Observer design for 1-D boundary controlled port-Hamiltonian systems with different boundary measurements ⋆

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Abstract: This paper investigates the observer design for the 1D boundary controlled port-Hamiltonian systems (BC-PHS) using the late lumping approach. Different observers are proposed for BC-PHS with different measured boundary variables. Based on the passivity propriety of the BC-PHS, sufficient conditions of the observer error convergence are provided for the different proposed observers. The wave equation is used to illustrate the effectiveness of the proposed observers with different boundary sensing.

Keywords: Distributed port-Hamiltonian systems; Observer design; Boundary measurements; Exponential stability; Asymptotic stability.

1. INTRODUCTION

The observer design is always an important research topic in the control theory. An observer is a dynamic extension that can reconstruct the system states using the limited information from sensors, actuators, and input signals. Since the work of (Luenberger, 1964), the observer design methods have been well studied for finite-dimensional linear systems or lumped parameter systems (LPS). Since the end of the last century, the observer development for the infinite-dimensional linear systems or distributed parameter systems (DPS) has been widely investigated (Hidayat et al., 2011). Different studies have investigated on the observer design for the wave equation (Guo and Xu, 2007; Krstic et al., 2008; Guo and Guo, 2009; Smyshlyaev and Krstic, 2009; Meurer and Kugi, 2011; Feng and Guo, 2016) or the diffusion-convection-reaction processes (Smyshlyaev and Krstic, 2005; Meurer and Kugi, 2009), among others. This paper investigates the observer design method for a class of linear infinite dimensional system using the port Hamiltonian system (PHS). The PHS is suitable for the modeling and controller design of complex multi-physical systems, considering the energy exchange between different system components. It has been proposed for the finite dimensional system (Maschke and van der Schaft, 1992) and generalized to the infinite dimensional case (van der Schaft and Maschke, 2002). In (Le Gorrec et al., 2005) a new class of boundary controlled infinite dimensional system has been defined as *boundary controlled port-Hamiltonian systems* (BC-PHSs) and its well-posedness is investigated using semi-group theory. Due to its passivity propriety, different control design strategies have been proposed for the BC-PHS (Villegas et al., 2005; Ramirez et al., 2014; Macchelli and Califano, 2018; Macchelli et al., 2020). The observer design for the infinite dimensional PHS has been less investigated in the literature (Toledo et al., 2020; Malzer et al., 2020; Wu et al., 2020) with different approaches.

The main contribution of this paper is to provide the observer design methods for a class of the BC-PHS based on different boundary measurements. (i) We propose the observer structure assuming that the conjugated output variables at the boundary are fully or partially measurable, for instance, the boundary force and velocity for the wave equation. The sufficient conditions for the exponential convergence of the error dynamics are given. (ii) The observer is proposed and the sufficient condition of the asymptotic convergence of the error dynamics is given when the time integration of the boundary conjugated output variables is measurable (as the boundary displacement of the wave equation). This paper is organized as follows: Section 2 gives some preliminaries of the BC-PHS and defines the structure of the observer and error dynamics. In Section 3, we assume the conjugated outputs are fully or partially available and the exponentially stable observer is designed based on these measured variables. Section 4 presents the asymptotically stable observer design of the BC-PHS when only the time integration of the conjugated output is measurable. The numerical simulations with the clamped free vibrating string are shown in Section 5 to illustrate different proposed observers and the final conclusion and the future work are given in Section 6.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this paper, we consider the following BC-PHS:

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$$\mathcal{P}\begin{cases}\dfrac{\partial x}{\partial t}(\zeta,t)=P_1\dfrac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)x(\zeta,t))+P_0\mathcal{H}(\zeta)x(\zeta,t),\\ W_{\mathcal{B}}\begin{pmatrix}f_\partial(t)\\ e_\partial(t)\end{pmatrix}=u(t),\quad x(\zeta,0)=x_0(\zeta),\\ y(t)=W_{\mathcal{C}}\begin{pmatrix}f_\partial(t)\\ e_\partial(t)\end{pmatrix},\quad y_m(t)=\mathcal{C}_m x(\zeta,t),\end{cases} \tag{1}$$

where $\zeta\in[a,b]$ is the spatial variable and $t\geq 0$ is the time variable. $x(\zeta,t)\in\mathbb{R}^n$ is the state variable with initial condition $x_0(\zeta)$. $P_1=P_1^T\in\mathbb{R}^{n\times n}$ is a non-singular matrix, $P_0=-P_0^T\in\mathbb{R}^{n\times n}$, $\mathcal{H}(\cdot)\in L_2([a,b];\mathbb{R}^{n\times n})$ is a bounded and continuously differentiable matrix-valued function satisfying for all $\zeta\in[a,b]$, $\mathcal{H}(\zeta)=\mathcal{H}^T(\zeta)$ and $mI<\mathcal{H}(\zeta)<MI$ with $0<m<M$ both scalars independent of ζ. The Hamiltonian of the BC-PHS (1) is given by

$$H(t)=\frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{2}$$

The *boundary port variables* (Le Gorrec et al., 2005) associated to the BC-PHS (1) are obtained such that $\dot{H}=f_\partial(t)^Te_\partial(t)$ as follows:

$$\begin{pmatrix}f_\partial(t)\\ e_\partial(t)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}P_1 & -P_1\\ I & I\end{pmatrix}\begin{pmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{pmatrix}. \tag{3}$$

$W_{\mathcal{B}}$ is defined such that it has full rank and satisfies $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^T=0$, with $\Sigma=\left(\begin{smallmatrix}0_n & I_n\\ I_n & 0_n\end{smallmatrix}\right)$. $W_{\mathcal{C}}$ is defined such that (1) is an impedance energy preserving system, *i.e.* $W_{\mathcal{C}}\Sigma W_{\mathcal{C}}^T=0$ and $W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^T=I$. Finally, $u(t)\in\mathbb{R}^n$ is the input, $y(t)\in\mathbb{R}^n$ is the **conjugated output**, and $y_m(t)\in\mathbb{R}^p$ is the measured output, with $\mathcal{C}_m$ a boundary operator which maps the state into the measured variables located at the spatial boundaries of the domain of ζ.

Definition 2.1. The following infinite-dimensional system:

$$\hat{\mathcal{P}}\begin{cases}\dfrac{\partial\hat{x}}{\partial t}(\zeta,t)=P_1\dfrac{\partial}{\partial\zeta}(\mathcal{H}\hat{x}(\zeta,t))+P_0(\mathcal{H}\hat{x}(\zeta,t)),\\ W_{\mathcal{B}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right)=\hat{u}(t),\quad \hat{x}(\zeta,0)=\hat{x}_0(\zeta)\\ \hat{y}(t)=W_{\mathcal{C}}\left(\begin{smallmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{smallmatrix}\right),,\quad \hat{y}_m(t)=\mathcal{C}_m\hat{x}(\zeta,t),\end{cases} \tag{4}$$

is an observer of the BC-PHS (1) if $\hat{x}(\zeta,t)$ converges to $x(\zeta,t)$ for some initial condition $\hat{x}_0(\zeta)\neq x_0(\zeta)$. P_1, P_0, $\mathcal{H}$, $W_{\mathcal{B}}$, $W_{\mathcal{C}}$, and $\mathcal{C}_m$ are defined in (1), and the observer boundary port variables $\begin{pmatrix}\hat{f}_\partial(t)\\ \hat{e}_\partial(t)\end{pmatrix}$ are defined in the same way as in (3). □

Since the system $\hat{\mathcal{P}}$ in (4) is virtual, the input $\hat{u}(t)$ is designed with all the available information, i.e. $\hat{u}(t)=f(u(t),y_m(t),\hat{x}(\zeta,t))$, where $u(t)$ and $y_m(t)$ are considered known from (1) and $f(\cdot)$ is a function to be designed. In the following section, we design this function for different types of measurements $y_m(t)$. To analyze the convergence of the observer, it is convenient to analyze the error between the state of (1) and the state of (4). To this end, we define the error between the BC-PHS (1) and the observer (4) as:

$$\tilde{x}(\zeta,t):=x(\zeta,t)-\hat{x}(\zeta,t). \tag{5}$$

Then, from (1) and (4), we obtain the error dynamics equations as follows:

$$\tilde{\mathcal{P}}\begin{cases}\dfrac{\partial\tilde{x}}{\partial t}(\zeta,t)=P_1\dfrac{\partial}{\partial\zeta}(\mathcal{H}\tilde{x}(\zeta,t))+P_0(\mathcal{H}\tilde{x}(\zeta,t)),\\ W_{\mathcal{B}}\left(\begin{smallmatrix}\tilde{f}_\partial(t)\\ \tilde{e}_\partial(t)\end{smallmatrix}\right)=\tilde{u}(t),\quad \tilde{x}(\zeta,0)=\tilde{x}_0(\zeta),\\ \tilde{y}(t)=W_{\mathcal{C}}\left(\begin{smallmatrix}\tilde{f}_\partial(t)\\ \tilde{e}_\partial(t)\end{smallmatrix}\right).\end{cases} \tag{6}$$

We define the Hamiltonian of the error system as: $\tilde{H}(t)=\dfrac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}}=\dfrac{1}{2}\int_a^b\tilde{x}(\zeta,t)^T\mathcal{H}(\zeta)\tilde{x}(\zeta,t)d\zeta$. Since $W_{\mathcal{B}}$ and $W_{\mathcal{C}}$ are such that $W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^T=I$, the time derivative of $\tilde{H}(t)$ satisfies

$$\dot{\tilde{H}}(t)=\tilde{u}(t)^T\tilde{y}(t). \tag{7}$$

An important property of BC-PHS is shown in the following theorem. This property is in general used for showing the exponential stability of BC-PHSs. In this paper, we use it for showing that the error system is exponentially stable for different kinds of observers. This theorem states that the Hamiltonian of the error system $\tilde{H}(t)$ is bounded by the integration over time of the co-energy variables evaluated at the spatial boundaries.

Theorem 2.1. Consider the error system (6) with $W_{\mathcal{B}}$ such that $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^T\geq 0$ and $\Sigma=\left(\begin{smallmatrix}0_n & I_n\\ I_n & 0_n\end{smallmatrix}\right)$. If $W_{\mathcal{B}}\left(\begin{smallmatrix}\tilde{f}_\partial(t)\\ \tilde{e}_\partial(t)\end{smallmatrix}\right)=0$, for all $t\geq 0$, then the Hamiltonian of the error system $\tilde{H}(t)=\frac{1}{2}\|\tilde{x}(t)\|^2_{\mathcal{H}}$ satisfies for τ large enough

$$\begin{aligned}\tilde{H}(\tau)&\leq c(\tau)\int_0^\tau\|\mathcal{H}(b)\tilde{x}(b,t)\|^2_{\mathbb{R}}dt,\quad \textit{and}\\ \tilde{H}(\tau)&\leq c(\tau)\int_0^\tau\|\mathcal{H}(a)\tilde{x}(a,t)\|^2_{\mathbb{R}}dt\end{aligned} \tag{8}$$

where $c(\tau)$ is a constant that only depends on τ. □

Proof. This result is a direct application of (Villegas, 2007, Theorem 5.17) to the error system (6). ■

Remark 2.1. We consider the BC-PHS (1) that is an impedance energy preserving system, *i.e.* $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^T=W_{\mathcal{C}}\Sigma W_{\mathcal{C}}^T=0$ and $W_{\mathcal{C}}\Sigma W_{\mathcal{B}}^T=I$. Then, the conditions of Theorem 2.1 are satisfied. Moreover, even if $W_{\mathcal{B}}\Sigma W_{\mathcal{B}}^T\geq 0$, we can use Theorem 2.1. □

3. OBSERVER DESIGN WITH BOUNDARY CONJUGATED OUTPUT MEASUREMENT

In this section, we investigate the observer design method of the BC-PHS when the power conjugated output $y(t)$ (co-energy variables) from (1) is fully or partially measured. For both measurement cases, the sufficient conditions for exponential convergence of the error dynamics will be given in Subsection 3.1 and 3.2, respectively.

3.1 Full measurement of the conjugated output

In the following proposition, we give the observer structure and provide the sufficient condition for the exponential convergence of the error dynamics when the power conjugated output $y(t)$ is fully measurable *i.e.*, $y_m(t)=y(t)$.

Proposition 3.1. Consider the BC-PHS (1). Assume that the full conjugated output is measurable, *i.e.* $y_m(t)=y(t)$. The state of the observer (4) with

$$\hat{u}(t)=u(t)+L(y_m(t)-\hat{y}_m(t)), \tag{9}$$

converge exponentially to the state of the BC-PHS (1) if $0<L+L^T\in\mathbb{R}^{n\times n}$. □

Proof. Substituting (9) in the observer (4), the input of the error system (6) reads $\tilde{u}(t)=-L\tilde{y}(t)$, and the balance equation (7) becomes

$$\dot{\tilde{H}}(t)=-\tilde{y}(t)^TL\tilde{y}(t). \tag{10}$$

Finally, the error system can be written as

$$\begin{cases} \frac{\partial \tilde{x}}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_L \left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix} \right) = 0, \quad W_L = W_\mathcal{B} + LW_\mathcal{C}, \\ \tilde{y}(t) = W_\mathcal{C} \left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix} \right), \quad \tilde{x}(\zeta,0) = \tilde{x}_0(\zeta). \end{cases} \tag{11}$$

According to (Le Gorrec et al., 2005, Theorem 4.1), the error system (11) is well-posed if $W_L \Sigma W_L^T \geq 0$. Since $W_L = W_\mathcal{B} + LW_\mathcal{C}$, $W_\mathcal{B}\Sigma W_\mathcal{B}^T = W_\mathcal{C}\Sigma W_\mathcal{C}^T = 0$, $W_\mathcal{B}\Sigma W_\mathcal{C}^T = I_n$, and $L = L^T > 0$, the inequality $W_L\Sigma W_L^T \geq 0$ is satisfied. Since $W_L\Sigma W_L^T \geq 0$ and $W_L \left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix} \right) = 0$, the estimations (8) of the Hamiltonian error are satisfied (see Theorem 2.1).

We use the estimations (8) to show that the error system converges to zero exponentially. To this end, we show that, for some τ large enough and some positive constants c_τ, l_1 and m_1, the Hamiltonian of the error system is such that

$$\tilde{H}(\tau) \leq \frac{c_\tau}{c_\tau + l_1 m_1} \tilde{H}(0). \tag{12}$$

To find the estimation (12), we write the boundary condition and the output $\tilde{y}(t)$ of (11) as follows:

$$\begin{pmatrix} 0 \\ \tilde{y}(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} W_L \\ W_\mathcal{C} \end{pmatrix} \begin{pmatrix} P_1 & -P_1 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} \mathcal{H}(b)\tilde{x}(b,t) \\ \mathcal{H}(a)\tilde{x}(a,t) \end{pmatrix}. \tag{13}$$

We define the matrix

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} W_L \\ W_\mathcal{C} \end{pmatrix} \begin{pmatrix} P_1 & -P_1 \\ I_n & I_n \end{pmatrix}. \tag{14}$$

One can prove the matrix M in (14) is invertible (Le Gorrec et al., 2005). This implies that $\|Mw\|^2_\mathbb{R} \geq m_1 \|w\|^2_\mathbb{R}$, for some vector w of appropriated dimension and a constant m_1 that can be the smallest eigenvalue of M, for instance. If we compute the norm at both sides of (13), we obtain the following:

$$\|\tilde{y}(t)\|^2_\mathbb{R} = \left\| M \begin{pmatrix} \mathcal{H}(b)\tilde{x}(b,t) \\ \mathcal{H}(a)\tilde{x}(a,t) \end{pmatrix} \right\|^2_\mathbb{R} \geq m_1 \left\| \begin{pmatrix} \mathcal{H}(b)\tilde{x}(b,t) \\ \mathcal{H}(a)\tilde{x}(a,t) \end{pmatrix} \right\|^2_\mathbb{R}$$

This implies that the norm of the error co-energy variables evaluated at the spatial boundaries are bounded by the norm of the output as follows:

$$\|\mathcal{H}(b)\tilde{x}(b,t)\|^2_\mathbb{R} \leq \frac{1}{m_1} \|\tilde{y}(t)\|^2_\mathbb{R} \tag{15}$$

(similar with $\|\mathcal{H}(b)\tilde{x}(a,t)\|^2_\mathbb{R}$). Moreover, since L is positive definite, the norm of the output can be also bounded as follows

$$\|\tilde{y}(t)\|^2_\mathbb{R} \leq \frac{1}{l_1} \tilde{y}(t)^T L \tilde{y}(t), \tag{16}$$

with l_1 a positive scalar that can be for instance the smallest eigenvalue of L. Then, from (15) and (16), one can conclude

$$\|\mathcal{H}(b)\tilde{x}(b,t)\|^2_\mathbb{R} \leq \frac{1}{m_1 l_1} \tilde{y}(t)^T L \tilde{y}(t). \tag{17}$$

Since the error system (11) satisfies the conditions of Theorem 2.1, thus one can use (8) and (17) to obtain the following bound for the Hamiltonian of the error system:

$$\tilde{H}(\tau) \leq \frac{c_\tau}{m_1 l_1} \int_0^\tau \tilde{y}(t)^T L \tilde{y}(t) dt. \tag{18}$$

Finally, from (10) we obtain

$$\tilde{H}(\tau) - \tilde{H}(0) = -\int_0^\tau \tilde{y}(t)^T L \tilde{y}(t) dt. \tag{19}$$

Replacing (19) into the estimation (18), we obtain the estimation (12), concluding that the error state $\tilde{x}(\zeta,t)$ converges to zero exponentially. ■

3.2 Partial measurement of the conjugated output

In the previous subsection we have considered the full conjugated output measurement is available for the observer design. However, the conjugate output is not always fully measurable. This is the case, when the sensors are restricted to be at one side of the spatial domain. For instance, one of the two conjugated outputs of the wave equation is measured (the force at $\zeta = a$ or the velocity at $\zeta = b$). In these cases, by showing an extra condition, we also can design exponentially convergent observers for the BC-PHS (1). In the following proposition, we give the sufficient condition that guarantees the exponential convergence of the observer (4) when the conjugated output is partially measured.

Proposition 3.2. Consider the BC-PHS (1). Assume that the conjugated output is partially measurable, *i.e.* $y_m(t) = C_m y(t)$, with $C_m = \left(\begin{smallmatrix} I_p & 0_{p\times n-p} \end{smallmatrix} \right) \in \mathbb{R}^{p\times n}$ and $0 < p < n$. The states of the observer (4) with

$$\hat{u}(t) = u(t) + C_m^T L(y_m(t) - \hat{y}_m(t)) \quad \text{and} \quad L \in \mathbb{R}^{p\times p} \tag{20}$$

converges exponentially to the state of the BC-PHS (1) if L is such that $C_m^T L^T C_m + C_m^T L C_m \geq 0$, and one of the following conditions is satisfied

$$\begin{array}{c} \|\mathcal{H}(b)\tilde{x}(b,t)\|^2_\mathbb{R} \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) \;\; \textit{or} \\ \|\mathcal{H}(a)\tilde{x}(a,t)\|^2_\mathbb{R} \leq \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t), \end{array} \tag{21}$$

for some scalar $\gamma > 0$. □

Proof. Using (20) in the observer (4), the input of the error system (6) becomes

$$\tilde{u}(t) = -C_m^T L C_m \tilde{y}(t), \tag{22}$$

and the balance equation (7) becomes

$$\dot{\tilde{H}}(t) = -\tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) = -\tilde{y}_m(t)^T L \tilde{y}_m(t). \tag{23}$$

Finally, the error system can be written as

$$\begin{cases} \frac{\partial \tilde{x}}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}\tilde{x}(\zeta,t)) + P_0(\mathcal{H}\tilde{x}(\zeta,t)), \\ W_L \left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix} \right) = 0, \quad W_L = W_\mathcal{B} + C_m^T L C_m W_\mathcal{C}, \\ \tilde{y}(t) = W_\mathcal{C} \left(\begin{smallmatrix} \tilde{f}_\partial(t) \\ \tilde{e}_\partial(t) \end{smallmatrix} \right), \quad \tilde{x}(\zeta,0) = \tilde{x}_0(\zeta). \end{cases} \tag{24}$$

According to (Le Gorrec et al., 2005, Theorem 4.1), the error system (24) is well-posed if $W_L \Sigma W_L^T \geq 0$. Since $W_L = W_\mathcal{B} + C_m^T L C_m W_\mathcal{C}$, $W_\mathcal{B}\Sigma W_\mathcal{B}^T = W_\mathcal{C}\Sigma W_\mathcal{C}^T = 0$, $W_\mathcal{B}\Sigma W_\mathcal{C}^T = I_n$, and $C_m^T L^T C_m + C_m^T L C_m \geq 0$, the inequality $W_L \Sigma W_L^T \geq 0$ is satisfied.

The exponential convergence of the error system (24) is a direct application of (Villegas, 2007, Corollary 5.19). In fact, we use (8), (23), and (21) to show that the Hamiltonian of the error system decreases exponentially. First, we use (21) to bound the Hamiltonian estimation from (8) as follows

$$\begin{aligned} \tilde{H}(\tau) &\leq c(\tau) \textstyle\int_0^\tau \|\mathcal{H}(b)\tilde{x}(b,t)\|^2_\mathbb{R} dt \\ &\leq c(\tau) \textstyle\int_0^\tau \gamma \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) dt, \end{aligned} \tag{25}$$

$$\Rightarrow \tilde{H}(\tau) \leq c(\tau)\gamma \int_0^\tau \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) dt. \tag{26}$$

Then, we integrate along time both sides of equation (23)

$$\tilde{H}(0)-\tilde{H}(\tau)=\int_0^\tau \tilde{y}(t)^T C_m^T L C_m \tilde{y}(t)dt,$$

and finally, we replace the last equation in (26)

$$\tilde{H}(\tau)\le c(\tau)\gamma\left(\tilde{H}(0)-\tilde{H}(\tau)\right)\Leftrightarrow \tilde{H}(\tau)\le \frac{c(\tau)\gamma}{c(\tau)\gamma+1}\tilde{H}(0).$$

This concludes the proof. ■

4. OBSERVER DESIGN WITH BOUNDARY DISPLACEMENT MEASUREMENTS

The last section presents the observer design for the BC-PHS (1) with the conjugated output measurement. However, the conjugated output is not easy or sometimes impossible to measure in practice. In this section, we investigate the observer design when the time integration of the conjugated output is available for the measurement. For instance, if we consider the boundary velocity as conjugated output, its time integration is the displacement of this boundary which is easy to measure. We show in the following proposition the observer design with a dynamics extension for the BC-PHS when the time integration of the conjugated output measurement is available. The sufficient condition for the asymptotic convergence of the error dynamics is given in the proposition.

Proposition 4.1. Consider the BC-PHS (1). Assume that the measurement is on the following form:

$$y_m(t)=\int_0^t C_m y(\tau)d\tau + y_m(0), \text{ with } C_m=\left(0_{p\times n-p}\ I_p\right). \tag{27}$$

Assume that the BC-PHS is approximately observable (Curtain and Zwart, 2012, Corollary 4.1.14) with respect to the output $C_m y(t)$. The state of the observer (4) with

$$\begin{aligned}\hat{u}(t)&=u(t)+C_m^T L_1(y_m(t)-\hat{y}_m(t)+\theta(t)),\\ \dot{\theta}(t)&=-L_2(y_m(t)-\hat{y}_m(t)+\theta(t)),\quad \theta(0)=\theta_0.\end{aligned}\tag{28}$$

converges asymptotically to the state of the BC-PHS (1) if $L_1,\ L_2\in\mathbb{R}^{p\times p}$ are both positive definite matrices. □

Proof. Since we assume that the BC-PHS (1) is approximately observable with respect to the output $C_m y(t)$, the error system (6) is approximately observable with respect to the output $C_m\tilde{y}(t)$. We use this property to show the asymptotic stability of the error system by using LaSalle's invariance principle.

From (28) and (6) we obtain

$$\tilde{u}(t)=-C_m^T L_1(y_m(t)-\hat{y}_m(t)+\theta(t)). \tag{29}$$

We define the following auxiliary variable:

$$x_o(t)=\tilde{y}_m(t)+\theta(t)\quad\text{with}\quad \tilde{y}_m(t)=y_m(t)-\hat{y}_m(t) \tag{30}$$

and $\hat{y}_m(t)=\int_0^t C_m\hat{y}(\tau)d\tau+\hat{y}_m(0)$. The dynamic equation of $x_o(t)$ in (30) is obtained from (27) and (28) as follows:

$$\begin{aligned}\dot{x}_o(t)&=\dot{\tilde{y}}_m(t)+\dot{\theta}(t),\\ &=C_m\tilde{y}(t)-L_2 x_o(t).\end{aligned}\tag{31}$$

We define the following auxiliary finite-dimensional system:

$$P_o\begin{cases}\dot{x}_o(t)=A_o x_o(t)+B_o u_o(t),\\ y_o(t)=C_o x_o(t),\end{cases}\tag{32}$$

with

$$A_o=-L_2,\quad B_o=C_m,\quad C_o=C_m^T L_1.$$

Then, the input of the error system (29) is equivalently obtained with the passive interconnection

$$\begin{pmatrix}\tilde{u}(t)\\ u_o(t)\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ y_o(t)\end{pmatrix}\tag{33}$$

between the error system (6) and the auxiliary system (32).

Now, we use LaSalle's invariance principle to show that the closed loop system represented by the passive interconnection of the error system (6) and the auxiliary system (32) converges asymptotically to zero. To this end, the reader is refer to (Villegas, 2007, Theorem 5.8) and (Villegas, 2007, Theorem 5.9) for the well-posedness and the compactness of the solutions of the system. We consider the following Lyapunov function: $V(t)=\frac{1}{2}\int_a^b \tilde{x}(\zeta,t)^T\mathcal{H}(\zeta)\tilde{x}(\zeta,t)d\zeta+\frac{1}{2}x_o(t)^T L_1 x_o(t)$. It follows from LaSalle's invariance principle that all solutions of the interconnected system tend to the maximal invariant set of

$$\vartheta_o=\{\tilde{x}\in L_2([a,b],\mathbb{R}^n),\ x_o\in\mathbb{R}^p \mid \dot{V}(t)=0\}. \tag{34}$$

We define the maximal invariant subset of ϑ_o as $\mathcal{I}$, and we show that $\mathcal{I}$ only contains the zero state, *i.e.* $\mathcal{I}=\{\tilde{x}(\zeta,t)=0,\ x_o(t)=0\}$. From (6), and (32), we obtain the following balance for the Lyapunov function:

$$\dot{V}(t)=-x_o(t)^T L_1 R_o L_1 x_o(t), \tag{35}$$

with $R_o=L_2L_1^{-1}>0$ and $L_1>0$ by definition. Then, $\dot{V}(t)=0$ implies $x_o(t)=0$, which implies $\dot{x}_o(t)=0$. Then, from (32) and (33) $B_o u_o(t)=C_m\tilde{y}(t)=0$. Since $x_o(t)=0$, from (32) $y_o(t)=0$. Since $y_o(t)=0$, from the interconnection (33) we can conclude $\tilde{u}(t)=0$. Then, the maximal invariant set $\mathcal{I}$ contains $x_o=0$ and the solution of the following BC-PHS

$$\begin{cases}\frac{\partial\tilde{x}}{\partial t}(\zeta,t)=P_1\frac{\partial}{\partial\zeta}(\mathcal{H}(\zeta)\tilde{x}(\zeta,t))+P_0\mathcal{H}(\zeta)\tilde{x}(\zeta,t),\\ W_{\mathcal{B}}\begin{pmatrix}\tilde{f}_\partial(t)\\ \tilde{e}_\partial(t)\end{pmatrix}=0,\quad \tilde{x}(\zeta,0)=\tilde{x}_0(\zeta),\\ C_m\tilde{y}(t)=C_m W_{\mathcal{C}}\begin{pmatrix}\tilde{f}_\partial(t)\\ \tilde{e}_\partial(t)\end{pmatrix}=0.\end{cases}$$

The latter being approximately observable with respect to $C_m\tilde{y}(t)$ implies that if $C_m\tilde{y}(t)=0$ for an interval of time, then the state is such that $\tilde{x}(\zeta,t)=0$ (See (Curtain and Zwart, 2012, Corollary 4.1.14)). Then, the maximal invariant set $\mathcal{I}$ only contains the states $\tilde{x}(\zeta,t)=0$ and $x_o(t)=0$. Thus, by LaSalle's invariance principle, the error system is asymptotically stable. ■

5. NUMERICAL ILLUSTRATIONS ON THE WAVE EQUATIONS

In this section, we use the vibrating string with the Young's modulus and the mass density equal to one ($T(\zeta)=\rho(\zeta)=1$), and unitary length ($a=0,\ b=1$) to illustrated proposed observers. We consider as input the velocity at $\zeta=a$, and the force at $\zeta=b$. The power conjugated outputs are the force at $\zeta=a$, and the velocity at $\zeta=b$, respectively. The system is written as an impedance energy preserving BC-PHS (1) as follows:

$$\mathcal{P}\begin{cases}\frac{\partial}{\partial t}\begin{pmatrix}q(\zeta,t)\\ p(\zeta,t)\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial\zeta}\begin{pmatrix}q(\zeta,t)\\ p(\zeta,t)\end{pmatrix},\\ \begin{pmatrix}p(a,t)\\ q(b,t)\end{pmatrix}=u(t),\ \begin{pmatrix}q(\zeta,0)\\ p(\zeta,0)\end{pmatrix}=\begin{pmatrix}q_0(\zeta)\\ p_0(\zeta)\end{pmatrix},\\ y(t)=\begin{pmatrix}-q(a,t)\\ p(b,t)\end{pmatrix},\quad y_m(t)=C_m y(t),\end{cases}\tag{36}$$

where $y_m(t)$ represents the measurements. The plant and the observer are simulated using the discretization method proposed in (Trenchant et al., 2018) with 200 state variables. The midpoint time discritisation method is used with the time step $\delta t = 0.1ms$. The initial conditions are $q_0(\zeta) = \dfrac{dw_0}{d\zeta}(\zeta)$, $w_0(\zeta) = e^{-16(\zeta-0.5)^2}$. and $p_0(\zeta) = \hat{q}_0(\zeta) = \hat{p}_0(\zeta) = 0$, Since $q(\zeta,t) = \frac{\partial w}{\partial \zeta}(\zeta,t)$, the string deformation $w(\zeta,t)$ and the observed one $\hat{w}(\zeta,t)$ are numerically obtained by $w(\zeta,t) = w(0,t) + \int_0^\zeta q(z,t)dz$, $\hat{w}(\zeta,t) = \hat{w}(0,t) + \int_0^\zeta \hat{q}(z,t)dz$ considering $w(0,t) = \hat{w}(0,t) = 0$.

We consider firstly the case where the conjugated output is fully measured, *i.e.* $y_m(t) = y(t)$. Using Proposition 3.1 with $u(t) = 0$ (the string is attached at $\zeta = a$ and free at $\zeta = b$), the state of the following observer

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix}, \begin{pmatrix}\hat{q}(\zeta,0)\\ \hat{p}(\zeta,0)\end{pmatrix} = \begin{pmatrix}\hat{q}_0(\zeta)\\ \hat{p}_0(\zeta)\end{pmatrix},$$
$$\begin{pmatrix}\hat{p}(a,t)\\ \hat{q}(b,t)\end{pmatrix} = \begin{pmatrix}l_1 & 0\\ 0 & l_2\end{pmatrix}\begin{pmatrix}-[q(a,t)-\hat{q}(a,t)]\\ p(b,t)-\hat{p}(b,t)\end{pmatrix}$$

converges exponentially to the state of the system if l_1, $l_2 > 0$. In this case, $L = \left(\begin{smallmatrix}l_1 & 0\\ 0 & l_2\end{smallmatrix}\right) > 0$, and for simplicity, we use $l_1 = l_2 = 1$.

The deformation error between the plant and the observer (Fig. 1(a)), starts from a non zero initial condition and reaches zero approximately at $t = 1$. In Fig. 1(b), we show the Hamiltonians of the plant, the observer and the error system. We can see the Hamiltonian of the string ($H(t)$) is constant since there is no dissipation. The Hamiltonian of the error system converges to zero exponentially ($\tilde{H}(t)$) and the Hamiltonian of the observer ($\hat{H}(t)$) reaches the Hamiltonian of the string.

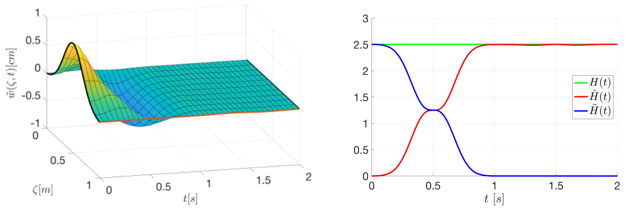

Fig. 1. Fully conjugated output measurement based observer design: (a) Estimation error; (b) Hamiltonian of the plant (green), observer (red) and error system (blue).

We consider that the conjugated output is partially measured with the following measured output $y_m = C_m y(t) = (1\ 0)\, y = -q(a,t)$. Consider the following infinite-dimensional observer

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix}, \begin{pmatrix}\hat{q}(\zeta,0)\\ \hat{p}(\zeta,0)\end{pmatrix} = \begin{pmatrix}\hat{q}_0(\zeta)\\ \hat{p}_0(\zeta)\end{pmatrix},$$
$$\begin{pmatrix}\hat{p}(a,t)\\ \hat{q}(b,t)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} l_1(y_m(t) - \hat{y}_m(t)),$$

with the scalar $l_1 > 0$. The error system is

$$\frac{\partial}{\partial t}\begin{pmatrix}\tilde{q}(\zeta,t)\\ \tilde{p}(\zeta,t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix}\tilde{q}(\zeta,t)\\ \tilde{p}(\zeta,t)\end{pmatrix}, \begin{pmatrix}\tilde{q}(\zeta,0)\\ \tilde{p}(\zeta,0)\end{pmatrix} = \begin{pmatrix}\tilde{q}_0(\zeta)\\ \tilde{p}_0(\zeta)\end{pmatrix},$$
$$\begin{pmatrix}\tilde{p}(a,t)\\ \tilde{q}(b,t)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} l_1\,(1\ 0)\begin{pmatrix}-\tilde{q}(a,t)\\ \tilde{p}(b,t)\end{pmatrix}$$

and its exponential convergence is shown using Proposition 3.2. To this end, we compute both sides of the second inequality of (21) as follows:

$$\|\mathcal{H}(a)\tilde{x}(a,t)\|^2 = \tilde{q}(a,t)^2 + \tilde{p}(a,t)^2 = (l_1^2+1)\tilde{q}(a,t)^2,$$
$$\gamma\tilde{y}(t)^T C_m^T L C_m \tilde{y}(t) = \gamma l_1 \tilde{q}(a,t)^2,$$

where in the second line we have replaced the boundary condition of the error system $\tilde{p}(a,t) = l_1\tilde{q}(a,t)$. Finally, choosing $\gamma \geq (l_1^2+1)l_1^{-1}$, the condition $\|\mathcal{H}\tilde{x}(a,t)\|^2_{\mathbb{R}} \leq \gamma\tilde{y}(t)^T C_m^T L C_m \tilde{y}(t)$ is satisfied and the infinite-dimensional observer converges to the BC-PHS. For simplicity, the observer is designed using $l_1 = 1$. Similarly as before, Fig.2(a), and Fig.2(b) show the error and the Hamiltonians. The observer converges to the real values approximately at $t = 2$.

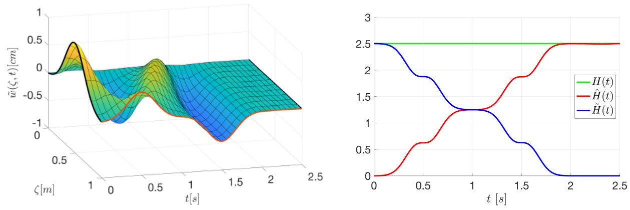

Fig. 2. Partially conjugated output measurement based observer design (a) Estimation error; (b) Hamiltonian of the plant (green), observer (red) and error system (blue).

We now assume that the displacement is measured at $\zeta = b$, *i.e.* $y_m(t) = w(b,t)$. Note that, the measured output (displacement at $\zeta = b$) is the integral over time of one of the conjugated outputs (velocity at $\zeta = b$). Using Proposition 4.1, the state of the following infinite-dimensional observer

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial \zeta}\begin{pmatrix}\hat{q}(\zeta,t)\\ \hat{p}(\zeta,t)\end{pmatrix}, \begin{pmatrix}\hat{q}(\zeta,0)\\ \hat{p}(\zeta,0)\end{pmatrix} = \begin{pmatrix}\hat{q}_0(\zeta)\\ \hat{p}_0(\zeta)\end{pmatrix},$$
$$\begin{pmatrix}\hat{p}(a,t)\\ \hat{q}(b,t)\end{pmatrix} = \begin{pmatrix}0\\ l_1\,[y_m(t) - \hat{y}_m(t) + \theta(t)]\end{pmatrix},$$
$$\dot{\theta}(t) = -l_2\,[y_m(t) - \hat{y}_m(t) + \theta(t)], \qquad \theta(0) = \theta_0,$$

with l_1, $l_2 > 0$, $\theta \in \mathbb{R}$ and $\hat{y}_m(t) = \hat{w}(b,t)$, converges asymptotically to the state of the system. Since the measured output is on the form $y_m(t) = \int_0^t C_m y(\tau)d\tau + y_m(0)$ with $C_m = (0\ 1)$, one can show that the plant is approximately observable with respect to $C_m y(t) = p(b,t)$ (velocity at $\zeta = b$). Indeed, by using (8), one can show that $\tilde{H}(\tau) \leq c(\tau)\int_0^\tau p(b,t)^2 dt$ satisfying the condition of exact observability (See (Curtain and Zwart, 2012, Corollary 4.1.14 a.(iii))), which is a stronger conditions than the approximate observability.

The observer is designed using $l_1 = l_2 = 1000 > 0$, and $\theta_0 = 0$. Similar as before Fig. 3(a) shows the spatial and temporal responses of the string deformation, the estimated one, and the estimation error. The observer converge to the real values approximately at $t = 4$. Finally, in Fig. 3(b), we show the Hamiltonians of the plant, the observer and the error system. The Hamiltonian of the error system converges to zero asymptotically ($\tilde{H}(t)$) and the Hamiltonian of the observer ($\hat{H}(t)$) reaches the Hamiltonian of the string.

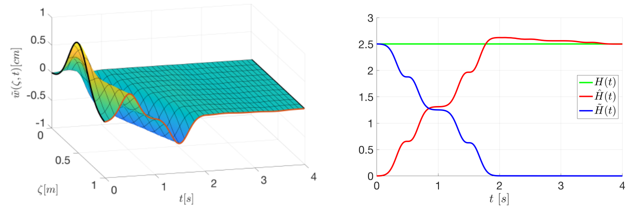

Fig. 3. Displacement measurement based observer design: (a) Estimation error; (b) Hamiltonian of the plant (green), observer (red) and error system (blue).

6. CONCLUSION AND FUTURE WORK

This paper presents different observer design methods for the BC-PHS with different boundary sensing. With fully or partially boundary co-energy variable measurement (the force and velocity measurement of the wave equation), we provide sufficient conditions of the observer gains such that the estimation error exponentially converges to zero. On the other hand, we suppose the measurement is the time integration of the boundary co-energy variable (the boundary displacement of wave equation). In that case, we propose an observer design with extended dynamics which can guarantee that the error converges asymptotically to zero. The wave equation is used to show the effectiveness of different observers. In the future, the state feedback control based on proposed observers will be investigated, taking advantage of the passivity of the BC-PHS.

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