)$$
$$-\frac{L}{2}M(\|w_x(t)\|^2)w_x^2(L,t) - LF(w_t(L,t))w_x(L,t)$$
$$+vLw_t(L,t)w_x(L,t). \quad (30)$$

According to $M(s) \ge a$ and Young's inequality, it follows that

$$-\frac{L}{2}M(\|w_x(t)\|^2)w_x^2(L,t) - LF(w_t(L,t))w_x(L,t)$$
$$+vLw_t(L,t)w_x(L,t)$$
$$\le -\frac{aL}{2}w_x^2(L,t) + \frac{L}{4\varepsilon}F^2(w_t(L,t))$$
$$+\frac{Lv}{4\varepsilon}w_t^2(L,t) + (L+Lv)\varepsilon w_x^2(L,t). \quad (31)$$

In view of the arbitrariness of parameters $\varepsilon > 0$, we let $\varepsilon = \frac{a}{2(1+v)}$. Then insert (31) into (30) to get

$$E(t) \le -\int_0^L [xw_x w_t + 2vxw_x^2]_t dx + \frac{L}{2}w_t^2(L,t)$$
$$+\frac{L(1+v)}{2a}F^2(w_t(L,t)) + \frac{Lv(1+v)}{2a}w_t^2(L,t), \quad (32)$$

which gives

$$E(t) \le -\int_0^L [xw_x w_t + 2vxw_x^2]_t dx$$
$$+C[w_t^2(L,t) + F^2(w_t(L,t))], \quad (33)$$

where $C = \max\{\frac{L(1+v)}{2a}, \frac{Lv(1+v)+La}{2a}\}$. Integrate simultaneously both sides of (33) from S to $T (S < T)$, and invoke (21) to obtain

$$\int_S^T E(t)dt \le C_1 E(S) + C\int_S^T [w_t^2(L,t) + F^2(w_t(L,t))]dt, \quad (34)$$

where C is the constant given by (33) and $C_1 = 2\max\{\frac{L^2+4vL}{a}, 1\}$. Letting $S = 0$ in (34) and $T > T_0$, we immediately see

$$\int_0^T E(t)dt \le C_1 E(0) + C\int_0^T [w_t^2(L,t) + F^2(w_t(L,t))]dt. \quad (35)$$

Thanks to (14), it follows from (35) that for $T > \hat{T} := \max\{T_0, C_1\}$,

$$E(T) \le C_T\int_0^T [w_t^2(L,t) + F^2(w_t(L,t))]dt, \quad (36)$$

where $C_T > 0$ is a constant depending on T. Denote $\Xi_N := \{t \in [0,T]; |w_t(L,t)| \le N\}$, with the constant $N \ge 1$ given by (6). By (3), it is easy to verify that

$$\int_{[0,T]\setminus\Xi_N} [w_t^2(L,t) + F^2(w_t(L,t))]dt$$
$$\le C_3\int_{[0,T]\setminus\Xi_N} w_t(L,t)F(w_t(L,t))dt, \quad (37)$$

where $C_3 = \frac{1+k_2^2}{k_1^2}$. On the other hand, from (6) we have

$$\int\limits_{\Xi_N} [w_t^2(L,t) + F^2(w_t(L,t))]dt$$

$$\leq \int\limits_{\Xi_N} W(w_t(L,t)F(w_t(L,t)))dt. \tag{38}$$

In view of Jensen's inequality, it holds that

$$\int\limits_{\Xi_N} W(w_t(L,t)F(w_t(L,t)))dt$$

$$\leq T \cdot W\left(\int\limits_0^T \frac{w_t(L,t)F(w_t(L,t))}{T}dt\right)$$

$$\leq T \cdot \hat{W}\left(\int\limits_0^T w_t(L,t)F(w_t(L,t))dt\right). \tag{39}$$

From (36), we can show

$$E(T) \leq C_T T \cdot \hat{W}\left(\int\limits_0^T w_t(L,t)F(w_t(L,t))dt\right)$$

$$+ C_T C_3 \int\limits_0^T w_t(L,t)F(w_t(L,t))dt. \tag{40}$$

Set $\hat{\delta} = \frac{1}{C_T T}$ and $\delta = \frac{C_3}{T}$ in (8), the following inequality

$$P(E(T)) + E(T) \leq E(0), \tag{41}$$

can be deduced from (40), where $P(s)$ is defined as (8). Applying this result repeatedly, we get

$$P(E((n+1)T)) + E((n+1)T) \leq E(nT), n = 0, 1, \cdots \tag{42}$$

Following the Lemma 3.3 in Lasiecka (1993) with $s_n = E(nT)$, $s_0 = E(0)$, $n = 1, 2, \cdots$, then we obtain $E(nT) \leq S(n)$, where S is the solution of the ODE system (9). For $t \geq T$, let $t = nT + \eta$, with $0 \leq \eta < T$ and $n = 0, 1, 2, \cdots$. Thus, we can derive

$$E(t) \leq E(nT) \leq S(n) = S(\frac{t-\eta}{T}) \leq S(\frac{t}{T} - 1). \tag{43}$$

The proof of Theorem 1 is complete.

Proof of Corollary 2.

The core of the proof is to construct a concave function $W(s)$ satisfying property (6). Due to (11), we have $k_1|s|^p \leq |F(s)|$ and $|F(s)|^p \leq k_2^p|s|$ for any $|s| < 1$. Then we can take $W(s) = (k_1^{\frac{-2}{p+1}} + k_2^{\frac{2p}{p+1}})s^{\frac{2}{p+1}}$. Thus, the map $P(s) = (\delta\mathcal{I} + \hat{W})^{-1}(\hat{\delta}s)$, i.e., $\delta P(s) + C_{k_1,k_2}P(s)^{\frac{2}{p+1}} = \hat{\delta}s$ where C_{k_1,k_2} is a suitable constant related to k_1 and k_2. Recalling the map $Q(s) = s - (\mathcal{I} + P)^{-1}(s)$ and when s is very small, one has $P(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ and $Q(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ for some constant $\mathcal{C} > 0$. Consequently, our desired estimates (12) and (13) follows by solving (9) with $Q(s)$ as above and invoking Theorem 1.

4. NUMERICAL SIMULATION

In this section, a simulation example is shown for the closed-loop system (4) to address the effectiveness of the

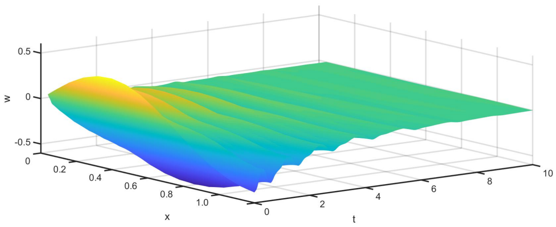

Fig. 1. Transverse displacements of the closed-loop system with the nonlinear feedback function (44).

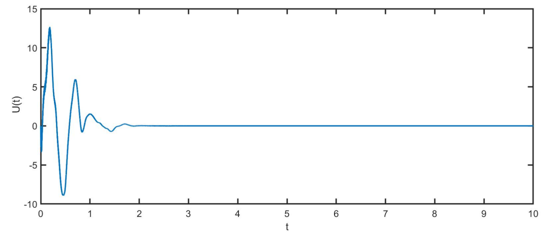

Fig. 2. Control input of the closed-loop system with the nonlinear feedback function (44).

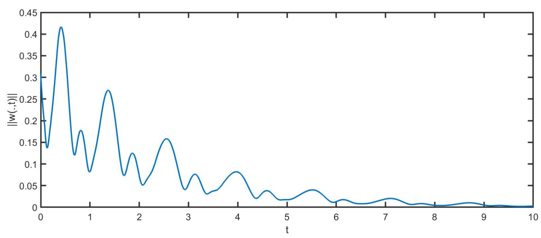

Fig. 3. Norm $\|w(\cdot,t)\|$ of the closed-loop system with the nonlinear feedback function (44).

proposed control law (2). The simulation is carried out by utilizing the finite element method, where the the quadratic Lagrange basis of the finite element equidistant meshes is applied.

To show the numerical results, the parameters of closed-loop system (4) are assigned as follows: $M(\|w_x(t)\|) = 2 + 3\|w_x(\cdot,t)\|$, $v = 0.5$, $L = 1$ and $\lambda = 0.03$. The initial displacement and velocity of the moving beam presented in simulation are $f_1(x) = 0.5\sin(5x)$ and $f_2(x) = 0.5\cos(3x)$. A nonlinear feedback function satisfying the restrictive conditions near infinity and zero is present in simulation as follows:

$$F(s) = \begin{cases} 2s - 8, & s \leq -1, \\ 10s^2, & 0 < s \leq 1, \\ -10s^2, & -1 < s \leq 0, \\ 3s + 7, & 1 < s. \end{cases} \tag{44}$$

The transverse vibration $w(x,t)$ of the closed-loop system (5) with the nonlinear feedback function (44) is illustrated in Fig. 1, and the corresponding control input $U(t)$ and the norm $\|w(\cdot,t)\|$ of the closed-loop system (5) are depicted in Fig. 2 and Fig. 3, which indicates that the transverse vibration of axially moving Kirchhoff-type beam has been suppressed.

5. CONCLUSION

In this paper, the energy decay rates of the axially moving Kirchhoff beam is considered. The asymptotic stability of the energy function of the closed-loop system is guaranteed by an ODE system. Under the specific growth of nonlinear boundary function near zero and at infinity, the specific decay rates of energy for the axially moving Kirchhoff beam can be obtained. If a time-delay controller is applied at the boundary, the boundary stabilization of the axially moving Kirchhoff beam is still an open problem and will be the focus of future work.

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