Linear Matrix Inequality Design of Observer-Based Controllers for port-Hamiltonian Systems

J. Toledo, H. Ramírez, Y. Wu, and Y. Le Gorrec.

***Abstract*—The design of an observer-based state feedback controller for port-Hamiltonian (pH) systems is addressed using linear matrix inequalities (LMIs). The controller is composed of the observer and the state feedback. By passivity, the asymptotic stability of the closed-loop system is guaranteed even if the controller is implemented on complex physical systems such as the ones defined by infinite-dimensional or nonlinear models. An infinite-dimensional Timoshenko beam model and a micro-electromechanical system are used to illustrate the achievable performances using such an approach under simulations.**

***Index Terms*—Distributed port-Hamiltonian systems, State feedback, Luenberguer observer, Linear Matrix Inequalities.**

I. Introduction

The port-Hamiltonian (pH) framework has been introduced in [1] and has shown to be well suited for the modelling and control of multi physical systems [2, 3]. It has been widely studied for finite-dimensional systems in [2, 4, 5, 6] and it has been generanized to infinite-dimensional systems in [7, 8]. The main idea of the pH approach is to describe physical systems in terms of the energy and its exchanges between each internal component and the environment.

Stabilization of pH systems using interconnection and damping assignment (IDA) has been proposed in [4, 5] and extensively developed for linear system in [6], where a linear matrix inequality (LMI) approach has been employed to obtain a solution of the IDA control problem. This LMI problem allows designing a static feedback matrix to have desired closed-loop performances. It can be seen as an alternative to traditional approaches as pole-placement, LQ-control or H_∞-control. This result is also implemented for the *dual problem*, i.e., for the observer design in [9]. Further works on observer design for linear and nonlinear pH systems have been reported in [10, 11, 12] where the properties of the system are used to ensure the observer convergence. Nevertheless, no results are reported regarding observer based control design.

An observer-based controller designed for pH systems is proposed in [9], where the observer-based state feedback is designed from the linearization of the system and used to stabilize the non linear system by means of a feedforward term. However, the passivity of the system is not preserved in closed-loop since the observer-based controller is not passive, thus the closed-loop stability is not guaranteed using the passivity properties of pH systems. In [13], an observer-based state feedback design is proposed such that the controller is on the pH form and in [14] the same authors proposed a similar controller for infinite-dimensional port-Hamiltonian system with distributed actuation. Nevertheless, the closed-loop performances can only be modified through damping injection. Recently in [15, 16], this result has been improved allowing to modify the whole structure of the plant in closed-loop and then, having more degrees of freedom in terms of control design.

In this work an observer-based state feedback design based on LMIs is proposed for linear pH system developing the LMIs presented in [6] for IDA control design. The feedback consists of a Luenberger observer and a negative feedback on the observed states. The novelty and main contribution of this paper is to recast the feedback and the Luenberger observer as a pH control system interconnected with the system to be controlled in a power preserving manner. This reinterpretation of the observer-based controller allows to use the passivity properties of the system to guarantee the closed-loop stability. A second contribution of this work is to explicitly give the conditions such that the observer based control system is strictly positive real, output strictly passive and zero state detectable. This result allows to use the proposed controller to asymptotically stabilize a large class of boundary controlled infinite dimensional pH systems [8, 17] and non-linear pH system [2] when using a linear approximation of these systems to design the controller.

The paper is organized as follows. Section II presents the main result of the paper, namely the pH observer based control system and its design parameters in terms of a set of LMIs. Section III presents two examples. An infinite dimensional Timoshenko beam model on a one dimensional spatial domain and a non-linear microelectromechanical system (MEMS), which are used to show the design procedure and the achieved closed-loop performances by means of numerical simulations. Finally, Section IV gives some final remarks and discussions on possible future work related to this topic.

II. Observer-based state feedback design

Consider the following linear pH system

$$P\begin{cases}\dot{x}(t) & = (J-R)Qx(t)+Bu(t), \quad x(0)=x_0\\ y(t) & = B^\top Qx(t)\end{cases} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$ is defined for all $t \geq 0$, $x_0 \in \mathbb{R}^n$ is the unknown initial condition, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^m$ is the power conjugated output of $u(t)$, which in this work is considered to be measurable. $J = -J^\top$, $R = R^\top \geq 0$ and $Q = Q^\top > 0$ all known real matrices of size $n \times n$ and $B \in \mathbb{R}^{n\times m}$. For simplicity, we refer to system (1) as the system (A, B, C), with $A = (J - R)Q$ and $C = B^\top Q$, and we assume that it is controllable and observable.

Define the following Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0 \quad (2)$$

for the pH system (1), where $\hat{x} \in \mathbb{R}^n$ is the estimation of x, $\hat{x}_0$ is a known initial condition and $L \in \mathbb{R}^{n\times m}$ is a matrix to design.

In this work, we use the results from [6] to design the matrix L such that (2) converge asymptotically to (1). Then, we design the state feedback matrix K such that the observer based control law

$$u(t) = r(t) - K\hat{x}(t), \quad r(t) \in \mathbb{R}^m, \quad K \in \mathbb{R}^{m\times n} \quad (3)$$

leads to a closed-loop system (2)-(3) on a pH form with inputs $r(t)$ and $y(t)$. The importance of guaranteeing this closed-loop property is that it is instrumental to assure asymptotic stability of the closed-loop system [18].

A. Observer design by LMIs

Define the error of the state as $\tilde{x}(t) = x(t) - \hat{x}(t)$. The error dynamics is obtained from (1) and (2):

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0 = x_0 - \hat{x}_0, \quad (4)$$

where $\tilde{x}_0$ is a unknown initial condition.

We recall the following proposition from [6], which is instrumental for the design of the matrix L such that $A - LC$ is Hurwitz.

Proposition 1: [6] Denote by $B_\perp$ a full rank $(n - m) \times n$ matrix that annihilates B, i.e. $B_\perp B = 0$. Let us also denote $E_\perp = B_\perp A$. There exist matrices $J_d = -J_d^\top$, $R_d = R_d^\top \geq 0$, $Q_d = Q_d > 0$ and F such that $(J_d - R_d)Q_d = A + BF$ if and only if there exists a solution $\mathbf{X} = \mathbf{X}^\top \in \mathbb{R}^{n\times n}$ to the LMIs:

$$\begin{aligned} \mathbf{X} &> 0, \\ -[E_\perp \mathbf{X} B_\perp^T + B_\perp \mathbf{X} E_\perp^T] &\geq 0. \end{aligned} \quad (5)$$

Given such an $\mathbf{X}$, compute S_d as follows:

$$S_d = \begin{pmatrix} B_\perp \\ B^T \end{pmatrix}^{-1} \begin{pmatrix} E_\perp \mathbf{X} \\ -B^T \mathbf{X} E_\perp^T (B_\perp B_\perp^T)^{-1} B_\perp \end{pmatrix}, \quad (6)$$

then the following matrices

$$\begin{array}{ll} J_d = \frac{1}{2}(S_d - S_d^T), & R_d = -\frac{1}{2}(S_d + S_d^T), \\ Q_d = \mathbf{X}^{-1}, & F = (B^T B)^{-1} B^T (S_d \mathbf{X}^{-1} - A) \end{array} \quad (7)$$

satisfy $J_d = -J_d^\top$, $R_d = R_d^\top \geq 0$, $Q_d = Q_d > 0$ and $(J_d - R_d)Q_d = A + BF$.

Remark 1: Proposition 1 is related to the stabilizability of (1). In fact, the LMI (5) has a solution if and only if the pair (A, B) is stabilizable [Proposition 9 in [6]].

Remark 2: The dual problem consists in following Proposition 1, but replacing A by A^T, B by C^T and F by $-L^T$. The reader can also refer to Proposition 1 in [9].

Remark 3: Similar to Remark 1, the pair (A, C) is detectable if and only if the LMI (5) has a solution with $E_\perp = B_\perp A^T$ and $B_\perp \in \mathbb{R}^{(n-m)\times n}$ a left annihilator of C^T, i.e. $B_\perp C^T = 0$.

The performances obtained using Proposition 1 are in terms of Q_d (energy matrix) and R_d (dissipation matrix). As it is mentioned in [6], the LMI (5) can be slightly modified in order to keep the energy matrix in a desired interval and to have sufficient but not excessive damping. This is formalized in the following proposition.

Proposition 2: Under the same statements of Proposition 1, if the following LMIs:

$$\begin{aligned} \Lambda_2^{-1} - \mathbf{X} &< 0, \\ -\Lambda_1^{-1} + \mathbf{X} &< 0, \\ \Xi_1 + E_\perp \mathbf{X} B_\perp^T + B_\perp \mathbf{X} E_\perp^T + &\leq 0, \\ -\Xi_2 - E_\perp \mathbf{X} B_\perp^T - B_\perp \mathbf{X} E_\perp^T + &\leq 0, \end{aligned} \quad (8)$$

have a solution $\mathbf{X} = \mathbf{X}^\top$ for some symmetric matrices Λ_1, $\Lambda_2 \in \mathbb{R}^{n\times n}$, $\Xi_1, \Xi_2 \in \mathbb{R}^{(n-m)\times(n-m)}$, such that $0 < \Lambda_1 < \Lambda_2$ and $0 \leq \Xi_1 < \Xi_2$, then $\Lambda_1 < Q_d < \Lambda_2$. Moreover, choosing

$$S_d = \begin{pmatrix} B_\perp \\ B^T \end{pmatrix}^{-1} \begin{pmatrix} E_\perp \mathbf{X} \\ -B^T \mathbf{X} E_\perp^T (B_\perp B_\perp^T)^{-1} B_\perp - \gamma B^T \end{pmatrix}, \quad (9)$$

for some scalar $\gamma > 0$, and the matrices J_d, R_d and F as in (7), then $A + BF = (J_d - R_d)Q_d$ with $R_d > 0$.

Proof. The proof of Proposition 1 is a direct application of Proposition 7 and Remark 8 in [6]. See also Proposition 1 in [9]. ■

Remark 4: Matrices Λ_1 and Λ_2 allow to fix the lowest and highest eigenvalues of Q_d respectively. Matrices Ξ_1 and Ξ_2 bound the damp term, while the scalar $\gamma > 0$ implies $R_d > 0$ and then, the asymptotic behavior is ensured.

In the following section, we consider the Luenberger observer (2) already designed by Proposition 2 using the *dual problem*, i.e. replacing A by A^T, B by C^T, and $L = -F^T$, and then we design the matrix K in the control law (3) such that the system is equivalent to control by interconnection.

B. PH observer-based controller

Consider the Luenberger observer (2) combined with the state feedback (3). The aim is to formulate this observer-based state feedback as the power preserving interconnection

$$u(t) = r(t) - y_c(t), \quad u_c(t) = y(t) \quad (10)$$

of (1) with a pH dynamic control system, defined as

$$\hat{P} \begin{cases} \dot{\hat{x}}(t) &= (J_c - R_c)Q_c \hat{x}(t) + B_c u_c(t) + Br(t), \\ y_c(t) &= B_c^\top Q_c \hat{x}(t), \\ y_r(t) &= B^\top Q_c \hat{x}(t). \end{cases} \quad (11)$$

as depicted in Fig. 1. This is possible if the control gain is defined as $K = B_c^T Q_c$, $B_c = L$ and the following matching equation

$$A - LC - BK = (J_c - R_c)Q_c \quad (12)$$

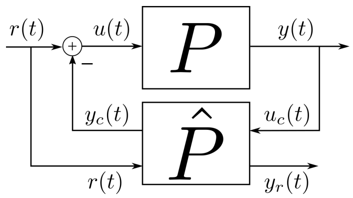

Fig. 1. Power preserving interconnection.

is satisfied for some $n \times n$ matrices $J_c = -J_c^\top$, $R_c = R_c^\top \geqslant 0$, $Q_c = Q_c^\top > 0$ and (A, B, C) defined in (1).

The following proposition, which presents a set of LMIs whose solution allows to define K, J_c, R_c and Q_c such that the observer-based controller (11) is a pH system, is the main contribution of this work.

Proposition 3: Given (A, B, C) (1), the power preserving interconnection (10) and a matrix L such that $A_L := A - LC$ is Hurwitz. Then (11) is a pH system if the LMIs

$$\begin{aligned} 2\Gamma_1 - BL^\top - LB^\top + A_L\mathbf{X} + \mathbf{X}A_L^\top &\le 0,\\ -2\Gamma_2 + BL^\top + LB^\top - A_L\mathbf{X} - \mathbf{X}A_L^\top &\le 0,\\ -\Delta_1^{-1} + \mathbf{X} &\le 0,\\ \Delta_2^{-1} - \mathbf{X} &\le 0, \end{aligned} \tag{13}$$

have a solution $\mathbf{X} = \mathbf{X}^\top$, for some $n \times n$ symmetric matrices $\Gamma_1, \Gamma_2, \Delta_1$ and Δ_2 such that $0 \le \Gamma_1 < \Gamma_2$ and $0 < \Delta_1 < \Delta_2$. $S_c = A_L\mathbf{X} - BL^\top$, we have $J_c = \frac{1}{2}(S_c - S_c^\top)$, $R_c = -\frac{1}{2}(S_c + S_c^\top)$, $Q_c = \mathbf{X}^{-1}$, $B_c = L$ and $K = B_c^\top Q_c$.

Corollary 1: The following results are direct consequences of Proposition 3.

(i) $\lim\limits_{t\to\infty}(x(t) - \hat{x}(t)) = 0$, characterized by the eigenvalues of A_L;

(ii) Matrices R_c and Q_c satisfy
  a) $\Gamma_1 \le R_c \le \Gamma_2$;
  b) $\Delta_1 \le Q_c \le \Delta_2$;

(iii) If $\Gamma_1 > 0$, (11) is strictly positive real (SPR), output strictly passive (OSP) and zero state detectable (ZSD) with respect to the input/output pair u_c/y_c.

Proof. The proof of Proposition 3 and Corollary 1 are shown here. $\mathbf{X}$ being the solution of the LMI (13), from $S_c = A_L\mathbf{X} - BL^\top$, $J_c = \frac{1}{2}(S_c - S_c^\top)$, $R_c = -\frac{1}{2}(S_c + S_c^\top)$, $Q_c = \mathbf{X}^{-1}$, $B_c = L$ and $K = B_c^\top Q_c$, one can verify that $J_c = -J_c^\top$, $R_c = R_c^\top$ and $Q_c = Q_c^\top$. To conclude that (11) is a pH system it has to be verified that $R_c \ge 0$ and $Q_c > 0$. From (13),

$$\begin{aligned} 2\Gamma_1 \le BL^\top + LB^\top - A_L\mathbf{X} - \mathbf{X}A_L^\top \le 2\Gamma_2,\\ {\Delta_2}^{-1} \le \mathbf{X} \le {\Delta_1}^{-1}. \end{aligned}$$

Replacing $\mathbf{X}$, $A_L\mathbf{X} - BL^T$ by their expression with respect to S_c and Q_c, and inverting the second inequality we obtain

$$\begin{aligned} 2\Gamma_1 \le -(S_c + S_c^\top) \le 2\Gamma_2,\\ \Delta_1 \le Q_c \le \Delta_2. \end{aligned} \tag{14}$$

Using $R_c = -(S_c + S_c^\top)$ we conclude that $Q_c > 0$ and $R_c \ge 0$ since $\Delta_1 > 0$ and $\Gamma_1 \ge 0$. Implication (i) of Corollary 1 is directly obtained from (4) and the assumption that A_L is Hurwitz. Implication (ii) is verified replacing $R_c = -(S_c + S_c^\top)$ in (14). The SPR property of implication (iii) is verified applying the Kalman-Yakubovich-Popov Lemma [19]. To this end, the existence of real matrices $P = P^T > 0$, S and a scalar $\varepsilon > 0$ such that $PA_c + A_c^T P = -S^T S - \varepsilon P$ and $C_c = B_c^T P$ is proved by choosing $P = Q_c$, which implies $S^T S = 2Q_cR_cQ_c - \varepsilon Q_c$, and since $\Gamma_1 > 0$ implies $R_c > 0$, we can always find a small enough ε such that $2Q_cR_cQ_c - \varepsilon Q_c$ is positive definite. Then, the matrix $2Q_cR_cQ_c - \varepsilon Q_c$ can always be decomposed as $S^T S$ using for instance Cholesky factorization. The OSP property follows noting that $\Gamma_1 > 0$ implies $R_c > 0$, and taking the time derivative of the Hamiltonian of the controller $H_c = \frac{1}{2}x_c^\top Q_c x_c$. It is not difficult to show that

$$\begin{aligned} \dot{H}_c &= -x_c^\top Q_c^\top R_c Q_c x_c + y_c^\top u_c\\ &= -x_c^\top Q_c(R_c - \epsilon B_c B_c^\top)Q_c x_c + y_c^\top u_c - \epsilon\|y_c\|^2 \end{aligned} \tag{15}$$

where we have added $\pm\epsilon y_c^\top y_c$, with $\epsilon > 0$, to the first line of (15) and used $y_c = B_c^\top Q_c x_c$ and $Q_c = Q_c^\top$ in the second line of (15). Hence it is always possible to find a small enough ϵ such that (11) is dissipative with respect to the supply rate $y_c^\top u_c - \epsilon\|y_c\|^2$, implying that (11) is OSP. The ZSD property is inferred from (15) setting $u_c = y_c = B_c^\top Q_c x_c = 0$ and noting that since $R_c > 0$, the states of (11) converge exponentially to zero. ■

Remark 5: Matrix L of Proposition 3 can be designed with Proposition 2 or with any other control design technique such as, for instance, Linear Quadratic Regulator (LQR) or pole-placement approaches.

Remark 6: A simple choice for designing matrices Γ_1, Γ_2, Δ_1 and Δ_2 is to use identity matrices modulated by a constant.

Proposition 3 permits to assure that the observer-based controller can be formulated as a pH system. This is instrumental to guarantee the asymptotic stability of the closed-loop system in some particular cases of interest. Indeed, if (1) is the finite-dimensional approximation of a boundary controlled pH system (BC-PHS) defined on a 1-dimensional spatial domain as in [8, Theorem 4.4], or the linear approximation of a finite dimensional non-linear system (see the appendix for the precise definition of the class of considered systems), then the controller (11) from Proposition 3 asymptotically stabilizes the non-approximated systems under some very general conditions. This is formalized in the following proposition.

Proposition 4: Let (1) be the finite-dimensional and linear approximation of

(i) a linear boundary controlled pH system (BC-PHS) defined on a 1-dimensional spatial domain, *or*

(ii) an output strictly passive (OSP) and zero-state detectable (ZTD) finite dimensional non-linear system as defined by (22),

then, (11) designed using Proposition 3 asymptotically stabilizes (i), respectively (ii), if $\Gamma_1 > 0$.

Proof. By Corollary 1 (11) is SPR, OSP and ZSD if $\Gamma_1 > 0$. Hence the proof of (i) follows by direct application of Theorem 5.10 in [20], concerning the power preserving interconnection of a BC-PHS defined on a 1-dimensional spatial domain and a SPR finite dimensional system and the proof of (ii) follows by direct application of Proposition 4.3.1 in [2], concerning the power preserving interconnection of OSP and ZSD systems. ■

III. Examples

In this section we illustrate the design approach on an infinite dimensional Timoshenko flexible beam model and on a non-linear model of a microelectromechanical optical switch.

A. Boundary control of a flexible beam

The Timoshenko beam model describes the behavior of a thick beam in a one dimensional spatial domain. It admits the following BC-PHS formulation (18)-(21) with

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{H}(\zeta) = \begin{bmatrix} T(\zeta) & 0 & 0 & 0 \\ 0 & \rho(\zeta)^{-1} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 0 \\ 0 & 0 & 0 & I_\rho(\zeta)^{-1} \end{bmatrix},$$

with state variables $z = (z_1, z_2, z_3, z_4)^\top$, where $z_1(\zeta,t) = w_\zeta(\zeta,t) - \phi(\zeta,t)$ is the shear displacement, $z_2(\zeta,t) = \rho(\zeta)w_t(\zeta,t)$ is the transverse momentum distribution, $z_3(\zeta,t) = \phi_\zeta(\zeta,t)$ the angular displacement and $z_4(\zeta,t) = I_\rho(\zeta)\phi_t(\zeta,t)$ is the angular momentum distribution. $w(\zeta,t)$ and $\phi(\zeta,t)$ are respectively the transverse displacement of the beam and the rotation angle of neutral fiber of the beam. Note that we have used the lower indexes ζ and t to refer to the partial derivative with respect to that index. $T(\zeta)$ is the shear modulus, $\rho(\zeta)$ is the mass per unit length, $EI(\zeta)$ is the Youngs modulus of elasticity E multiplied by the moment of inertia of a cross section I, and $I_\rho(\zeta)$ is the rotational momentum of inertia of a cross section. Note that, $T(\zeta)z_1(\zeta,t)$ is the shear force, $\rho(\zeta)^{-1}z_2(\zeta,t)$ the longitudinal velocity, $EI(\zeta)z_3(\zeta,t)$ the torque and $I_\rho(\zeta)^{-1}z_4(\zeta,t)$ the angular velocity. With the following inputs and outputs

$$u(t) = \begin{pmatrix} \rho(a)^{-1}z_2(a,t) \\ I_\rho(a)^{-1}z_4(a,t) \\ T(b)z_1(b,t) \\ EI(b)z_3(b,t) \end{pmatrix}, \; y(t) = \begin{pmatrix} -T(a)z_1(a,t) \\ -EI(a)z_3(a,t) \\ \rho(b)^{-1}z_2(b,t) \\ I_\rho(b)^{-1}z_4(b,t) \end{pmatrix}$$

the energy balance is given by $\dot{H}(t) = u(t)^\top y(t)$. The reader is refereed to [21] for more details on the model, to [8, 17] for the well-posedness of this class of systems and to [20] for stability analysis. The parameters of the model are shown in Table I.

TABLE I
Plant Parameters.

	Value	Measurement unit
T	1	Pa
ρ	1	kg.m^{-1}
EI	1	Pa.m^4
I_ρ	1	Kg.m^2
a	0	m
b	1	m

To design the passive observer-based controller using Proposition 3, the infinite-dimensional model is first approximated by a finite-dimensional system using the finite difference discretization scheme on staggered grids proposed in [22]. This is a structure preserving spatial approximation method which preserves the pH structure of the system. The matrices of the finite-dimensional approximation on the form (1) are

$$J = \begin{bmatrix} 0 & D & 0 & -F \\ -D^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & D \\ F^\top & 0 & -D^\top & 0 \end{bmatrix}, \quad R = 0,$$

$$Q = \begin{bmatrix} hQ_1 & 0 & 0 & 0 \\ 0 & hQ_2 & 0 & 0 \\ 0 & 0 & hQ_3 & 0 \\ 0 & 0 & 0 & hQ_4 \end{bmatrix},$$

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & b_{23} & 0 \\ 0 & b_{32} & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} \end{bmatrix}$$

where

$$D = \frac{1}{h^2} \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

$$F = \frac{1}{2h} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

$Q_i, i \in \{1, \cdots, 4\}$ are diagonal matrices containing the evaluation of $T(\zeta)$, $\rho(\zeta)^{-1}$, $EI(\zeta)$ and $I_\rho(\zeta)^{-1}$ respectively, at the specific discretization points and

$$b_{11} = \frac{1}{h} \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad b_{12} = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad b_{32} = b_{11},$$

$$b_{23} = \frac{1}{h} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad b_{43} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad b_{44} = b_{23}.$$

The state variables of the approximated model are $x(t) = (x_1^d, x_2^d, x_3^d, x_4^d)^\top$, where $x_i^d(t) \in \mathbb{R}^{n_d}$, $i \in \{1, \cdots, 4\}$ and the $i-th$ component of x_1^d, x_2^d, x_3^d and x_4^d corresponds to the approximation of $z_1((i-0.5)h,t)$, $z_2(ih,t)$, $z_3((i-0.5)h,t)$ and $z_4(ih,t)$ respectively, with $h = 2\frac{b-a}{2*n_d+1}$, $b-a$ being the length of the beam and n_d the number of element. In this example, we choose $n_d = 5$ and hence the complete state is composed of 20 elements. The reader is refereed to [22] for further details about this discretization method. The observer design is done following Propositions 1, 2 and Remark 2. The design parameters for the observer are shown in Table II and

TABLE II
OBSERVER DESIGN PARAMETERS

Matrix	Value
Λ_1	$0.1I_{20}$
Λ_2	$5000I_{20}$
Ξ_1	$1I_{18}$
Ξ_2	$1000I_{18}$
γ	10

TABLE III
CONTROLLER DESIGN PARAMETERS

Matrix	Design 1	Design 2
Γ_1	$1 \times 10^{-15} I_{20}$	$1 \times 10^{-15} I_{20}$
Γ_2	$1 \times 10^{15} I_{20}$	$1 \times 10^{15} I_{20}$
Δ_1	$0.1 \times 10^{-1} I_{20}$	$0.18 \times 10^{-1} I_{20}$
Δ_2	$1 \times 10^{15} I_{20}$	$1 \times 10^{15} I_{20}$

the eigenvalues of the matrix $A_L = A - LC$ are shown in Figure 2. For the state feedback design, we follow Proposition 3 with the design matrices given in Table III varying only the matrix Δ_1. The eigenvalues of both closed-loop matrices are shown in Figure 2, where K_1 and K_2 refer to design 1 and 2, respectively. Since for both controllers $\Gamma_1 > 0$, the

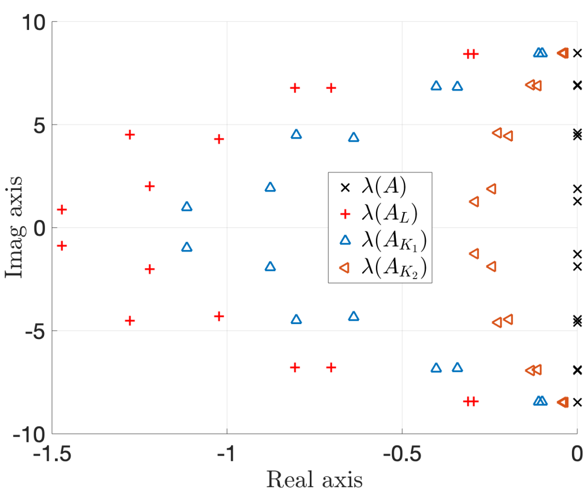

Fig. 2. $\lambda(A)$: Eigenvalues of A, $\lambda(A_L)$: Eigenvalues of $A - LC$, $\lambda(A_{K_i})$: Eigenvalues of $A - BK_i$, with $i = \{1, 2\}$.

closed-loop between the low order observer-based controller and the infinite-dimensional system is asymptotically stable by Proposition 4. For the simulation we use a time interval $t = [0, 10s]$ with step time $\delta_t = 0.1\ ms$ and mid point temporal discretization [22]. The simulation is done taking 100 elements per state variable for the infinite-dimensional system (in order to approach the infinite dimensional system over a large set of frequencies), 400 in total, while for the observer we only take 5 elements per state variable, *i.e.* 20 in total. The initialization is such that the beam is in equilibrium position with a force of $0.01\ N$ applied at the end tip, which gives the following initial conditions for the plant: $z_1(\zeta, 0) = 0.01$, $z_2(\zeta, 0) = 0$, $z_3(\zeta, 0) = -0.01(\zeta - 1)$ and $z_4(\zeta, 0) = 0$. The observer is initialized with null initial conditions, i.e. $\hat{x}(0) = 0$. The deformation of the beam is reconstructed from the state variables $z(\zeta, t)$ and $\hat{z}(\zeta, t)$, taking into account that the beam is clamped at the left side. Figure 3 shows the end tip responses in open-loop and closed-loop for design 1 and 2. The settling time is improved when increasing Δ_1. Figure

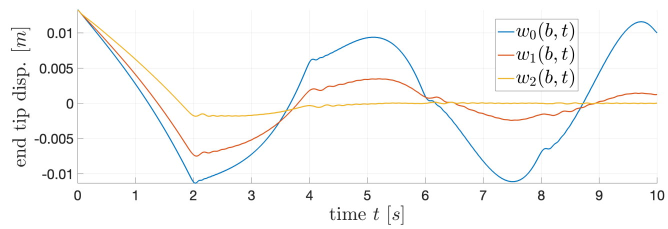

Fig. 3. End tip displacement in open-loop ($w_0(b, t)$), in closed-loop using K_1 ($w_1(b, t)$), and in closed-loop using K_2 ($w_2(b, t)$).

4 shows the observer convergence at the end tip displacement in the design 2 case. The deformation and its estimation along

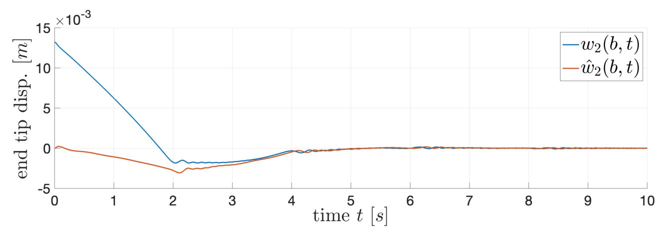

Fig. 4. End tip displacement in closed-loop $w_2(b, t)$ and its estimations $\hat{w}_w(b, t)$ under design 2.

space and time are shown in Figure 5, where the first row of sub-plots shows, from left to right, the deformation in open-loop and its estimation, the second row the deformation in closed-loop and its estimation when K_1 is applied, and the third row the deformation in closed-loop and its estimation when K_2 is applied. Notice that, the observer convergence is ensured when applying the controller to the discretized model and not when applying to the BC-PHS. However, stability is preserved when applying the controller to the BC-PHS and performances will be closer to the desired one as long as the discretization is closer to the BC-PHS.

B. Microelectromechanical optical switch

Microelectromechanical systems (MEMS) are micro robots with an electronic actuation part. Due to the miniaturization of technology, MEMS are being an important tool in the micro-robotic industry. In optics for instance [23], using tiny mirrors MEMS allows to connect two optical devices without converting continuous signals into electronic ones. A dynamical model of this system can be found in [23] and its

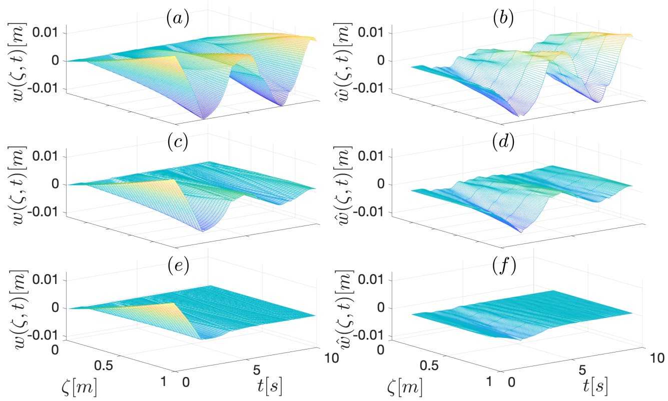

Fig. 5. (a) and (b): Deformation and its estimation in open-loop. (c) and (d): Deformation and its estimation in closed-loop under design 1. (e) and (f): Deformation and its estimation in closed-loop under design 2.

port-Hamiltonian representation in [24]

$$\begin{aligned}
\begin{pmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & -b & 0 \\ 0 & 0 & \frac{-1}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} u \\
y &= \tfrac{1}{r}\tfrac{\partial H}{\partial Q} \\
H &= \frac{p^2}{2m} + \frac{1}{2}k_1 q^2 + \frac{1}{4}k_2 q^4 + \frac{Q^2}{2C(q)} \\
C(q) &= \frac{\varepsilon A_s}{q_{max} - q}
\end{aligned} \tag{16}$$

where $q(t)$, $p(t)$ and $Q(t)$ are respectively, the position, the momentum, and the charge in the capacitor, k_1 and k_2 are the spring coefficients, m is the mass of the moving part, $C(q)$ is the non-linear capacitance which depends on the gap of the MEMS, $b > 0$ and $r > 0$ are the damping and resistance constant parameters, respectively, ε is the dielectric constant, A_s is the surface of the MEMS and q_{max} is such that $q < q_{max}$. The input of the system $u(t)$ is the input voltage and $y(t)$ is the supplied current. The balance equation of the Hamiltonian is

$$\dot{H}(t) = -b\left(\frac{p(t)}{m}\right)^2 - ry(t)^2 + y(t)u(t)$$

which implies that the system is OSP. Under realistic operation conditions we can assume that the state space of the system is such that $Q(t) > 0$ for all $t > 0$, allowing to conclude that the system is ZSD. The parameters of the plant are shown in Table IV. The linearization of (16) around an equilibrium operation point is given by

$$\begin{aligned}
A &= \begin{pmatrix} 0 & \frac{1}{m} & 0 \\ -3k_2(q^*)^2 - k_1 & -\frac{b}{m} & \frac{Q^*}{A_s\varepsilon} \\ \frac{Q^*}{A_s\varepsilon r} & 0 & \frac{q^* - q_{max}}{A_s\varepsilon r} \end{pmatrix} \\
B &= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \end{pmatrix}, \qquad C = \begin{pmatrix} \frac{-Q^*}{A_s\varepsilon r} & 0 & \frac{-q^* - q_{max}}{A_s\varepsilon r} \end{pmatrix}
\end{aligned} \tag{17}$$

In the current example the studied equilibrium is given in Table V. Following the design procedure of section II the linearized model (17) is used for the synthesis of an observer-based controller. For the observer design Proposition 2 is used with the parameters given in Table VI. The eigenvalues of the matrix $A_L = A - LC$ are shown in Figure (6). Two state feedbacks are designed using Proposition 3 with the parameters given in Table VII. Note that the first and second controller only differ by Δ_1. Since (16) is OSP and for booth controllers $\Gamma_1 > 0$, the closed-loop system in asymptotically stable by Proposition 4. The feedback matrices are for each controller denoted by K_1 and K_2, respectively, and the closed-loop eigenvalues are shown in Figure 6. For the simulation, time $t = [0, 0.01s]$ is used with a step time $\delta t = 1\ \mu s$. The initial conditions are set equal to $q(0) = q^*$, $p(0) = p^*$, $Q(0) = 0.9Q^*$ for the non linear system, while for the observer all initial conditions are set exactly at the equilibrium

TABLE IV
PLANT PARAMETERS.

	Value	Measurement unit
k_1	0.46	Nm^{-1}
k_2	0.46	Nm^{-3}
m	2.4×10^{-8}	kg
ε	8.854×10^{-12}	Fm^{-1}
A_s	4×10^{-4}	m^2
q_{max}	10^{-5}	m
b	10^{-7}	Ns
r	0.5×10^{6}	Ω

TABLE V
LINEARIZATION POINT.

	Value	Measurement unit
q^*	0.5×10^{-6}	m
p^*	0	$\mathrm{kg\ m\ s}^{-1}$
Q^*	4.0363×10^{-11}	C
u^*	0.1083	V
y^*	2.1654×10^{-8}	A

TABLE VI
OBSERVER DESIGN PARAMETERS

Matrix	Value
Λ_1	$1 \times 10^{-2} \times diag([1, 200, 1])$
Λ_2	$1 \times 10^{10} I_3$
Ξ_1	$1 \times 10^{-1} I_2$
Ξ_2	$1 \times 10^{4} I_2$
γ	30×10^{4}

TABLE VII
CONTROLLER DESIGN PARAMETERS

Matrix	Design 1	Design 2
Γ_1	$1 \times 10^{-15} I_3$	$1 \times 10^{-15} I_3$
Γ_2	$1 \times 10^{15} I_3$	$1 \times 10^{15} I_3$
Δ_1	$0.5 \times 10^{-1} I_3$	$1.5999 \times 10^{-1} I_3$
Δ_2	$1 \times 10^{15} I_3$	$1 \times 10^{15} I_3$

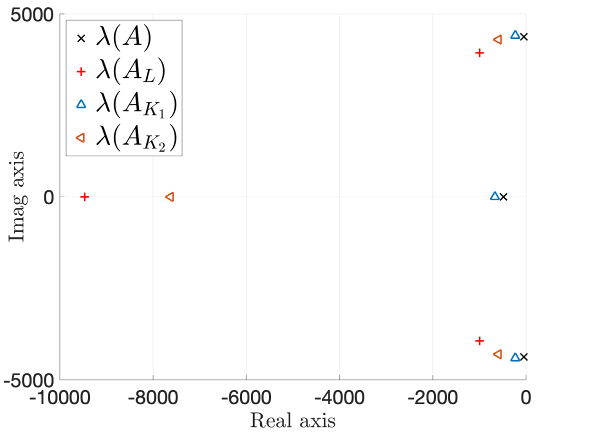

Fig. 6. $\lambda(A)$: Eigenvalues of A, $\lambda(A_L)$: Eigenvalues of $A - LC$, $\lambda(A_{Ki})$: Eigenvalues of $A - BK_i$, with $i = \{1, 2\}$.

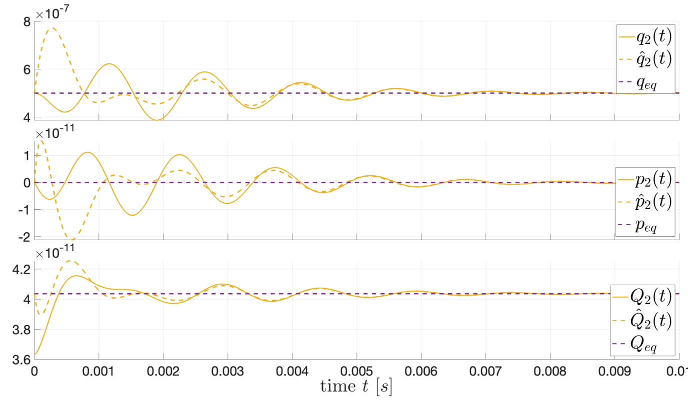

Fig. 8. Top: q_2 is the displacement of the moving part, $\hat{q}_2$ its estimations and q_{eq} the equilibria. Middle: p_2 is the momenta of the moving part, $\hat{p}_2$ its estimations and p_{eq} the equilibria. Bottom: Q_2 is the displacement of the moving part, $\hat{Q}_2$ its estimations and Q_{eq} the equilibria. This simulation is under the design 2.

point. Figure 7 shows the open-loop response and the closed-loop responses when applying the two different controllers on the non linear system. Figure 8 shows the closed-loop response for the second controller together with the observed variables. In both cases, the mechanical oscillations have been reduced by increasing the electrical ones. We observe that changing the lower bound of Q_c, *i.e.* Δ_1, better performances for the mechanical part of the micro robot are obtained.

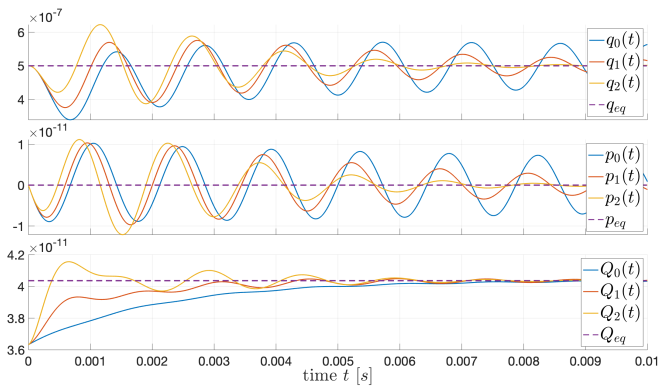

Fig. 7. Top: q_i is the displacement of the moving part. Middle: p_i is the momentum of the moving part. Bottom: Q_i is the electric charge. In all sub-plots $i = \{0, 1, 2, eq\}$ refers respectively to the open-loop response, the response under design 1, the response under design 2 and the equilibria.

IV. Conclusion

An observer-based state feedback controller design based on LMIs is proposed for linear pH systems. The feedback consists on a Luenberger observer and a negative feedback on the observed states. The novelty and main contribution of this paper is to cast the feedback and the Luenberger observer as a pH control system interconnected in a power preserving manner with the system to be controlled. This reinterpretation of the observer based controller allows to use the passivity properties of the system to guarantee the closed-loop stability. The second contribution of this work is to explicitly give the conditions such that the observer-based control system is strictly positive real, output strictly passive, and zero state detectable. This result allows to use the proposed controller to asymptotically stabilize a large class of linear boundary controlled infinite dimensional pH systems and non-linear pH systems when using a linear approximation of these system to design the controller. An infinite dimensional Timoshenko beam model and a finite dimensional non-linear model of a microelectromechanical actuator are used to illustrate the effectiveness of the proposed approach.

Appendix

Boundary controlled PHS on 1D domain

In this subsection the definition of boundary controlled port-Hamiltonian (BC-PHS) system is given. The reader is refereed to [8, 17, 20] for further details and definitions. A BC-PHS is a dynamical system governed by the following partial differential equation

$$\frac{\partial z}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta) z(\zeta, t)) + P_0 \mathcal{H}(\zeta) z(\zeta, t), \quad (18)$$

$$z(\zeta, 0) = z_0(\zeta), \quad (19)$$

$$W_{\mathcal{B}} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = u(t), \quad (20)$$

$$y(t) = W_{\mathcal{C}} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \quad (21)$$

where the initial condition is given by (19), the boundary input by (20) and the boundary output by (21). Here $z(\zeta, t) \in \mathbb{R}^n$ is the state variable with initial condition $z_0(\zeta)$. $\zeta \in [a, b]$ is the 1D domain and $t \geq 0$ is the time. $P_1 = P_1^T \in \mathbb{R}^{n \times n}$ is a non-singular matrix, $P_0 = -P_0^T \in \mathbb{R}^{n \times n}$, $\mathcal{H}(\zeta)$ is a bounded and continuously differentiable matrix-valued function satisfying for all $\zeta \in [a, b]$, $\mathcal{H}(\zeta) = \mathcal{H}^T(\zeta)$ and $mI < \mathcal{H}(\zeta) < MI$ with $0 < m < M$ both scalars independent on ζ. The Hamiltonian energy function of (18) is given by

$$H(t) = \frac{1}{2} \int_a^b z(\zeta, t)^T \mathcal{H}(\zeta) z(\zeta, t) d\zeta.$$

$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}$ are the *boundary port variables* defined as

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} \mathcal{H}(b)z(b,t) \\ \mathcal{H}(a)z(a,t) \end{pmatrix}.$$

$W_\mathcal{B}$, $W_\mathcal{C} \in \mathbb{R}^{n\times 2n}$ are two matrices such that if $W_\mathcal{B}\Sigma W_\mathcal{B}^T = W_\mathcal{C}\Sigma W_\mathcal{C}^T = 0$ and $W_\mathcal{C}\Sigma W_\mathcal{B}^T = I$, with $\Sigma = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, then $\dot{H}(t) = u(t)^T y(t)$.

ZSD and OSP non-linear control system

In this subsection the definition of zero-state detectable (ZSD) and output strictly passive (OSP) non-linear systems is given. The reader is refereed to [2] for further details and definitions. Consider a non-linear controlled system

$$\dot{x} = f(x,u), \quad y = h(x,u) \tag{22}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and $f(\cdot)$ and $h(\cdot)$ sufficiently smooth differentible mappings, then (22) is

- OSP if there exists $\epsilon > 0$ such that it is dissipative with respect to the supply rate $s(u,y) = u^\top y - \epsilon\|y\|^2$,
- ZSD if $u(t) = 0$, $y(t) = 0$, $\forall t \geq 0$, implies $\lim_{t\to\infty} x(t) = 0$.

A (non-linear) PHS is a dissipative system with storage function $H(x)$ [2].

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