Energy shaping control of 1D Distributed Parameter Systems

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Abstract In this chapter we give an overview on energy shaping control for Distributed Parameter Systems defined on a 1D spatial domain using the port Hamiltonian framework. We consider two different cases : when actuators and sensors are located within the spatial domain and when the actuator is situated at the boundary of the spatial domain, leading to a boundary control system (BCS). In the first case we show how dynamic extensions and structural invariants can be used to change the internal properties of the system when the system is fully actuated, and how it can be done in an approximate way when the system is actuated using piecewise continuous actuators stemming from the use of patches. Asymptotic stability is achieved using damping injection. In the boundary controlled case we show how the closed loop energy function can be partially shaped, modifying the minimum and a part of the shape of this function and how damping injection can be used to guarantee asymptotic convergence.

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1 Introduction

Control of distributed parameter systems driven by Partial Differential Equations (PDEs) has raised a major attention in the last decades due to the increasing use of compliant structures, system's information networks, smart materials and structures, multiscale and multiphysics systems etc... in engineering applications. Port Hamiltonian formulations which are an extension of Hamiltonian formulations to open multiphysics systems have shown to be particularly suited for the modelling and control of non linear systems [1, 2, 3]. They have been extended to distributed parameter systems in [4] using differential geometry, and in [5, 6] in the case of one dimensional linear systems using functional analysis. This approach has shown to be very efficient to prove existence of solution, asymptotic or exponential stability in the case the system is controlled at the boundary of its spatial domain with a linear or non-linear static or dynamic feedback [7, 8, 9, 10]. The extension of these results to 2D and 3D systems has been initiated in [11] and is still subject to many ongoing studies.

Even if the stabilization of non linear and infinite dimensional systems is quite well established in the literature, the control design consisting in assigning the closed loop performances of the system is much less developed even if it is of particular interest from an engineering point of view. In this respect, the natural control design technique that has been proposed for non linear finite dimensional port Hamiltonian systems is Energy Shaping [2, 3, 12, 13, 14]. This control design technique consists in modifying the closed loop energy function using control by interconnection with a dynamic controller and structural invariants or a state feedback. This approach has been improved in order to cope with systems with pervasive diffusion and the so called dissipation obstacle leading to the Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) methodology. These results have been generalized, at least partially, to one dimensional port Hamiltonian systems in [9].

In this chapter we give an overview on energy shaping control for Distributed Parameter Systems defined on a 1D spatial domain using the port Hamiltonian framework. We consider two different cases : when actuators and sensors are located within the spatial domain and when the actuator is situated at the boundary of the spatial domain, leading to a boundary control system (BCS). In the first case we show how dynamic extensions and structural invariants can be used to change the internal properties of the system when the system is fully actuated, and how it can be done in an approximate way when the system is actuated using piecewise continuous actuators stemming from the use of patches. Asymptotic stability is achieved using damping injection. In the boundary controlled case we show how the closed loop energy function can be partially shaped, modifying the minimum and a part of the shape of this function and how damping injection can be used to guarantee asymptotic convergence.

The chapter is organized as follows. In Section 2 we define the considered class of systems. The problem formulation is recalled in Section 3 and the control design is detailed in Section 4 when the system is actuated at the boundary of its spatial

domain and in Section 5 when the system is actuated in the domain. The chapter ends with some conclusions and perspectives.

2 Considered class of systems

In this chapter we consider one dimensional distributed parameter systems ($\zeta \in [a, b]$) with distributed and/or boundary control and observation of the form

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -R \end{bmatrix} \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_d(\zeta, t) \tag{1}$$

$$y_d(\zeta, t) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix} \tag{2}$$

$$u_\partial = \mathcal{B} \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix}, \quad y_\partial = C \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix} \tag{3}$$

where $x = [x_1^T, x_2^T]^T \in X := L^2([a, b], \mathbb{R}^n) \times L^2([a, b], \mathbb{R}^n)$, $\mathcal{L} = diag(\mathcal{L}_1, \mathcal{L}_2)$ is a bounded and Lipschitz continuous matrix-valued function such that $\mathcal{L}(\zeta) = \mathcal{L}^T(\zeta)$ and $\mathcal{L}(\zeta) \geq \eta$ with $\eta > 0$ for all $\zeta \in [a, b]$, $R \in \mathbb{R}^{(n,n)}$, $R = R^T > 0$, $\mathcal{B}(\cdot)$ and $C(\cdot)$ are some boundary input and boundary output mapping operators that will be defined later. The state space X is endowed with the inner product $\langle x | \tilde{x} \rangle_{\mathcal{L}} = \langle x | \mathcal{L}\tilde{x} \rangle$ and norm $\|x\|_{\mathcal{L}}^2 = \langle x | x \rangle_{\mathcal{L}}$ where $\langle \cdot | \cdot \rangle$ denotes the natural L^2-inner product. $X \ni x$ is the space of energy variables and $\mathcal{L}x$ denotes the co-energy variable associated to the energy variable x. Furthermore

$$\mathcal{G} = \sum_{i=0}^{N} G_i \frac{\partial^i}{\partial \zeta^i}, \text{ and } \mathcal{G}^* = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i}{\partial \zeta^i}$$

with $G_i \in \mathbb{R}^{(n,n)}$. For a sake of compactness we shall use the following notation

$$P_i = \begin{bmatrix} 0 & G_i \\ (-1)^{i+1}G_i^T & 0 \end{bmatrix}, \quad R_0 = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \tag{4}$$

and the formulation of (1)

$$\frac{\partial x}{\partial t}(\zeta, t) = \sum_{i=0}^{N} P_i \frac{\partial^i}{\partial \zeta^i}(\mathcal{L}(\zeta)x(\zeta, t)) - R_0\mathcal{L}(\zeta)x(\zeta, t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_d(\zeta, t) \tag{5}$$

$$y_d(\zeta, t) = \begin{bmatrix} 0 & I \end{bmatrix} \mathcal{L}(\zeta)x(\zeta, t) \tag{6}$$

$$u_\partial = \mathcal{B}(\mathcal{L}(\zeta)x(\zeta, t)), y_\partial = C(\mathcal{L}(\zeta)x(\zeta, t)) \tag{7}$$

The total energy of the system $H(x)$ is defined by

$$H(x) = \int_a^b \left(x^T(\zeta,t)\mathcal{L}(\zeta)x(\zeta,t) \right) d\zeta$$

Definition 1 (Mixed in-domain / boundary controlled port Hamiltonian systems (IDBC-PHS)) A mixed in-domain / boundary controlled port Hamiltonian system is an infinite dimensional system of the form (5-7) where the input mapping $u_\partial = \mathcal{B}(\mathcal{L}(\zeta)x(\zeta,t))$ and the output mapping $y_\partial = C(\mathcal{L}(\zeta)x(\zeta,t))$ are defined by

$$u_\partial = W_B \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix}, \text{ and } y_\partial = W_C \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix} \quad (8)$$

with

$$W_B = \left[\tfrac{1}{\sqrt{2}}(\Xi_2 + \Xi_1 P_e) \quad \tfrac{1}{\sqrt{2}}(\Xi_2 - \Xi_1 P_e) \right], \quad (9)$$

$$W_C = \left[\tfrac{1}{\sqrt{2}}(\Xi_1 + \Xi_2 P_e) \quad \tfrac{1}{\sqrt{2}}(\Xi_1 - \Xi_2 P_e) \right], \quad (10)$$

where

$$P_e = \begin{bmatrix} P_1 & \cdots & (-1)^{N-1}P_N \\ \vdots & \ddots & 0 \\ (-1)^{N-1}P_N & 0 & 0 \end{bmatrix} \quad (11)$$

and Ξ_1 and Ξ_2 in $\mathbb{R}^{k \times k}$ satisfy

$$\Xi_2^\top \Xi_1 + \Xi_1^\top \Xi_2 = 0, \text{ and } \Xi_2^\top \Xi_2 + \Xi_1^\top \Xi_1 = I \quad (12)$$

The energy balance associated the system reads

$$\frac{dH}{dt} = \int_a^b y_d^T u_d d\zeta - \int_a^b \left(x_2^T(\zeta,t)\mathcal{L}_2^T(\zeta)R\mathcal{L}_2(\zeta)x_2(\zeta,t) \right) d\zeta + y_\partial^T u_\partial \quad (13)$$

$$\leq \int_a^b y_d^T u_d d\zeta + y_\partial^T u_\partial \quad (14)$$

The existence of solutions of (5-7) under the conditions (9-10) and (12) has been investigated in [5, 15, 16] as stated in Theorem 1.

Theorem 1 ([15]) *The operator*

$$\mathcal{J} = \sum_{i=0}^N P_i \frac{\partial^i}{\partial\zeta^i}(\mathcal{L}(\zeta)x(\zeta,t)) - R_0\mathcal{L}(\zeta)x(\zeta,t)$$

with domain

$$D(\mathcal{J}) = \left\{ \mathcal{L} \in H^N\,(a,b;\mathbb{R}^n) \mid \begin{bmatrix} \mathcal{L}(b)x(b,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(b,t) \\ \mathcal{L}(a)x(a,t) \\ \vdots \\ \frac{\partial^{N-1}(\mathcal{L}x)}{\partial\zeta^{N-1}}(a,t) \end{bmatrix} \in Ker W_B \right\}$$

where W_B is defined by (9) and Ξ_1 and Ξ_2 satisfy (12), generates a contraction semigroup on X. Furthermore the system (5-7) with (9-10) and (12) defines a boundary control system.

The general formulation (1) allows to model a large class of systems. For example:

- The 1D wave equation where $n = 1$, $N = 1$, $G_0 = 0$, $G_1 = 1$.
- The Euler Bernouilli beam equation. In this case $n = 1$, $N = 2$, $G_0 = 0$, $G_1 = 0$, $G_2 = 1$.
- The Timoshenko beam equation. In this case $n = 2$, $N = 1$, and

$$G_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 1 (The Timohenko beam) The Timoshenko beam model describes the behavior of a thick beam in a one dimensional spatial domain. We choose as state variables $x = (x_1, x_2, x_3, x_4)^\top$, where $x_1(\zeta,t) = w_\zeta(\zeta,t) - \phi(\zeta,t)$ is the shear strain, $x_2(\zeta,t) = \phi_\zeta(\zeta,t)$ the angular strain, $x_3(\zeta,t) = \rho(\zeta)w_t(\zeta,t)$ the transverse momentum distribution, and $x_4(\zeta,t) = I_\rho(\zeta)\phi_t(\zeta,t)$ the angular momentum distribution where $w(\zeta,t)$ and $\phi(\zeta,t)$ are respectively the transverse displacement of the beam and the rotation angle of the neutral fiber of the beam [1]. The balance equations on these state variables leads to the dynamic model

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \\ x_3(\zeta,t) \\ x_4(\zeta,t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial\zeta} & -1 \\ 0 & 0 & 0 & \frac{\partial}{\partial\zeta} \\ \frac{\partial}{\partial\zeta} & 0 & 0 & 0 \\ 1 & \frac{\partial}{\partial\zeta} & 0 & 0 \end{bmatrix}\begin{bmatrix} T(\zeta)x_1(\zeta,t) \\ EI(\zeta)x_2(\zeta,t) \\ \rho(\zeta)^{-1}x_3(\zeta,t) \\ I_\rho(\zeta)^{-1}x_4(\zeta,t) \end{bmatrix} \tag{15}$$

where $T(\zeta)$ is the shear modulus, $\rho(\zeta)$ is the mass per unit length, $EI(\zeta)$ is the Youngs modulus of elasticity E multiplied by the moment of inertia of a cross

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[1] Note that we have used the lower indexes ζ and t to refer to the partial derivative with respect to that index.

section I, and $I_\rho(\zeta)$ is the rotational momentum of inertia of a cross section. Note that, $T(\zeta)x_1(\zeta,t)$ is the shear force, $EI(\zeta)x_2(\zeta,t)$ the torque, $\rho(\zeta)^{-1}x_3(\zeta,t)$ the longitudinal velocity, and $I_\rho(\zeta)^{-1}x_4(\zeta,t)$ the angular velocity. This formulation is equivalent to (1) with $N = 1, n = 2$

$$
G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \mathcal{L}(\zeta) = \begin{bmatrix} T(\zeta) & 0 & 0 & 0 \\ 0 & EI(\zeta) & 0 & 0 \\ 0 & 0 & \rho(\zeta)^{-1} & 0 \\ 0 & 0 & 0 & I_\rho(\zeta)^{-1} \end{bmatrix}.
$$

Using $\Xi_1 = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$ and $\Xi_2 = \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix}$ satisfying the condition (12) we obtain the following inputs and outputs

$$
u_\partial(t) = \begin{pmatrix} \rho(a)^{-1}x_2(a,t) \\ I_\rho(a)^{-1}x_4(a,t) \\ T(b)x_1(b,t) \\ EI(b)x_3(b,t) \end{pmatrix}, \quad y_\partial(t) = \begin{pmatrix} -T(a)x_1(a,t) \\ -EI(a)x_3(a,t) \\ \rho(b)^{-1}x_2(b,t) \\ I_\rho(b)^{-1}x_4(b,t) \end{pmatrix} \tag{16}
$$

and $u_d = 0$. The energy balance is given by $\dot{H}(t) = u_\partial(t)^\top y_\partial(t)$. The reader is refered to [17] for more details on the model, to [6, 16] for the well-posedness of this class of systems and to [18] for stability analysis.

3 Control by interconnection and energy shaping

In what follows we consider control by interconnection [14], meaning that the system is connected to a dynamic controller in a power preserving way, as shown in Figure 1.
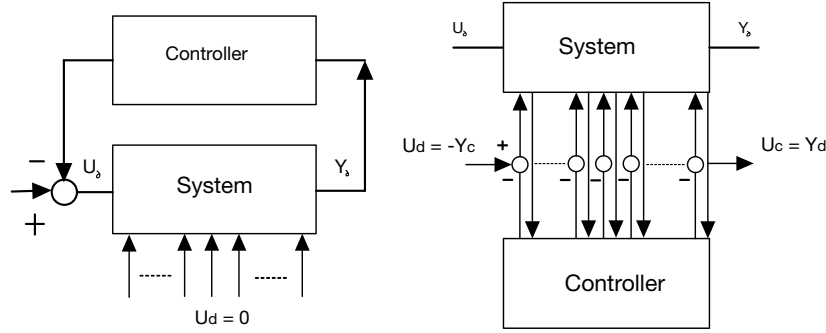
Fig. 1 Control by interconnection. Boundary control (left), in domain control (right).

If the input and output of the controller are denoted u_c and y_c respectively, the interconnection reads $u = u' - y_c$ and $u_c = y$ where u' is a reference signal. This controller can be very simple and may reduce to a simple positive gain in order to dissipate energy. In this case the total energy function can be used as a Lyapunov function and the controller can only be used for stabilization purposes with limited performances. Damping injection only allows to stabilize the system with the main drawback of slowing it down. This is the reason why in general the controller is searched for as a dynamic system of state x_c and energy $H(x_c)$. The nature and dimensions of x_c depends on the nature of the control and will be made precise later on. From the power preserving interconnection the closed loop system is the sum of the energy of the open loop system and the one of the controller

$$H_{cl}(x, x_c) = H(x) + H_c(x_c)$$

Energy shaping control design consists in modifying the closed loop energy function by choosing appropriately the parameters of the controller. For that purpose, we first look for structural invariants $C(x, x_c)$, *i.e.* functions of the system and controller's state variables that do not vary along the system trajectories, $\dot{C} = 0$. They are looked for under the form

$$C(x, x_c) = x_c + F(x) = \kappa$$

where F is a smooth function. If these functions exist, it is possible to choose $\kappa = 0$ by an appropriate choice of the initial conditions. In this case the closed loop energy function reads

$$H_{cl}(x, x_c) = H_{cl}(x) = H(x) + H_c(-F(x))$$

and the shape of the energy function can be modified by choosing appropriately the function $H_c(.)$. At the end asymptotic stability of the closed loop system in x^* is achieved using damping injection such that

$$\frac{dH_{cl}}{dt} < 0, \forall x \neq x^*.$$

In this chapter two different cases are considered, when the control is applied at the boundary of the spatial domain (boundary control) and when the control is applied within the spatial domain through a network of actuators and sensors. For a sake of simplicity we restrict the considered studies to the case $N = 1$, *i.e.*

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{L}(\zeta)x(\zeta, t)) + (P_0 - R_0)\,\mathcal{L}(\zeta)x(\zeta, t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_d(\zeta, t) \quad (17)$$

$$y_d(\zeta, t) = \begin{bmatrix} 0 & I \end{bmatrix} \mathcal{L}(\zeta)x(\zeta, t) \quad (18)$$

$$u_\partial = \mathcal{B}\left(\mathcal{L}(\zeta)x(\zeta, t)\right), y_\partial = C\left(\mathcal{L}(\zeta)x(\zeta, t)\right) \quad (19)$$

4 Boundary control design

We first consider boundary control design, *i.e.* $u_d = 0$. The system is connected to the controller at the boundaries of the spatial domain in the following way

$$\begin{pmatrix} u_\partial \\ y_\partial \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C \\ y_C \end{pmatrix} + \begin{pmatrix} u' \\ 0 \end{pmatrix}. \quad (20)$$

We consider the controller as a passive linear control system in port-Hamiltonian form [19]

$$\begin{cases} \dot{x}_C = (J_C - R_C)\,Q_C x_C + (G_C - P_C)\,u_C \\ y_C = (G_C + P_C)^T\,Q_C x_C + (M_C + S_C)\,u_C \end{cases} \quad (21)$$

where $x_C \in \mathbb{R}^{n_C}$ and $u_C,\ y_C \in \mathbb{R}^n$, while $J_C = -J_C^T$, $M_C = -M_C^T$, $R_C = R_C^T$, $Q_C = Q_C^T > 0$ and $S_C = S_C^T$. The passivity is guaranteed as soon as

$$\begin{pmatrix} R_C & P_C \\ P_C^T & S_C \end{pmatrix} \geq 0. \quad (22)$$

For the sake of compactness, this system can be easily written in standard (A_C, B_C, C_C, D_C) form, being

$$\begin{aligned} A_C &= (J_C - R_C)\,Q_C & B_C &= G_C - P_C \\ C_C &= (G_C + P_C)^T\,Q_C & D_C &= M_C + S_C. \end{aligned} \quad (23)$$

The closed loop energy function reads

$$H_{cl}(x(t), x_C(t)) = \frac{1}{2}\,\|x(t)\|_{\mathcal{L}}^2 + \underbrace{\frac{1}{2}x_C^T(t)Q_C x_C(t)}_{=:H_C(x_C(t))} \quad (24)$$

and the closed loop system can be written

$$\begin{cases} \dot{z} = \mathcal{J}_{cl}z \\ u' = \left(\mathcal{B} + D_C C \; C_C\right) z =: \mathcal{B}'z, \end{cases} \tag{25}$$

where

$$z = \begin{pmatrix} x \\ x_C \end{pmatrix} \in Z := X \times \mathbb{R}^{n_C}$$

is the state variable of the closed loop system and $\mathcal{J}_{cl} : D(\mathcal{J}_{cl}) \subset Z \to Z$ is the following linear operator

$$\mathcal{J}_{cl}z := \begin{pmatrix} \mathcal{J} & 0 \\ B_C C & A_C \end{pmatrix} \begin{pmatrix} x \\ x_C \end{pmatrix} \tag{26}$$

with domain

$$D(\mathcal{J}_{cl}) = D(\mathcal{J}) \times \mathbb{R}^{n_C}. \tag{27}$$

Z is endowed with the inner product defined as $\langle z_1, z_2 \rangle_Z = \langle x_1, x_2 \rangle_{\mathcal{L}} + x_{C,1}^T Q_C x_{C,2}$ which means that $H_{cl}(z) = \frac{1}{2} \|z\|_Z^2$.

Proposition 1 ([20])

Consider the port-Hamiltonian system resulting from the power-conserving interconnection (20) of (17-19) and (21), which results in (25). Then, (25) with $\mathcal{J}_{cl}$ defined in (26) with domain (27) is a boundary control system. Moreover, the operator $\bar{\mathcal{J}}_{cl}$ given by

$$\bar{\mathcal{J}}_{cl}z := \begin{pmatrix} \mathcal{J} & 0 \\ B_C C & A_C \end{pmatrix} \begin{pmatrix} x \\ x_C \end{pmatrix}$$

with domain

$$D(\bar{\mathcal{J}}_{cl}) = \left\{ \begin{pmatrix} x \\ x_C \end{pmatrix} \in Z \mid x \in D(\mathcal{J}), \; and \; \mathcal{B}' \begin{pmatrix} x \\ x_C \end{pmatrix} = 0 \right\}$$

with $\mathcal{B}'$ defined in (25) generates a contraction semigroup.

Definition 2 (Casimir functions) Consider the boundary control system defined in Proposition 1 with $u' = 0$ in (20). A function $C : X \times \mathbb{R}^{n_C} \to \mathbb{R}$ is a Casimir function if $\dot{C} = 0$ along the (classical) solutions for every possible choice of $\mathcal{L}(\cdot)$ and Q_C, [21, 22, 23].

As stated in Section 3 we look for Casimir functions in the form

$$C(x(t), x_C(t)) = \Gamma^T x_C(t) + \int_a^b \Psi^T(z)x(t, z)\, \mathrm{d}z \tag{28}$$

with $\Gamma \in \mathbb{R}^{n_C}$ and $\Psi \in L^2(a, b; \mathbb{R}^n)$. Note that they are not (yet) in the form assumed above.

Proposition 2 ([20]) *Consider the boundary control system introduced in Proposition 1 with $u' = 0$ in (20). Then, (28) is a Casimir function for this system if and only*

if $\Psi \in H^1(a, b; \mathbb{R}^n)$,

$$P_1 \frac{d\Psi}{dz}(z) + (P_0 + R_0)\Psi(z) = 0 \tag{29}$$

$$(J_C + R_C)\Gamma + (G_C + P_C)\tilde{W}R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0 \tag{30}$$

$$(G_C - P_C)^T \Gamma + \left[W + (M_C - S_C)\tilde{W} \right] R \begin{pmatrix} \Psi(b) \\ \Psi(a) \end{pmatrix} = 0 \tag{31}$$

We now suppose there exist n_c independent Casimir functions. In this case the state of the controller is related to the state of the system through the relation

$$x_C(t) = -\hat{\Gamma}^{-T} \int_a^b \hat{\Psi}^T(z)x(t, z)\, dz + \kappa \tag{32}$$

It is then possible to verify that the closed loop dynamics is given by the boundary control system:

$$\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial}{\partial z} \frac{\delta H_{cl}}{\delta x}(x(t))(z) + (P_0 - G_0)\frac{\delta H_{cl}}{\delta x}(x(t))(z)$$

$$u'(t) = W'R \begin{pmatrix} \left(\frac{\delta H_{cl}}{\delta x}(x(t)) \right)(b) \\ \left(\frac{\delta H_{cl}}{\delta x}(x(t)) \right)(a) \end{pmatrix} \tag{33}$$

where δ denotes the Fréchet derivative and the closed loop energy function is given by:

$$H_{cl}(x(t)) = \frac{1}{2} \|x(t)\|_{\mathcal{L}}^2 + \frac{1}{2} \left(\int_a^b \hat{\Psi}^T(z)x(t, z)\, dz \right)^T \times$$

$$\times \hat{\Gamma}^{-1} Q_C \hat{\Gamma}^{-T} \int_a^b \hat{\Psi}^T(z)x(t, z)\, dz \tag{34}$$

and W' is a $n \times 2n$ full rank, real matrix such that $W'\Sigma W'^T \geq 0$.

The main drawback of such approach is that existence of Casimir function is subject to the fact that there is no dissipation in the considered coordinate. This is known as the dissipation obstacle. It is then not possible to use the dynamic extension and its reduction to shape the closed loop energy function. Yet, as stated in Proposition 3 it is possible to directly shape the closed loop energy by using a state feedback having a similar form than the one obtained using the immersion/reduction approach.

Proposition 3 (Energy-shaping [9]) *Consider the system (17) with boundary control given by (19). Denote by $H(x) = \frac{1}{2} \|x\|_{\mathcal{L}}^2$ its Hamiltonian function. Then, the feedback law $u = \beta(x) + u'$, with u' an auxiliary boundary input, maps (17), (19) into the target dynamical system*

$$\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial}{\partial z} \frac{\delta H_d}{\delta x}(x(t))(z) + (P_0 - G_0)\frac{\delta H_d}{\delta x}(x(t))(z)$$

$$u'(t) = W_B \left(\begin{pmatrix} \left(\frac{\delta H_d}{\delta x}(x(t)) \right)(b) \\ \left(\frac{\delta H_d}{\delta x}(x(t)) \right)(a) \end{pmatrix} \right) \tag{35}$$

with $H_d(x) = H(x) + H_a(x)$, provided that

$$P_1 \frac{\partial}{\partial z} \frac{\delta H_a}{\delta x}(x) + (P_0 - G_0)\frac{\delta H_a}{\delta x}(x) = 0 \tag{36}$$

$$\beta(x) + W_B \left(\begin{pmatrix} \left(\frac{\delta H_a}{\delta x}(x) \right)(b) \\ \left(\frac{\delta H_a}{\delta x}(x) \right)(a) \end{pmatrix} \right) = 0. \tag{37}$$

Once H_d is defined a natural choice for the output is

$$y'(t) = W_C \left(\begin{pmatrix} \left(\frac{\delta H_d}{\delta x}(x(t)) \right)(b) \\ \left(\frac{\delta H_d}{\delta x}(x(t)) \right)(a) \end{pmatrix} \right) \tag{38}$$

which implies that $\frac{d}{dt} H_d(x(t)) \le y'^T(t)u'(t)$. In order to get asymptotic stability one has to consider damping injection:

$$u'(t) = -\Xi y'(t), \qquad \Xi = \Xi^T \ge 0. \tag{39}$$

The additional Hamiltonian H_a is constructed in such a way that $\mathcal{L}^{-1}\frac{\delta H_a}{\delta x}(x)$ are equilibrium states of (17). Furthermore, since the system has to reach a non-zero state, H_d is chosen with a global minimum in this non-zero state. In the following lemma, a construction for H_a which achieves this is illustrated. Since, in this paper, the linear case is treated, the focus is on quadratic Hamiltonian functions.

Lemma 1 ([9]) *Let $\Phi_i \in H^1(a, b; \mathbb{R}^n)$, $i = 1, \ldots, n$ be independent solutions of*

$$P_1 \frac{d\Phi_i}{d\zeta}(\zeta) + (P_0 - G_0)\Phi_i(\zeta) = 0, \tag{40}$$

and define $\hat{\Phi}(\zeta) = \big(\Phi_1(\zeta), \ldots, \Phi_n(\zeta)\big)$. Furthermore, let $x_\star$ be an equilibrium state of (17), i.e. $\mathcal{L}x_\star \in H^1(a, b; \mathbb{R}^n)$ and

$$P_1 \frac{\partial(\mathcal{L}x_\star)}{\partial \zeta}(\zeta) + (P_0 - G_0)(\mathcal{L}x_\star)(\zeta) = 0. \tag{41}$$

Then

$$H_a(x) = \frac{1}{2} \left[\int_a^b \hat{\Phi}^T (x - x_\star) \, d\zeta \right]^T \times Q_a \left[\int_a^b \hat{\Phi}^T (x - x_\star) \, d\zeta \right] -$$

$$- \int_a^b x_\star^T \mathcal{L} x \, d\zeta + \kappa, \quad (42)$$

with $Q_a = Q_a^T > 0$ and $\kappa \in \mathbb{R}$ some constant, satisfies (36) and $H_d = H + H_a$ has a global minimum in $x_\star$.

Define the bounded linear operator $K_\Phi : X \to \mathbb{R}^n$ as

$$K_\Phi x = \int_a^b \hat{\Phi}^T (\zeta) x(\zeta) \, d\zeta, \quad (43)$$

and $\mathcal{L}_d$ as

$$\mathcal{L}_d = \mathcal{L} + K_\Phi^* Q_a K_\Phi, \quad (44)$$

where $K_\Phi^* : \mathbb{R}^n \to X$ is the adjoint operator of K_Φ. Clearly, $K_\Phi^* = \hat{\Phi}$, and $\mathcal{L}_d$ is a bounded, coercive operator on $L^2(a, b; \mathbb{R}^n)$. Furthermore, H_a being given by (42) we find

$$H_d(x) = \frac{1}{2} \langle (x - x_\star) \mid \mathcal{L}_d (x - x_\star) \rangle_{L^2} + H_d(x_\star). \quad (45)$$

In order to prove the asymptotic stability (Proposition 2) one has first to prove existence and uniqueness of solutions as well as precompactness of trajectories (Propositions 4 and 5) in order to use Lassalle's invariance principle

Proposition 4 ([9]) *The closed loop system (35), (38) with (39) in which H_a is defined by (42) admits a unique solution. Furthermore, the mapping from the initial error state at time $t = 0$, $x_0 - x_\star$ to the error state at time t, $x(t) - x_\star$ defines a contraction semigroup in the norm $\frac{1}{2} \langle (x - x_\star) \mid \mathcal{L}_d (x - x_\star) \rangle_{L^2}$.*

Proposition 5 *The operator $\mathcal{J}_d$ defined as*

$$\mathcal{J}_d x := P_1 \frac{\partial (\mathcal{L}_d x)}{\partial \zeta} + (P_0 - G_0)(\mathcal{L}_d x) \quad (46)$$

with domain

$$D(\mathcal{J}_d) = \left\{ x \in L^2(a, b; \mathbb{C}^n) \mid \mathcal{L}_d x \in H^1(a, b; \mathbb{C}^n) \right.$$

$$\left. \text{and } 0 = [W_B + \Xi W_C] \begin{pmatrix} (\mathcal{L}_d x)(b) \\ (\mathcal{L}_d x)(a) \end{pmatrix} \right\} \quad (47)$$

is the infinitesimal generator of a contraction semigroup and has a compact resolvent.

Theorem 2 (Asymptotic stability [9]) *Consider the linear, infinite dimensional, port-Hamiltonian system (17) and the equilibrium state $x_\star$ satisfying (41). Then, the control action $u = \beta(x) + u'$ with β defined in (37), H_a chosen as in (42), and with u' defined in (39) with $\Xi > 0$, makes $x_\star$ asymptotically stable.*

It has been shown for the vibrating string example [9] that the energy shaping control plus damping injection using closed loop invariants or integral feedback forms is equivalent to a proportional derivative control acting at the boundary. The overall stability is achieved while the performances at the end point of the string are assignable. The main advantage of using this control technique lies in its physical interpretation and its potential extension to the case of a PDE connected to a non linear ODE.

5 In domain distributed control

In this section the distributed parameter system is controlled within its spatial domain meaning that $u_\partial = 0$

$$\begin{pmatrix} u_d(\zeta,t) \\ y_d(\zeta,t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} u_C(\zeta,t) \\ y_C(\zeta,t) \end{pmatrix} + \begin{pmatrix} u'(\zeta,t) \\ 0 \end{pmatrix}, \tag{48}$$
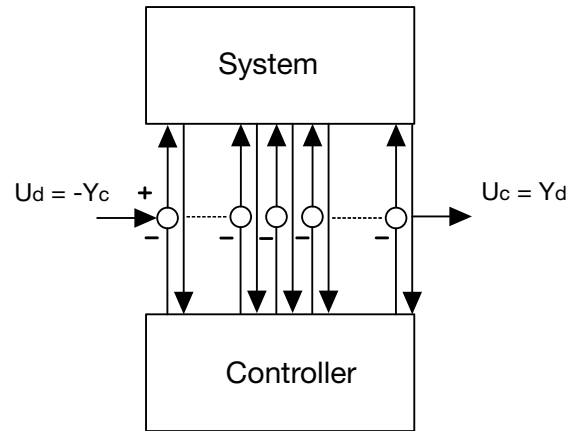
as depicted in Figure 2.

Fig. 2 Distributed control.

We consider two cases. The ideal case where the control is infinite dimensional and uniformly distributed along the spatial domain. In this case we show how the dynamic controller parameters and structural invariants can be used to perfectly shape the closed loop energy function. This control strategy being not realistic from an engineering point of view, the existence of solution and the precompactness of the closed loop trajectories are not further investigated to the benefit to the second case

when the control is achieved through a continuous piecewise network of actuators. In this case an early lumping approach is proposed and the shaping of the closed loop energy function is achieved in an approximate and optimal way. The finite dimensional controller is shown to achieve closed loop asymptotic stability.

5.1 Ideal case

We first consider the ideal case where the distributed parameter system is actuated at any point of its spatial domain. In this case the state of the controller can be distributed in space and the controller has the general formulation

$$
\begin{cases}
\dfrac{\partial x_C}{\partial t}(\zeta,t) = \mathcal{J}_c \mathcal{Q}_c x_C(\zeta,t) + \mathcal{B}_c u_C(\zeta,t) \\
y_C(\zeta,t) = -\mathcal{B}_c^* \mathcal{Q}_c x_C(\zeta,t) + \mathcal{S}_c u_C(\zeta,t)
\end{cases}
\tag{49}
$$

where $\mathcal{Q}_c$ is a bounded and Lipschitz continuous matrix-valued function such that $Q_c(\zeta) = Q_c^T(\zeta)$ and $Q_c(\zeta) \geq \eta_c$ with $\eta_c > 0$ for all $\zeta \in [a,b]$, $\mathcal{S}_c$ is a bounded and Lipschitz continuous function such that $\mathcal{S}_c(\zeta) = \mathcal{S}_c^T(\zeta)$ and $\mathcal{S}_c(\zeta) \geq \eta_s$ with $\eta_s > 0$ for all $\zeta \in [a,b]$. $\mathcal{B}_c$ and $\mathcal{J}_c$ are differential operators of the form:

$$
\mathcal{B}_c = B_{c0} + B_{c1}\frac{\partial}{\partial \zeta}
\tag{50}
$$

$$
\mathcal{J}_c = J_{c0} + J_{c1}\frac{\partial}{\partial \zeta}
\tag{51}
$$

with $B_{c0}, B_{c1} \in \mathbb{R}^{(n_c,1)}, J_{c0} = -J_{c0}^T, J_{c1} = J_{c1}^T \in \mathbb{R}^{(n_c,n_c)}$. The closed loop system corresponding to (17) interconnected to (2) through (48) is given by:

$$
\frac{\partial x_e}{\partial t} := \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_c}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{G} & 0 \\ -\mathcal{G}^* & -(\mathcal{S}_c + R) & \mathcal{B}_c^* \\ 0 & \mathcal{B}_c & \mathcal{J}_c \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 x_1 \\ \mathcal{L}_2 x_2 \\ \mathcal{Q} x_c \end{pmatrix}
\tag{52}
$$

Proposition 6 *The closed loop system* (52) *admits structural invariants of the form*

$$
\kappa_0 = C(x_e) = \int_a^b \Psi^T x_e \, d\zeta
\tag{53}
$$

with $\Psi = (\psi_1, \psi_2, \psi_3)$ *if and only if*

$$-\mathcal{G}\psi_2(\zeta) = 0 = \mathcal{B}_c\psi_2(\zeta) + \mathcal{J}_c^*\psi_3(\zeta) \tag{54}$$

$$(\mathcal{S} + R)\,\psi_2(\zeta) = 0 \tag{55}$$

$$\mathcal{G}\psi_1(\zeta) + \mathcal{B}_c^*\psi_3(\zeta) = 0 \tag{56}$$

$$\begin{pmatrix} 0 & G_1 & 0 \\ -G_1^T & 0 & -B_{c1} \\ 0 & B_{c1}^T & J_{c1} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \\ \psi_3(\zeta) \end{pmatrix}\Bigg|_{a,b} = 0 \tag{57}$$

Proposition 7 ([20]) *Choosing $\mathcal{B}_c = \mathcal{G}$ the closed loop system* (52) *admits as structural invariants the function $C(x_e)$ defined by* (53) *and*

$$\Psi = (\Psi_1, 0, \Psi_1)$$

In this case the hyperbolic system (1) *connected to the dynamic controller* (63) *of the form*

$$\begin{cases} \dfrac{\partial x_C}{\partial t}(\zeta, t) = \mathcal{G}u_C(\zeta, t) \\ y_C(\zeta, t) = -\mathcal{G}^*\mathcal{Q}_c x_C(\zeta, t) + \mathcal{S}_c u_C(\zeta, t) \end{cases} \tag{58}$$

is equivalent to the system

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & -(R + \mathcal{S}) \end{bmatrix} \begin{bmatrix} (\mathcal{L}_1(\zeta) + \mathcal{Q}_c(\zeta))\,x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix} \tag{59}$$

$$u_\partial = \mathcal{B} \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix}, \quad y_\partial = C \begin{bmatrix} \mathcal{L}_1(\zeta)x_1(\zeta, t) \\ \mathcal{L}_2(\zeta)x_2(\zeta, t) \end{bmatrix} \tag{60}$$

One can see that using an ideal distributed feedback the equivalent stiffness and the overall damping coefficient of the closed loop system can be modified using $\mathcal{Q}_c$ and $\mathcal{S}$ respectively. The former allows to modify the closed loop performances of the system while the latter guarantees its stability. In this respect existence and precompactness of solutions as well as closed loop asymptotic stability can be proven using passivity arguments. The proof is omitted here and will be proposed in the more realistic case when the control is implemented using a network of actuators.

5.2 Under actuated case

We consider now that the distributed parameter system is actuated through piecewise constant elements (cf. Fig. 3).

The controller is in this case finite dimensional and the control design is implemented using an early lumping approach. The first step is to discretize the system (1) using a structure preserving discretization method. We apply the mixed finite element method [24] such that the approximation of (1) is again a PHS with n elements:
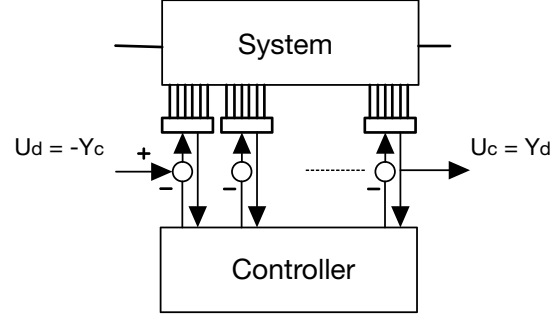
Fig. 3 In domain control using piecewise constant elements.

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = (J_n - R_n) \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + B_b u_b + \begin{pmatrix} 0 \\ B_{0d} M \end{pmatrix} \mathbf{u}_d, \tag{61a}$$

$$y_b = B_b^T \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix} + D_b u_b, \tag{61b}$$

$$\mathbf{y}_d = \begin{pmatrix} 0 & M^T B_{0d}^T \end{pmatrix} \begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{61c}$$

where $x_{id} = \begin{pmatrix} x_i^1 & \cdots & x_i^n \end{pmatrix}^T \in \mathbb{R}^{np \times 1}$ for $i \in \{1, \cdots, 2p\}$,

$$J_n = \begin{pmatrix} 0 & J_i \\ -J_i^T & 0 \end{pmatrix} \qquad \text{and} \qquad R_n = \begin{pmatrix} 0 & 0 \\ 0 & R_d \end{pmatrix},$$

are the discretized matrices of the operators $\mathcal{J}$ and $\mathcal{R}$ with J_i and R_d the discretized matrices of the operators $\mathcal{G}$ and R. $B_{0d} \in \mathbb{R}^{np \times n}$, $Q_1 \in \mathbb{R}^{np \times np}$ and $Q_2 \in \mathbb{R}^{np \times np}$ are the discretized matrices of B_0, $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The input u_b denotes the boundary input which corresponds to the boundary actuation or/and conditions. Since the distributed actuation of the system is considered, we assume that there is no energy changes (actuation) at the boundary of the spatial domain, *i.e.* $u_b = 0$ and the discretized system (61) can therefore be simplified. The Hamiltonian of the discretized model (61) writes:

$$H_d(x_{1d}, x_{2d}) = \frac{1}{2} \left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} \right). \tag{62}$$

Remark 1 In what follows the patches are supposed to cover the entire spatial domain of the system in an homogeneous way. Yet the network of patched may present discontinuities. In this case the high frequency modes are not controlled by the actuators but have a guaranteed stability due to the internal dissipation of the open loop system.

5.2.1 Control design

The controller in Fig. 3 is designed to be a finite dimensional PHS, which is expressed as follows:

$$\dot{x}_c = (J_c - R_c) Q_c x_c + B_c u_c,$$
$$y_c = B_c^T Q_c x_c + D_c u_c, \tag{63}$$

where $x_c \in \mathbb{R}^{m \times 1}$, $J_c = -J_c^T \in \mathbb{R}^{m \times m}$, $R_c = R_c^T \geq 0$ and $Q_c = Q_c^T \geq 0$, $B_c \in \mathbb{R}^{m \times m}$, $\mathbb{R}^{m \times m} \ni D_c > 0$, $u_c \in \mathbb{R}^{m \times 1}$ and $y_c \in \mathbb{R}^{m \times 1}$. Matrices Q_c and D_c work as the energy shaping and the damping injection or the diffusion operator, respectively. Without considering external signals, the interconnection between the discretized plant system (61) and the controller (63) is:

$$\begin{pmatrix} u_d \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} y_d \\ y_c \end{pmatrix}, \quad \text{where } M = \mathbb{I}_m \otimes \mathbf{1}_{k \times 1} \in \mathbb{R}^{n \times m}, \tag{64}$$

as illustrated in Fig. 3, k being the number of elements sharing the same input. [2] The passive interconnection (64) keeps the passivity of the closed loop. It results in a new PHS in closed loop:

$$\dot{x}_{cl} = (J_{cl} - R_{cl}) Q_{cl} x_{cl}, \tag{65}$$

where $x_{cl} = \left(x_{1d}^T, x_{2d}^T, x_c^T \right)^T$, $Q_{cl} = \text{diag} \left(Q_1, Q_2, Q_c \right)$,

$$J_{cl} = \begin{pmatrix} O & J_i & 0 \\ -J_i^T & 0 & -B_{0d} M B_c^T \\ 0 & B_c M^T B_{0d}^T & J_c \end{pmatrix}, \quad R_{cl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_d + B_{0d} M D_c M^T B_{0d}^T & 0 \\ 0 & 0 & R_c \end{pmatrix}.$$

The Hamiltonian of the controller (63) is:

$$H_c(x_c) = \frac{1}{2} x_c^T Q_c x_c. \tag{66}$$

Therefore, the closed loop Hamiltonian function reads:

$$H_{cld}(x_{1d}, x_{2d}, x_c) = H_d(x_{1d}, x_{2d}) + H_c(x_c). \tag{67}$$

The next step is to design controller matrices J_c, R_c, B_c, Q_c, and D_c in order to shape the closed loop Hamiltonian (67). We first give the following Proposition 8 that links the state variables of the discretized plant (61) and that of the controller (63) through a Casimir function.

Proposition 8 ([25]) *Choosing* $J_c = 0$, *and* $R_c = 0$, *the closed loop system* (65) *admits the Casimir function* $C(x_{1d}, x_c)$ *defined by:*

$$C(x_{1d}, x_c) = B_c M^T B_{0d}^T J_i^{-1} x_{1d} - x_c \tag{68}$$

---

[2] $\otimes$ is the Kronecker product and $\mathbf{1}_{k \times 1}$ the vector of dimension k containing only ones.

as structural invariant, i.e. $\dot{C}(x_{1d}, x_c) = 0$ *along the closed loop trajectories. If the initial conditions of* $x_{1d}(0)$ *and* $x_c(0)$ *satisfy* $C(x_{1d}(0), x_c(0)) = 0$, *the controller is a proportional-integral control, and the control law* (64) *is equivalent to the state feedback:*

$$\mathbf{u}_d = -B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} x_{1d} - D_c M^T B_{0d}^T Q_2 x_{2d}. \tag{69}$$

Therefore, the closed loop system yields:

$$\begin{pmatrix} \dot{x}_{1d} \\ \dot{x}_{2d} \end{pmatrix} = \begin{pmatrix} 0 & J_i \\ -J_i^T & -\left(R_d + B_{0d} M D_c M^T B_{0d}^T\right) \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 x_{1d} \\ Q_2 x_{2d} \end{pmatrix}, \tag{70}$$

where

$$\tilde{Q}_1 = Q_1 + Q_c, \ with \ Q_c = J_i^{-T} B_{0d} M B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} \tag{71}$$

is the new closed loop energy matrix associated to x_{1d}.

From a physical point of view, (68) implies that with the dynamic controller (63) equivalent to the state feedback (69), it is possible to change, at least partially (depending on p and the range of B_{0d}), the energy matrix related to x_{1d}. Actually, one can only shape the energy matrix related to the first n elements of x_{1d}, i.e. $(\tilde{Q}_1)_{n \times n}$. For a given number of distributed inputs m, the objectives of the energy shaping is to find matrices B_c and Q_c such that the *distance* (considered here in the Frobenius norm, see Definition 6.4 of [26]) between the desired energy matrix $\tilde{Q}_{1d}$ and the closed loop one $\tilde{Q}_1$ is minimal:

$$\min_{B_c^T Q_c B_c} \left\| J_i^{-T} B_{0d} M B_c^T Q_c B_c M^T B_{0d}^T J_i^{-1} + Q_1 - \tilde{Q}_{1d} \right\|_F. \tag{72}$$

If we consider p and eliminate B_{0d}, (72) is equivalent to:

$$\min_{B_c^T Q_c B_c} \left\| (J_i)_{n \times n}^{-T} M B_c^T Q_c B_c M^T (J_i)_{n \times n}^{-1} - Q_m \right\|_F, \tag{73}$$

where $(J_i)_{n \times n}$ is the first n lines n columns of J_i and $Q_m = \left(\tilde{Q}_{1d} - Q_1\right)_{n \times n} \geq 0$. Furthermore, (73) can be formalized by the optimization Problem 1.

Problem 1 The energy related to first n elements of x_{1d} in closed loop is shaped in an optimal way if and only if $X = B_c^T Q_c B_c \in SR_0^{m \times m}$ minimizes the criterion

$$f(X) = \left\| AXA^T - Q_m \right\|_F, \tag{74}$$

where $A = (J_i)_{n \times n}^{-T} M \in \mathbb{R}^{n \times m}$ and $SR_0^{m \times m}$ represents the set of symmetric and positive semi-definite matrices.

The solution of Problem 1 depends on the number of actuators. The system can be fully actuated ($m = n$) or under-actuated ($m < n$). In what follows we consider the realistic under-actuated case. In this case A in (74) is not invertible and one has to consider the numerical optimization of $f(X)$ in Problem 1. The solution of the optimization Problem 1 is given in Proposition 9.

Proposition 9 $f(X)$ *is convex and the minimization of $f(X)$ is equivalent to the minimization of $f^2(X)$, which has a unique minimum given by*

$$\hat{X} = V\Sigma_0^{-1}U_1^T Q_m U_1 \Sigma_0^{-1} V^T \tag{75}$$

with V, Σ_0 and U_1 the matrices of the singular value decomposition (SVD) of the matrix A i.e.

$$A = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_0 \\ 0 \end{pmatrix} V^T, \tag{76}$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are unitary matrices, $U_1 \in \mathbb{R}^{n\times m}$, $U_2 \in \mathbb{R}^{n\times q}$, $q = n - m$, and $\Sigma_0 = \Sigma_0^T \geq 0$ is the diagonal matrix of singular values of A.

The choice of the controller matrices B_c and Q_c is not unique and only has to satisfy the condition (75). It is done in order to modify the shape of the closed loop energy function of the system in the x_1 coordinate. The choice of the controller matrix D_c follows a similar procedure, with the optimization of the difference between the approximate dissipation and the desired one.

5.2.2 Stability analysis

We now consider the closed loop stability analysis when the finite-dimensional controller (63) elaborated on the finite dimensional approximation of the system is connected to the infinite-dimensional system (17). In this case the closed loop system reads:

$$\dot{\mathcal{X}} = \underbrace{\begin{pmatrix} (\mathcal{J} - \mathcal{R} - \mathcal{B}D_c\mathcal{B}^*) & -\mathcal{B}B_c^T \\ B_c\mathcal{B}^* & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & Q_c \end{pmatrix}}_{\mathcal{A}_{cl}} \mathcal{X}, \tag{77}$$

where $\mathcal{X} = \begin{pmatrix} x^T & x_c^T \end{pmatrix}^T \in X_s$ is the state defined on the state space $X_s = L_2\left([0,L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.

Theorem 3 *The operator $\mathcal{A}_{cl}$ defined in (77) generates a contraction semigroup on $X_s = L_2\left([0,L], \mathbb{R}^{2p}\right) \times \mathbb{R}^m$.*

Theorem 4 *The operator $\mathcal{A}_{cl}$ has a compact resolvent.*

Theorem 5 *For any $\mathcal{X}(0) \in L_2\left([0,L], \mathbb{R}^{2n}\right) \times \mathbb{R}^m$, the unique solution of (77) tends to zero asymptotically, and the closed loop system (77) is globally asymptotically stable.*

Example 2 The discretization of the Timoshenko beam with dissipation in Example 1 in n elements with clamped-free boundary conditions leads to the finite dimensional system:

$$\frac{d}{dt}\underbrace{\begin{pmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \\ x_{4d} \end{pmatrix}}_{x_d} = (J_n - R_n)\begin{pmatrix} Q_1 x_{1d} \\ Q_2 x_{2d} \\ Q_3 x_{3d} \\ Q_4 x_{4d} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ M \end{pmatrix}\mathbf{u}_d, \tag{78a}$$

$$\mathbf{y}_d = b_d^T Q_4 x_{4d}, \tag{78b}$$

where $x_{id} = \begin{pmatrix} x_i^1 & \cdots & x_i^n \end{pmatrix}^T$ for $i \in \{1, 2, 3, 4\}$,

$$J_n = \begin{pmatrix} 0 & J_i & 0 & -S^T \\ -J_i^T & 0 & 0 & 0 \\ 0 & 0 & 0 & J_i \\ S & 0 & -J_i^T & 0 \end{pmatrix}, S = \text{diag}\,(L_{ab})_{n\times n},$$

$$J_i = \begin{pmatrix} \frac{1}{\beta} & & & \\ -\frac{1}{\beta^2} & \frac{1}{\beta} & & \\ \vdots & \ddots & \ddots & \\ (-1)^{n-1}\frac{(\beta')^{n-2}}{\beta^n} & \cdots & -\frac{1}{\beta^2} & \frac{1}{\beta} \end{pmatrix}_{n\times n},$$

$$R_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & R_{td} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{rd} \end{pmatrix}, \quad L_{ab} = \frac{L}{n},$$

$Q_1 = \text{diag}\left(\frac{GA_r}{L_{ab}}\right) \in \mathbb{R}^{n\times n}$, $Q_2 = \text{diag}\left(\frac{1}{\rho A_r L_{ab}}\right) \in \mathbb{R}^{n\times n}$, $Q_3 = \text{diag}\left(\frac{EI}{L_{ab}}\right) \in \mathbb{R}^{n\times n}$, $Q_4 = \text{diag}\left(\frac{1}{I\rho L_{ab}}\right) \in \mathbb{R}^{n\times n}$, $R_{td} = \text{diag}\left(R_t L_{ab}\right) \in \mathbb{R}^{n\times n}$, and $R_{rd} = \text{diag}\left(R_r L_{ab}\right) \in \mathbb{R}^{n\times n}$. β denotes the effort mapping parameter and $\beta' = 1 - \beta$ [24]. They are chosen in this case equal to $\frac{1}{2}$ in order to get a centered scheme. The Hamiltonian of the discretized model (78) is given as follows:

$$H_d(x_d) = \frac{1}{2}\left(x_{1d}^T Q_1 x_{1d} + x_{2d}^T Q_2 x_{2d} + x_{3d}^T Q_3 x_{3d} + x_{4d}^T Q_4 x_{4d}\right). \tag{79}$$

It is important to notice that the input matrix M depends on the considered case, whether the system is fully, *i.e. m = n*, or under-actuated, *i.e. m < n*. We consider that the control is achieved using m patches as depicted in Fig.3. The aim of the control design is to modify as far as possible the closed loop Young's modulus $\tilde{E}$ of the beam. $J_m \in \mathbb{R}^{m\times m}$ stems from the discretization of $\partial/\partial\zeta$, B_c is chosen to be $B_c = J_m$. According to (8), $Q_c = J_m^{-T} V \Sigma_0^{-1} U_1^T Q_m U_1 \Sigma_0^{-1} V^T J_m^{-1}$. D_c is chosen such that the time derivative of the Hamiltonian behaves similarly than in the fully-actuated case, *i.e.* in order to satisfy $\min_{D_c \in \mathbb{R}^{m\times m}} \left\| M D_c M^T - \text{diag}\,(\alpha L_{ab}) \right\|_F$, where M is given in (64). This optimization problem is similar to Problem 1, and the optimal D_c is given by $\hat{D}_c = \text{diag}\left(\frac{\alpha L_{ab}}{k}\right)$.

We first consider the case with 10 patches, *i.e.* $m = 10$, $n = 50$ and $k = 5$. The initial conditions are set to a spatial distribution $x_3(\zeta, 0) = 0.1e^{-4\zeta}$ for the angular strain and to zero for other state variables. In this case the angular strain evolution is quite similar to that obtained in the fully-actuated case as depicted in Fig.4(a). This indicates that if the controller matrices B_c, Q_c and D_c are adequately selected, the achievable performances in the under-actuated case can be optimized in order to be close to the ones obtained in the fully-actuated case. When the number of patches is reduced to 5 and to 2, *i.e.* $n = 50$, $k = 10$ and $n = 50$, $k = 25$ respectively, these performances are slightly deteriorated in the high frequencies as shown in Fig. 4(b).
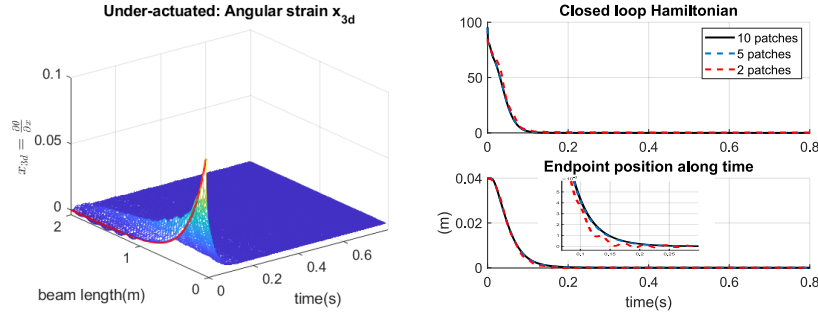

Fig. 4 Closed loop evolution of the angular strain for $m = 10$ (a), Hamiltonian function and endpoint position (b) in the under-actuated case for $m = 10$, $m = 5$ and $m = 2$.

In order to illustrate the effect of the neglected dynamics on the achievable performances we implement the controller designed considering 10 patches on the discretized system where $n = 50$ in a more precise model of the beam derived using $n = 200$. In Fig. 5 we can see that, due to the damping injection and the associated closed loop bandwidth, the neglected dynamics do not impact significantly the closed loop response of the system.
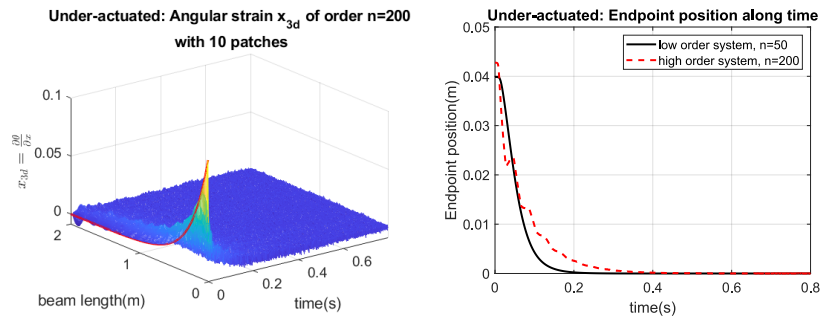

Fig. 5 Closed loop evolution of the angular strain of the high order system (a), and comparison of the endpoint position of the low order and high order systems using the same controller (b).

6 Conclusions, future works

In this chapter we give an overview on energy shaping control design techniques for linear infinite dimensional port Hamiltonian systems defined on a one dimensional spatial domain. This allows to cope with a large class of physical systems with spatially varying parameters such as the wave equation, the Timoshenko beam or the elastic string. Regarding the controller implementation, two cases are considered, when the system is actuated/sensed at the boundaries of its spatial domain and when the system is actuated/sensed through a network of actuators/sensors situated along its spatial domain. In the two cases we show how a dynamic extension and structural invariants can be used to shape, at least partially in the case of boundary control, the closed loop energy function of the system. This means that through the use of a dynamic feedback with proper initialization or of a state feedback one can modify the dynamic properties of the closed loop system, *i.e.* its stiffness from the shaping of its potential energy for example. In the distributed control case an early lumping approach is used and the closed loop properties are modified in an optimal way. The passivity of the controller is used to achieve the asymptotic stability on the infinite dimensional system. This work has been illustrated on the boundary control and in domain control of a clamped-controlled Timoshenko beam. Future works will consist in implementing the dynamic controller using an observer in order to get rid of the constraint of perfect initialization. Other extensions are the generalization of the control strategy and stability results to the 2D case and to a class of non linear systems.

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