Heteroclinic orbits for a system of amplitude equations for orthogonal domain walls

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Abstract

Using a variational method, we prove the existence of heteroclinic solutions for a 6-dimensional system of ordinary differential equations. We derive this system from the classical Bénard-Rayleigh problem near the convective instability threshold. The constructed heteroclinic solutions provide first order approximations for domain walls between two orthogonal convective rolls.

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1. Introduction

We consider the following system of ordinary differential equations

$$\frac{d^4 A_0}{dx^4} = A_0(1-|A_0|^2 - g|B_0|^2), \tag{1.1}$$

$$\frac{d^2 B_0}{dx^2} = \varepsilon^2 B_0(-1+g|A_0|^2+|B_0|^2), \tag{1.2}$$

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in which A_0 and B_0 are complex-valued functions defined on $\mathbb{R}$, and the parameters ε and g are real. The purpose of this work is twofold: to rigorously derive this system from the Bénard-Rayleigh convection problem and to prove that it possesses heteroclinic orbits. In the system derived in this way ε is a small positive bifurcation parameter and $g \geqslant g_0$ for some $g_0 > 1$. Beyond these values, the existence of heteroclinic solutions is a mathematically interesting question in itself and it could also be relevant in other applications.

The Bénard-Rayleigh convection problem is a classical problem in fluid mechanics. It concerns the flow of a three-dimensional viscous fluid layer situated between two horizontal parallel plates and heated from below. Upon increasing the difference of temperature between the two plates, the simple conduction state looses stability at a critical value of the temperature difference. In terms of nondimensional parameters this instability occurs at a critical value $\mathcal{R}_c$ of the Rayleigh number. Beyond the instability threshold, a convective regime develops in which patterns are formed, such as convective rolls, hexagons, or squares. Observed patterns are often accompanied by defects, as for instance domain walls which occur between rolls with different orientations (see Figure 1.1). These patterns and defects are exten-

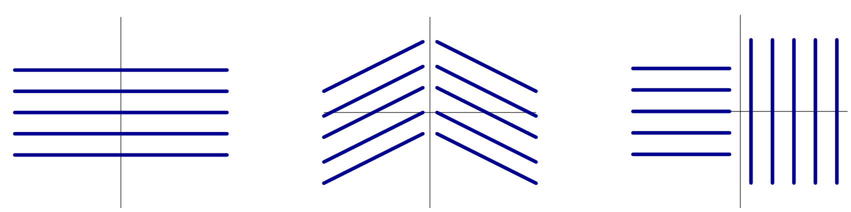

Figure 1.1: From left to right, schematic plots of the projections on the horizontal plane of: convective rolls, a symmetric domain wall between two sets of rolls rotated by opposite angles, and an orthogonal domain wall.

sively studied in the Bénard-Rayleigh convection problem, but also in other pattern-forming systems. We refer to the works [2, 13], and the references therein, for experimental and analytical results, and detailed descriptions of these patterns and defects.

Mathematically, the governing equations are the Navier-Stokes equations coupled with an equation for the temperature, and completed by boundary conditions at the two plates (e.g., see [11]). Observed patterns are then found as particular steady solutions of these equations. Since the pioneering works of Yudovich [17, 19, 20, 21], Rabinowitz [14], and Görtler et al [6] in the sixties, the existence of patterns was studied in various works by different authors (e.g., see [5, 11, 3] and the references therein). Very recently, the existence of symmetric domain walls has been shown in [7, 8], whereas the existence of asymmetric domain walls, and in particular of orthogonal domain walls, are open questions.

Handling the full governing equations being often technically challenging, alternative studies rely on simpler amplitude equations which provide approximate descriptions of solutions in particular parameter regimes. For instance, the amplitude equations describing symmetric domain walls are a particular case of the system considered in [18]. We adopt this type of approach for the existence problem for orthogonal domain walls.

As a first step, we rigorously derive the system of amplitude equations (1.1)-(1.2) in the parameter regime of Rayleigh numbers $\mathcal{R}$ slightly above the threshold of convective instability $\mathcal{R}_c$. We apply the reduction procedure used in [7, 8] for the analysis of symmetric domain walls. Starting from a formulation of the steady governing equations as an infinite-dimensional dynamical system in which the horizontal coordinate x plays the role of evolutionary variable, we apply a center manifold reduction and obtain a 12-dimensional reduced dynamical system. Then, we compute a normal form for this reduced system and find the system (1.1)-(1.2) to leading order after an appropriate rescaling of the normal form. The unknowns A_0 and B_0 in the system (1.1)-(1.2) represent the rescaled amplitudes of the two critical eigenmodes at the instability threshold, the power ε^4 of the small parameter ε is proportional to the positive difference $\mathcal{R}^{1/2}-\mathcal{R}_c^{1/2}$, which is the bifurcation parameter, and g depends on the physical parameters. A computation of g shows that $g > g_0$ for some $g_0 > 1$. This first step is carried out in Section 2.

Solutions of the system (1.1)-(1.2) provide leading order approximations of solutions of the full governing equations. In particular, the equilibrium $(A_0, B_0) = (0, 1)$ of the system (1.1)-(1.2) gives an approximation of convection rolls bifurcating for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to $\mathcal{R}_c$, whereas the equilibrium $(A_0, B_0) = (1, 0)$ of the system (1.1)-(1.2) gives the same convection rolls but rotated by an angle $\pi/2$. A heteroclinic orbit connecting these two equilibria provides then an approximation of orthogonal domain walls. Our main result shows the existence of such heteroclinic orbits for the system (1.1)-(1.2).

Theorem 1. *For any $\varepsilon > 0$ and $g > 1$, the system* (1.1)-(1.2) *possesses* *a smooth real-valued heteroclinic solution* $(A_0, B_0) = (A_{\varepsilon,g}, B_{\varepsilon,g})$ *with the following properties:*

(i) $\lim_{x\to-\infty}(A_{\varepsilon,g}(x), B_{\varepsilon,g}(x)) = (1,0)$ *and* $\lim_{x\to\infty}(A_{\varepsilon,g}(x), B_{\varepsilon,g}(x)) = (0,1)$*;*

(ii) $B_{\varepsilon,g}(x) \geqslant 0$*, for all* $x \in \mathbb{R}$*;*

(iii) *for fixed* $\varepsilon > 0$*,* $\lim_{g\to 1^+} \sup_{x\in\mathbb{R}} |A_{\varepsilon,g}(x)^2 + B_{\varepsilon,g}(x)^2 - 1| = 0.$

After some rescaling, the limit $\varepsilon = 0$ is also considered, as it could give indications of how the heteroclinic orbits look like for small $\varepsilon > 0$. Our analysis of the system (1.1)-(1.2) is only valid for $g > 1$ and we don't know whether such heteroclinic solutions exist for $g \leqslant 1$. The result in Theorem 1(iii) indicates that these heteroclinic solutions cease to exist at $g = 1$.

The proof of Theorem 1 is given in Section 3. The heteroclinic solution being real-valued, it is a solution of the 6-dimensional system obtained by restricting the system (1.1)-(1.2) to real-valued functions A_0 and B_0. We use a variational method in which the heteroclinic solution is found as a minimizer of a rescaled functional. A compactness by concentration type argument is used to prove the convergence of minimizing sequences towards the heteroclinic solution.

Our analysis also provides the first necessary steps towards an existence proof for orthogonal domain walls for the Rayleigh-Bénard convection problem. For a complete proof it remains to show that the heteroclinic orbit found in Theorem 1 persists as a perturbed heteroclinic solution for the full 12-dimensional reduced system, hence without restricting to the leading order system (1.1)-(1.2). Relying upon Implicit Function Theorem arguments such a proof was given for symmetric domain walls in [7, 8], but we were not able to obtain a full proof in the case of orthogonal domain walls so far. The main obstacle in the present case is the analysis of the kernel of the linear operator obtained by linearizing the (1.1)-(1.2) about the heteroclinic solutions found in Theorem 1. This is also related to the local uniqueness question which remains open as well.

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2. Derivation of the amplitude equations

Relying upon a center manifold reduction and a normal forms analysis, we derive the system of amplitude equations (1.1)-(1.2) from the Bénard-

Rayleigh convection problem. This derivation being similar to that of the leading order systems in [7, 8], we recall the main steps, focus on differences, and refer to these works for further details.

2.1. Formulation of the hydrodynamic problem

We consider the formulation as a dynamical system of the governing equations for the steady convection problem from [7]. In Cartesian coordinates $(x, y, z) \in \mathbb{R}^3$, where (x, y) are the horizontal coordinates and z is the vertical coordinate, after rescaling variables, the fluid occupies the domain $\mathbb{R}^2 \times (0, 1)$. The physical variables are the particle velocity $\mathbf{V} = (V_x, V_y, V_z)$, the deviation θ of the temperature from the conduction profile, and the pressure p. There are two dimensionless parameters, the Rayleigh number $\mathcal{R}$ and the Prandtl number $\mathcal{P}$. We refer to [11] for more details on the governing equations.

Taking the horizontal coordinate x as evolutionary variable, the governing equations are written as a system of the form

$$\partial_x \mathbf{U} = \mathcal{L}_\mu \mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{2.1}$$

with $\mathbf{U} = (V_x, V_\perp, W_x, W_\perp, \theta, \phi)$ an 8-components vector, in which $V_\perp = (V_y, V_z)$, $W_\perp = (W_y, W_z)$, and $\mathbf{W} = (W_x, W_\perp)$ and ϕ are additional variables defined by

$$\mathbf{W} = \mu^{-1}\partial_x \mathbf{V} - p\mathbf{e}_x, \quad \phi = \partial_x \theta, \tag{2.2}$$

where $\mathbf{e}_x = (1, 0, 0)$. The parameter μ is the square root of the Rayleigh number, $\mu = \mathcal{R}^{1/2}$, and $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ in the right hand side of (2.1) are linear and quadratic operators, respectively, defined by

$$\mathcal{L}_\mu \mathbf{U} = \begin{pmatrix} -\nabla_\perp \cdot V_\perp \\ \mu W_\perp \\ -\mu^{-1}\Delta_\perp V_x \\ -\mu^{-1}\Delta_\perp V_\perp - \theta \mathbf{e}_z^\perp - \mu^{-1}\nabla_\perp(\nabla_\perp \cdot V_\perp) - \nabla_\perp W_x \\ \phi \\ -\Delta_\perp \theta - \mu V_z \end{pmatrix},$$

$$\mathcal{B}_\mu(\mathbf{U}, \mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_x - V_x(\nabla_\perp \cdot V_\perp)\big) \\ \mathcal{P}^{-1}\big((V_\perp \cdot \nabla_\perp)V_\perp + \mu V_x W_\perp\big) \\ 0 \\ \mu\big((V_\perp \cdot \nabla_\perp)\theta + V_x \phi\big) \end{pmatrix},$$

where $\Delta_\perp = \partial_{yy} + \partial_{zz}$, $\nabla_\perp = (\partial_y, \partial_z)$, and here $\mathbf{e}_z^\perp = (0, 1)$.

The phase space $\mathcal{X}$ for the dynamical system (2.1) and the domain of definition $\mathcal{Z}$ of the linear operator $\mathcal{L}_\mu$ include the boundary conditions and a condition on the flux. We consider periodic boundary conditions in y and the case of "rigid-rigid" boundary conditions in z:[1]

$$\mathbf{V}|_{z=0,1} = 0, \quad \theta|_{z=0,1} = 0. \tag{2.3}$$

Taking the period $2\pi/k$ in y, for some fixed $k > 0$, a direct calculation shows that the derivative of the flux

$$\mathcal{F}(x) = \int_{\Omega_{per}} V_x \, dy \, dz, \quad \Omega_{per} = (0, 2\pi/k_y) \times (0, 1), \tag{2.4}$$

vanishes, hence $\mathcal{F}(x)$ is a constant function (see [7, Section 3]).

Fixing the constant flux to 0, the phase space $\mathcal{X}$ is defined by

$$\mathcal{X} = \Big\{ \mathbf{U} \in \widetilde{\mathcal{X}} \; ; \; V_x = V_\perp = \theta = 0 \text{ on } z = 0, 1, \text{ and } \int_{\Omega_{per}} V_x \, dy \, dz = 0 \Big\},$$

where

$$\widetilde{\mathcal{X}} = (H^1_{per}(\Omega))^3 \times (L^2_{per}(\Omega))^3 \times H^1_{per}(\Omega) \times L^2_{per}(\Omega),$$

and the subscript *per* means that the functions are $2\pi/k$-periodic in y (for simplicity, we have written $V_x = V_\perp = \theta = 0$ although these vectors do not have the same dimension). The boundary conditions (2.3) and the flux (2.4) being well-defined on $\widetilde{\mathcal{X}}$, they are included in the definition of the phase space $\mathcal{X}$ which is a closed subspace of $\widetilde{\mathcal{X}}$. Equipped with the scalar product of $\widetilde{\mathcal{X}}$, the phase space $\mathcal{X}$ is a Hilbert space.

The domain of definition $\mathcal{Z}$ of the linear operator $\mathcal{L}_\mu$ is defined by

$$\begin{aligned} \mathcal{Z} = \Big\{ \mathbf{U} \in \mathcal{X} \cap (H^2_{per}(\Omega))^3 \times (H^1_{per}(\Omega))^3 \times H^2_{per}(\Omega) \times H^1_{per}(\Omega) \; ; \\ \nabla_\perp \cdot V_\perp = W_\perp = \phi = 0 \text{ on } z = 0, 1 \Big\}, \end{aligned}$$

such that $\mathcal{L}_\mu$ is a closed operator in $\mathcal{X}$ with dense and compactly embedded domain. An immediate consequence of the latter property is that the linear

[1]The subsequent analysis remains valid for other types of boundary conditions in z; see [7, Section 8] and [8, Section 2] for the definition of the spaces $\mathcal{X}$ and $\mathcal{Z}$ in the cases of "free-free" and "rigid-free" boundary conditions, respectively.

operator $\mathcal{L}_\mu$ has compact resolvent and therefore pure point spectrum consisting of isolated eigenvalues with finite algebraic multiplicity [10, p.187]. Finally, the quadratic operator $\mathcal{B}_\mu$ is well-defined on $\mathcal{Z}$ with values in $\mathcal{X}$.

As a consequence of the symmetries of the hydrodynamic problem, the dynamical system (2.1) is reversible with reversibility symmetry

$$\boldsymbol{S}_1\boldsymbol{U}(y,z) = (-V_x, V_\perp, W_x, -W_\perp, \theta, -\phi)(y,z), \quad \boldsymbol{U} \in \mathcal{X},$$

which anti-commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$, and $O(2)$-equivariant with discrete symmetry

$$\boldsymbol{S}_2\boldsymbol{U}(y,z) = (V_x, -V_y, V_z, W_x, -W_y, W_z, \theta, \phi)(-y,z), \quad \boldsymbol{U} \in \mathcal{X},$$

and continuous symmetry $(\boldsymbol{\tau}_a)_{a\in\, \mathbb{R}/2\pi\mathbb{Z}}$,

$$\boldsymbol{\tau}_a\boldsymbol{U}(y,z) = \boldsymbol{U}(y + a/k_y, z), \quad \boldsymbol{U} \in \mathcal{X},$$

which commute with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$ and satisfy

$$\boldsymbol{\tau}_a\boldsymbol{S}_2 = \boldsymbol{S}_2\boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_0 = \boldsymbol{\tau}_{2\pi} = \mathbb{I}.$$

The symmetries $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ follow from the reflections $x \mapsto -x$ and $y \mapsto -y$, respectively, whereas the continuous symmetry $\boldsymbol{\tau}_a$ is a consequence of the invariance under translations in y of the governing equations.[2] In addition, the system does not change when adding any constant to the new variable W_x, i.e., it is invariant under the action of the one-parameter family of maps $(\boldsymbol{T}_b)_{b\in\mathbb{R}}$ defined by

$$\boldsymbol{T}_b\boldsymbol{U} = \boldsymbol{U} + b\boldsymbol{\varphi}_0, \quad \boldsymbol{\varphi}_0 = (0,0,0,1,0,0,0,0)^t, \quad \boldsymbol{U} \in \mathcal{X}. \tag{2.5}$$

We keep track of these symmetries at each step of our reduction procedure, hence ensuring that they are correctly reproduced by the amplitude equations (1.1)-(1.2).

In this setting, the classical convection rolls are equilibria of the dynamical system (2.1). As explained in [7, Section 4], these rolls provide a circle of

[2]In the case of "rigid-rigid" and "free-free" boundary conditions, there is an additional vertical reflection symmetry $z \mapsto 1 - z$ leading to the symmetry $\boldsymbol{S}_3\mathbf{U}(y,z) = (V_x, V_y, -V_z, W_x, W_y, -W_z, -\theta, -\phi)(y, 1-z)$ which commutes with $\mathcal{L}_\mu$ and $\mathcal{B}_\mu$. Aiming for a result which is also valid in the case of "rigid-free" boundary conditions, we do not make use of this symmetry.

equilibria $\boldsymbol{\tau}_a(\mathbf{U}^*_{k,\mu})$, for $a \in \mathbb{R}/2\pi\mathbb{Z}$, of the dynamical system (2.1) which bifurcate for $\mu > \mu_0(k)$ sufficiently close to a critical value $\mu_0(k)$, for any fixed wavenumber k. Due to the rotation invariance of the hydrodynamic problem, horizontally rotated rolls are solutions of the dynamical system (2.1). In particular, for the rotation angle of $\pi/2$, we obtain solutions which are $2\pi/k$-periodic in x and constant in y. Orthogonal domain walls could then be constructed as heteroclinic orbits connecting these latter periodic solutions with the equilibria $\mathbf{U}^*_{k,\mu}$. According to the classical theory, the map $k \mapsto \mu_0(k)$ is analytic in k and has a strict global minimum at $k = k_c$ where $\mu_0''(k_c) > 0$. The values k_c, $\mu_0(k_c)$ and $\mu_0''(k_c)$ depend on the imposed boundary conditions at $z = 0, 1$ and can be computed numerically. We refer to [7, Section 2.1] for a more detailed discussion of these properties.

2.2. Reduced dynamics

We consider the parameter regime with (k, μ) close to (k_c, μ_c), where $\mu_c = \mu_0(k_c)$. We set

$$\mu = \mu_c + \widetilde{\mu}, \quad k = k_c(1 + \widetilde{k}), \tag{2.6}$$

in which $\widetilde{\mu}$ and $\widetilde{k}$ are small parameters. We also eliminate the dependence on k of the phase space $\mathcal{X}$ of the dynamical system (2.1) by normalizing to $2\pi/k_c$ the period in y of the solutions. The resulting system is of the form (2.1) in which now $\Delta_\perp = (1 + \widetilde{k})^2\partial_{yy} + \partial_{zz}$, $\nabla_\perp = ((1 + \widetilde{k})\partial_y, \partial_z)$, and its phase space is $\mathcal{X}$ with $k = k_c$. We write this system in the form

$$\partial_x \mathbf{U} = \mathcal{L}_c \mathbf{U} + \mathcal{R}(\mathbf{U}, \widetilde{\mu}, \widetilde{k}), \tag{2.7}$$

where

$$\mathcal{L}_c = \mathcal{L}_{\mu_c}\big|_{\widetilde{k}=0}, \quad \mathcal{R}(\mathbf{U}, \widetilde{\mu}, \widetilde{k}) = (\mathcal{L}_\mu - \mathcal{L}_{\mu_c}\big|_{\widetilde{k}=0})\mathbf{U} + \mathcal{B}_\mu(\mathbf{U}, \mathbf{U}), \tag{2.8}$$

and $\mathcal{R}$ is a smooth map from $\mathcal{Z} \times (-\mu_c, \infty) \times \mathbb{R}$ into $\mathcal{X}$ satisfying

$$\mathcal{R}(0, \widetilde{\mu}, \widetilde{k}) = 0, \quad D_{\mathbf{U}}\mathcal{R}(0, 0, 0) = 0. \tag{2.9}$$

We apply a center manifold theorem to obtain a reduced system of ordinary differential equations which describes the dynamics of (2.7) in a neighborhood of the equilibrium $\mathbf{U} = 0$ for small $(\widetilde{\mu}, \widetilde{k})$. The arguments are the same as the ones from [7, Section 5], except for the center spectrum (the set of eigenvalues with zero real part) of the linear operator $\mathcal{L}_c$ which is different. The following result is obtained by taking the limit $\alpha = 0$ in the result from [7, Lemma 4.2].

Lemma 2.1. *The center spectrum of the linear operator $\mathcal{L}_c$ consists of the three eigenvalues $0, \pm ik_c$ with the following properties.*

(i) *The eigenvalue 0 has algebraic multiplicity 9 and geometric multiplicity 3, and the complex conjugated eigenvalues $\pm ik_c$ are algebraically double and geometrically simple.*

(ii) *For the eigenvalue 0, there are three linearly independent eigenvectors: $\boldsymbol{\varphi}_0$ given by (2.5), $\boldsymbol{\zeta}_0$ of the form $\boldsymbol{\zeta}_0(y,z) = \widehat{U}_{k_c}(z)e^{ik_c y}$, with $\widehat{U}_{k_c}(z) \in \mathbb{C}^8$, and the complex conjugated vector $\overline{\boldsymbol{\zeta}}_0$, and two chains of generalized eigenvectors: $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3$ associated to $\boldsymbol{\zeta}_0$,* [3]

$$\mathcal{L}_c\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_0, \quad \mathcal{L}_c\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_1, \quad \mathcal{L}_c\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_2,$$

and the conjugated vectors $\overline{\boldsymbol{\zeta}_1}, \overline{\boldsymbol{\zeta}_2}, \overline{\boldsymbol{\zeta}_3}$ associated to $\overline{\boldsymbol{\zeta}_0}$. The eigenvector $\boldsymbol{\varphi}_0$ is invariant under the actions of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{\tau}_a$, and the other generalized eigenvectors satisfy:

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_0 = \overline{\boldsymbol{\zeta}_0}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_0 = e^{ia}\boldsymbol{\zeta}_0,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_1 = -\boldsymbol{\zeta}_1, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_1 = \overline{\boldsymbol{\zeta}_1}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_1 = e^{ia}\boldsymbol{\zeta}_1,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_2, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_2 = \overline{\boldsymbol{\zeta}_2}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_2 = e^{ia}\boldsymbol{\zeta}_2,\\
&\boldsymbol{S}_1\boldsymbol{\zeta}_3 = -\boldsymbol{\zeta}_3, \quad \boldsymbol{S}_2\boldsymbol{\zeta}_3 = \overline{\boldsymbol{\zeta}_3}, \quad \boldsymbol{\tau}_a\boldsymbol{\zeta}_3 = e^{ia}\boldsymbol{\zeta}_3.
\end{aligned}$$

(iii) *For the eigenvalue ik_c, there is one eigenvector $\boldsymbol{\xi}_0$ of the form $\boldsymbol{\xi}_0(y,z) = \widehat{\mathbf{U}}_0(z) \in \mathbb{C}^8$, and an associated generalized eigenvector $\boldsymbol{\xi}_1$ with the properties*

$$(\mathcal{L}_c - ik_c)\boldsymbol{\xi}_1 = \boldsymbol{\xi}_0,$$

and

$$\begin{aligned}
&\boldsymbol{S}_1\boldsymbol{\xi}_0 = \overline{\boldsymbol{\xi}_0}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0,\\
&\boldsymbol{S}_1\boldsymbol{\xi}_1 = -\overline{\boldsymbol{\xi}_1}, \quad \boldsymbol{S}_2\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1, \quad \boldsymbol{\tau}_a\boldsymbol{\xi}_1 = \boldsymbol{\xi}_1.
\end{aligned}$$

The complex conjugated vectors $\overline{\boldsymbol{\xi}_0}$ and $\overline{\boldsymbol{\xi}_1}$ are eigenvector and generalized eigenvector, respectively, for the eigenvalue $-ik_c$.

[3] For our purposes, we do not need the explicit formulas for eigenvectors and generalized eigenvectors which can be obtained from [7, Section 4].

As a consequence of this lemma, the spectral subspace associated with the center spectrum of the linear operator $\mathcal{L}_c$ has dimension 13 and as a result of the center manifold theorem, the infinite-dimensional dynamical system (2.7) possesses a 13-dimensional local center manifold, for any sufficiently small $\widetilde{\mu}$ and $\widetilde{k}$. Solutions $\mathbf{U} : \mathbb{R} \to \mathcal{Z}$ of the dynamical system (2.7) which are bounded on $\mathbb{R}$ and sufficiently small belong to this local center manifold, and are of the form

$$\begin{aligned} \boldsymbol{U}(x) \;\; = \;\; & w(x)\boldsymbol{\varphi}_0 + A_0(x)\boldsymbol{\zeta}_0 + A_1(x)\boldsymbol{\zeta}_1 + A_2(x)\boldsymbol{\zeta}_2 + A_3(x)\boldsymbol{\zeta}_3 \\ & + \overline{A_0(x)\boldsymbol{\zeta}_0} + \overline{A_1(x)\boldsymbol{\zeta}_1} + \overline{A_2(x)\boldsymbol{\zeta}_2} + \overline{A_3(x)\boldsymbol{\zeta}_3} \\ & + B_0(x)\boldsymbol{\xi}_0 + B_1(x)\boldsymbol{\xi}_1 + \overline{B_0(x)\boldsymbol{\xi}_0} + \overline{B_1(x)\boldsymbol{\xi}_1} \\ & + \boldsymbol{\Phi}(w(x), X(x), \overline{X(x)}, \widetilde{\mu}, \widetilde{k}), \end{aligned} \tag{2.10}$$

in which the x-dependent functions w and $X = (A_0, A_1, A_2, A_3, B_0, B_1)$ take values in $\mathbb{R}$ and $\mathbb{C}^6$, respectively. The eigenvectors, except $\boldsymbol{\varphi}_0$, and the generalized eigenvalues in Lemma 2.1 being complex-valued, it is convenient to use the complex variables $(X, \overline{X})$, instead of 12 real variables, hence identifying $\mathbb{R}^{12}$ with the space $\mathbb{C}^6 \times \overline{\mathbb{C}^6} = \{(Z, \overline{Z})\ ;\ Z \in \mathbb{C}^6\}$. The map $\boldsymbol{\Phi}$ is defined on $\mathbb{R} \times \mathbb{C}^6 \times \overline{\mathbb{C}^6} \times (-\mu_c, \infty) \times \mathbb{R}$ and can be chosen of class C^m for any arbitrary, but fixed, $m \geqslant 1$.

The reduced 13-dimensional system for w, X, and $\overline{X}$ inherits the symmetries of the infinite-dimensional dynamical system (2.1) listed in Section 2.1. The invariance of (2.1) under the action of $\boldsymbol{T}_b$, implies that the reduced vector field is invariant under the action of the induced transformation $w \mapsto w + b$, for any $b \in \mathbb{R}$, and therefore does not depend on w. Consequently, the equations for w and $(X, \overline{X})$ in the reduced system are decoupled,

$$\frac{dw}{dx} = h(X, \overline{X}, \widetilde{\mu}, \widetilde{k}), \tag{2.11}$$

$$\frac{dX}{dx} = F(X, \overline{X}, \widetilde{\mu}, \widetilde{k}), \quad \frac{d\overline{X}}{dx} = \overline{F(X, \overline{X}, \widetilde{\mu}, \widetilde{k})}. \tag{2.12}$$

so that we can restrict to the system (2.12) for $(X, \bar{X})$, the component w being computed by directly integrating (2.11). Next, from the symmetry properties of the eigenvectors and generalized eigenvectors in Lemma 2.1, we deduce their actions on the variable $X = (A_0, A_1, A_2, A_3, B_0, B_1)$,

$$\boldsymbol{S}_1(A_0, A_1, A_2, A_3, B_0, B_1) = (A_0, -A_1, A_2, -A_3, \overline{B_0}, -\overline{B_1}), \tag{2.13}$$
$$\boldsymbol{S}_2(A_0, A_1, A_2, A_3, B_0, B_1) = (\overline{A_0}, \overline{A_1}, \overline{A_2}, \overline{A_3}, B_0, B_1), \tag{2.14}$$
$$\boldsymbol{\tau}_a(A_0, A_1, A_2, A_3, B_0, B_1) = (e^{ia}A_0, e^{ia}A_1, e^{ia}A_2, e^{ia}A_3, B_0, B_1). \tag{2.15}$$

Then, the vector field in the reduced system (2.12) anti-commutes with $\boldsymbol{S}_1$ and commutes with $\boldsymbol{S}_2$ and $\boldsymbol{\tau}_a$. Notice that the equivariance under the action of $\boldsymbol{S}_2$ implies that the reduced system leaves invariant the 8-dimensional subspace $\{(X, \overline{X})\ ;\ A_j = \overline{A_j},\ j = 0, 1, 2, 3\}$. Solutions in this subspace correspond to solutions of (2.1) which are even in y. There is a second invariant subspace $\{(X, \overline{X})\ ;\ A_j = 0,\ j = 0, 1, 2, 3\}$, which corresponds to solutions of (2.1) which do not depend on y.

Finally, from the properties (2.8)-(2.9) and the result in Lemma 2.1 we obtain that

$$F(0, 0, \widetilde{\mu}, \widetilde{k}) = 0, \quad D_X F(0, 0, 0, 0) = L_c, \quad D_{\overline{X}} F(0, 0, 0, 0) = 0, \tag{2.16}$$

where L_c is the 6×6 Jordan matrix

$$L_c = \begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} ik_c & 1 \\ 0 & ik_c \end{pmatrix}. \tag{2.17}$$

2.3. Leading order dynamics

The next step consists in a normal form transformation of the reduced system (2.12). In Appendix A we compute a normal form for 12-dimensional vector fields which satisfy the properties (2.16)-(2.17) and have the symmetries (2.13)-(2.15). Applying this result to the system (2.12), we then prove that the coefficients of the polynomials P_3, Q_0, and Q_1 appearing in the vector field N from Lemma A1(iii) satisfy

$$d_0' = \alpha_0' = \beta_0' = 0, \quad d_0 = -4k_c^2\beta_0 > 0, \quad d_1 = -4k_c^2\beta_5 < 0, \quad \frac{\beta_1}{\beta_5} = \frac{d_5}{d_1} > 1.$$

These properties are obtained after long, but standard, computations (see [7, Appendix B.2] and [8, Appendix A]).

Following [7, Section 6.3] and [8, Section 4], we assume that $\widetilde{\mu}$ given by (2.6) is positive and rescale variables in the normal form (A.1) by

$$x = \tfrac{1}{2\varepsilon k_c}\widetilde{x}, \quad \widetilde{\mu} = \tfrac{4k_c^2}{-\beta_0}\varepsilon^4, \quad \widetilde{k} = \varepsilon^2\widehat{k}, \tag{2.18}$$

$$A_0(x) = \tfrac{2k_c}{\sqrt{\beta_5}}\varepsilon^2\widetilde{A_0}(\widetilde{x}), \quad A_1(x) = \tfrac{4k_c^2}{\sqrt{\beta_5}}\varepsilon^3\widetilde{A_1}(\widetilde{x}), \tag{2.19}$$

$$A_2(x) = \tfrac{8k_c^3}{\sqrt{\beta_5}}\varepsilon^4\widetilde{A_2}(\widetilde{x}), \quad A_3(x) = \tfrac{16k_c^4}{\sqrt{\beta_5}}\varepsilon^5\widetilde{A_3}(\widetilde{x}), \tag{2.20}$$

$$B_0(x) = \tfrac{2k_c}{\sqrt{\beta_5}}\varepsilon^2 e^{\frac{i}{2\varepsilon}\widetilde{x}}\widetilde{B_0}(\widetilde{x}), \quad B_1(x) = \tfrac{4k_c^2}{\sqrt{\beta_5}}\varepsilon^3 e^{\frac{i}{2\varepsilon}\widetilde{x}}\widetilde{B_1}(\widetilde{x}). \tag{2.21}$$

Notice here the exponential factor $e^{\frac{i}{2\varepsilon}\widetilde{x}}$ in the formulas for B_0 and B_1. Then, taking into account the properties of the coefficients above we obtain the rescaled system

$$\begin{aligned}
\frac{d\widetilde{A_0}}{d\widetilde{x}} &= \widetilde{A_1} + O(|\varepsilon|^2(|\widehat{k}|^2 + |\varepsilon|^2)),\\
\frac{d\widetilde{A_1}}{d\widetilde{x}} &= \widetilde{A_2} + O(|\widehat{k}| + |\varepsilon|^2),\\
\frac{d\widetilde{A_2}}{d\widetilde{x}} &= \widetilde{A_3} + O(|\widehat{k}| + |\varepsilon|^2),\\
\frac{d\widetilde{A_3}}{d\widetilde{x}} &= \widetilde{A_0}(1 - |\widetilde{A_0}|^2 - g|\widetilde{B_0}|^2) + O(|\widehat{k}| + |\varepsilon|),\\
\frac{d\widetilde{B_0}}{d\widetilde{x}} &= \widetilde{B_1} + O(|\varepsilon|(|\widehat{k}| + |\varepsilon|^2)),\\
\frac{d\widetilde{B_1}}{d\widetilde{x}} &= \varepsilon^2\widetilde{B_0}(-1 + g|\widetilde{A_0}|^2 + |\widetilde{B_0}|^2) + O(|\varepsilon|^2(|\widehat{k}| + |\varepsilon|)).
\end{aligned}$$

Keeping only the leading order terms in each equation, the resulting system is equivalent to the system (1.1)-(1.2) in which

$$g := \frac{\beta_1}{\beta_5} = \frac{d_5}{d_1} > 1.$$

Interestingly, the computation of g shows that it is equal to the ratio g computed in [7] in the particular case of symmetric domain walls when the angle between the rotated rolls is equal to $\pi/2$ (rotation angle $\alpha = \pi/4$ in that work). As a result, we find that g is a function of the Prandtl number $\mathcal{P}$

and satisfies $g > g_0$ for some constant $g_0 > 1$.[4]

The real equilibrium $M_+ = (0, 1)$ of the system (1.1)-(1.2) corresponds to the roll solution $\mathbf{U}^*_{k,\mu}$ of the dynamical system (2.1), whereas the real equilibrium $M_- = (1, 0)$ corresponds to the same roll solution rotated by an angle $\pi/2$. Consequently, a domain wall connecting these two orthogonal rolls in the Bénard-Rayleigh problem corresponds to a heteroclinic connection between these two real equilibria of the system (1.1)-(1.2) (for further details, see [7, Section 6.3] and [8, Section 4.2]).

Remark 2.1. *An approximate qualitative, but also quantitative, description of orthogonal domain walls could be obtained from the formula* (2.10) *for the solutions* $\mathbf{U}(x)$ *on the center manifold provided the heteroclinic solution of the system* (1.1)-(1.2) *is known explicitly through analytical formulas or numerically. Indeed, taking into account the scaling* (2.18)-(2.21)*, a leading order approximation of the physical variables* $\boldsymbol{V} = (V_x, V_y, V_z)$ *and* θ *is given by the first three components and the seventh component, respectively, of the vector in the expression*

$$\frac{2k_c}{\sqrt{\beta_5}}\varepsilon^2\left(\widetilde{A}_0(2\varepsilon k_c x)\boldsymbol{\zeta}_0 + \widetilde{B}_0(2\varepsilon k_c x)e^{ik_c x}\boldsymbol{\xi}_0 + c.c.\right),$$

where c.c. stands for the complex conjugated terms, $(\widetilde{A}_0, \widetilde{B}_0)$ *is the heteroclinic solution of the system* (1.1)-(1.2)*, and the other quantities (*k_c*,* β_5 *and the eigenvectors* $\boldsymbol{\zeta}_0, \boldsymbol{\xi}_0$*) can be found from the results in [7, 8], after some more computations. This computation of domain walls is outside the scope of the present work.*

3. Existence of a heteroclinic orbit

In this section we prove the result in Theorem 1. We restrict to real-valued solutions and choose new scales by taking

$$\epsilon = \varepsilon^4 > 0, \quad \bar{x} = \epsilon^{1/4}x, \quad A_0(x) = \bar{A}(\bar{x}), \quad B_0(x) = \bar{B}(\bar{x}).$$

[4] The precise value of g_0 depends on the considered boundary conditions: $g_0 \approx 1.227$ for "rigid-rigid" boundary conditions, $g_0 = 673/473$ for "free-free" boundary conditions, and $g_0 \approx 1.332$ for "rigid-free" boundary conditions.

Then the system (1.1)-(1.2) becomes

$$\epsilon \frac{d^4 \bar{A}}{d\bar{x}^4} = \bar{A}(1 - \bar{A}^2 - g\bar{B}^2),$$
$$\frac{d^2 \bar{B}}{d\bar{x}^2} = \bar{B}(-1 + g\bar{A}^2 + \bar{B}^2),$$

and, after suppression of bars (for simplicity),

$$\epsilon \frac{d^4 A}{dx^4} = A(1 - A^2 - gB^2), \tag{3.1}$$
$$\frac{d^2 B}{dx^2} = B(-1 + gA^2 + B^2). \tag{3.2}$$

We construct the heteroclinic orbit as a minimizer of the functional

$$J_\epsilon(A,B) := \int_{\mathbb{R}} \Big(\frac{\epsilon}{2} A''^2 + \frac{1}{2} B'^2 + \frac{1}{4}(A^2 + B^2 - 1)^2 + \frac{1}{2}(g-1)A^2B^2 \Big) dx,$$

on the set X of real-valued functions $(A,B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ such that

$$\lim_{x\to-\infty} (A(x), B(x)) = (1,0) \quad \text{and} \quad \lim_{x\to\infty} (A(x), B(x)) = (0,1). \tag{3.3}$$

For any $\epsilon > 0$ and $g > 1$ this functional is nonnegative, $J_\epsilon(A,B) \in [0,\infty]$. In fact J_ϵ is more generally defined on $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ with values in $[0,+\infty]$, that is, without restricting ourselves to functions satisfying (3.3). A delicate issue will be, once a solution $(A,B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ is obtained with $J_\epsilon(A,B) < \infty$, to check that indeed (3.3) is satisfied.

Setting

$$P(A,B) = \frac{1}{4}\Big(A^2 + B^2 - 1\Big)^2 + \frac{1}{2}(g-1)A^2B^2\,,$$

a stationary point $(A,B) \in X$ of J_ϵ satisfies the system

$$\epsilon A'''' + \partial_A P(A,B) = 0, \quad -B'' + \partial_B P(A,B) = 0,$$

in the sense of distributions. This system is precisely the system (3.1)-(3.2). Notice that a standard bootstrap argument shows that any solution $(A,B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ is smooth if $\epsilon > 0$ (by 'smooth' we mean 'C^∞').

3.1. The case $\epsilon = 0$

Although the case $\epsilon = 0$ is not part of the statement of Theorem 1, we nevertheless mention this case, because it could give an additional insight on the problem when $\epsilon > 0$ is small. From our point of view, the main interest of the case $\epsilon = 0$ is the possibility to obtain an explicit heteroclinic orbit (the second order differential equation (3.5) for B has the explicit increasing solution (3.7), A being then given by (3.6)). Besides the attractive aspect of an explicit formula, there is also the hope that, from this explicit solution, a perturbative argument could be set up. Conversely, it is natural to ask whether the solutions we obtain for $\epsilon > 0$ converge in some sense to the explicit solution obtained for $\epsilon = 0$, but to start such a discussion here would be beyond the scope of the present paper.

For $\epsilon = 0$ and $g > 1$, we have the functional

$$J_0(A, B) = \int_{\mathbb{R}} \Big(\frac{1}{2}B'^2 + \frac{1}{4}(A^2 + B^2 - 1)^2 + \frac{1}{2}(g-1)A^2B^2\Big)dx \in [0, \infty],$$

For fixed B, one can minimize with respect to A. Differentiating the map

$$A \to f(A) := \frac{1}{4}(A^2 + B^2 - 1)^2 + \frac{1}{2}(g-1)A^2B^2\,,$$

one gets the equation for A:

$$(A^2 + gB^2 - 1)A = 0.$$

Hence critical points satisfy $A = 0$ or $A^2 = 1 - gB^2$ if $1 - gB^2 \geqslant 0$. As $f''(A) = 3A^2 - (1 - gB^2)$, we see that if $1 - gB^2 > 0$, then $f''(0) < 0$ and the minimum of f is reached at $A = \pm\sqrt{1 - gB^2}$. Consequently, $A = 0$ if $1 - gB^2 \leqslant 0$, and $A = \pm\sqrt{1 - gB^2}$ if $1 - gB^2 \geqslant 0$, or equivalently,

$$A^2 = \max\{0, 1 - gB^2\} = (1 - gB^2)_+. \tag{3.4}$$

Substituting A^2 above in $J_0(A, B)$, one gets the reduced functional

$$J_{\mathrm{red}}(B) = \int_{\mathbb{R}} \Big(\frac{1}{2}B'^2 + \frac{1}{4}((1-gB^2)_+ + B^2 - 1)^2 + \frac{1}{2}(g-1)(1-gB^2)_+B^2\Big)dx \in [0, \infty],$$

which depends on $B \in H^1_{loc}(\mathbb{R})$, only. Observe that A'' does no more appear in J_0 and this is why we let (A, B) be a priori in $C(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ when dealing with J_0. The explicit solution $(A,B) \in C(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ given below by (3.7) and (3.6) is such that A is not C^1.

A stationary point $(A,B) \in C(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ of J_0 satisfies

$$\partial_A P(A,B) = 0, \quad -B'' + \partial_B P(A,B) = 0,$$

or equivalently,

$$A(A^2 + gB^2 - 1) = 0, \quad -B'' + B(gA^2 + B^2 - 1) = 0,$$

in the sense of distributions for the second equation, together with the property (3.3) for the limits at $x = \pm\infty$. Consequently, $A = 0$ or $A^2 = 1 - gB^2$ if $B^2 \leqslant 1/g$, hence leading to the equation for B (we use more precisely (3.4))

$$B'' = \begin{cases} -B + B^3 & \text{if } B^2 \geqslant 1/g, \\ (g-1)B + (1-g^2)B^3 & \text{if } B^2 \leqslant 1/g. \end{cases} \tag{3.5}$$

Observe that the right-hand side is continuous. This problem has an increasing solution $B > 0$ of class C^2 such that

$$\lim_{x\to-\infty} B(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} B(x) = 1,$$

which gives a solution of (3.1)-(3.2) by taking

$$A = \sqrt{1 - gB^2} \ \text{ if } \ 0 < B \leqslant 1/\sqrt{g} \ \text{ and } \ A = 0 \ \text{ if } \ B \geqslant 1/\sqrt{g}. \tag{3.6}$$

Indeed $B = 1$ is an hyperbolic equilibrium of the first equation in (3.5) and $B = 0$ is an hyperbolic equilibrium of the second equation in (3.5) because $g > 1$. On the other hand, the first equation possesses the invariant $|B'|^2 + B^2 - \frac{1}{2}B^4$, which is $1/2$ at the equilibrium $B = 1$, and the second equation possesses the invariant $|B'|^2 + (1-g)B^2 - (1-g^2)\frac{1}{2}B^4$, which is 0 at the equilibrium $B = 0$. Let us study the curves for $B = \pm\sqrt{1/g}$ in the plane (B, B'). From $|B'|^2 + \frac{1}{g} - \frac{1}{2g^2} = 1/2$ that corresponds to the first equation, one gets $|B'|^2 = \frac{1}{2} - \frac{1}{g} + \frac{1}{2g^2}$. This is the same value of $|B'|^2$ that one gets by solving the second equation: $|B'|^2 + \frac{1-g}{g} - \frac{1-g^2}{2g^2} = 0$. This shows that B' is continuous (if its sign does not jump) at the junction of the two curves in the (B, B') plane. Hence there is a heteroclinic solution coming from $(B, B') = (0, 0)$, staying on the set $|B'|^2 + (1-g)B^2 - (1-g^2)\frac{1}{2}B^4 = 0$ for $B \in [0, 1/\sqrt{g}]$, then on the set $|B'|^2 + B^2 - \frac{1}{2}B^4 = 1/2$ for $B \in [1/\sqrt{g}, 1]$, and finally tending to $(B, B') = (1, 0)$, thus providing a solution $(A, B) \in C(\mathbb{R}) \times C^2(\mathbb{R})$ of our problem.

Under the additional condition $B(0) = 1/\sqrt{g}$, the solution is given explicitly by

$$\begin{aligned} B(x) &= \sqrt{\tfrac{2}{g+1}}\,\mathrm{sech}\left(\sqrt{g-1}\,(x+x_1)\right), \quad x \leqslant 0, \\ B(x) &= \tanh\left(\tfrac{x+x_2}{\sqrt{2}}\right), \quad x \geqslant 0, \end{aligned} \tag{3.7}$$

with the constants $x_1 < 0$ and $x_2 > 0$ such that

$$\frac{2}{g+1}\,\mathrm{sech}^2\left(\sqrt{g-1}\,x_1\right) = \frac{1}{g} = \tanh^2\left(\frac{x_2}{\sqrt{2}}\right).$$

3.2. Estimates

From now on we assume that $\epsilon > 0$ and $g > 1$. Let us first observe that, for all $(A, B) \in \mathbb{R}^2$,

$$P(A,B) \geqslant K\min\{(B-1)^2+A^2,\ (B+1)^2+A^2,\ B^2+(A-1)^2,\ B^2+(A+1)^2\}$$

for some constant $K > 0$. This is because the Hessian

$$P''(A,B) = \begin{pmatrix} 3A^2+gB^2-1 & 2gAB \\ 2gAB & gA^2+3B^2-1 \end{pmatrix}$$

is positive definite at $(A, B) \in \{(\pm 1, 0), (0, \pm 1)\}$ and the growth of P is quartic at infinity. Therefore

$$(\|(A,B)\|-1)^2 \leqslant \min\{(B\pm 1)^2+A^2,\ B^2+(A\pm 1)^2\} \leqslant \frac{P(A,B)}{K}$$

and thus

$$\|(A,B)\| \leqslant 1 + \sqrt{P(A,B)/K}\,. \tag{3.8}$$

Let I be a closed interval of length 1 and $\mathring{I}$ be its interior. For simplicity, we shall use the notation $H^m(I)$ for the Sobolev space $H^m(\mathring{I})$.

For all $(A, B) \in H^2(I) \times H^1(I)$, the functions A, B and the derivative A' are continuous.

Lemma 3.1. (i) *Assume that*

$$\int_I \frac{1}{2}|A''(x)|^2 dx \leqslant M_1 \quad \textit{and} \quad \int_I P(A(x),B(x))dx \leqslant M_2$$

for some closed interval I of length 1, $(A,B) \in H^2(I) \times H^1(I)$ and $M_1, M_2 \in [0,\infty)$. Then there exists $x \in I$ such that

$$P(A(x),B(x)) \leqslant M_2, \quad \|(A(x),B(x))\| \leqslant 1 + \sqrt{M_2/K}$$

and

$$|A'(x)| \leqslant 8\left(\sqrt{2M_1/3} + 2 + 2\sqrt{M_2/K}\right).$$

(ii) *Let $\kappa > 0$ be any given constant. For all $\mu > 0$, there exists $\nu > 0$ such that, for all closed interval I of length 1 and all $(A,B) \in H^2(I) \times H^1(I)$, the inequalities*

$$\int_I \left(\frac{\epsilon}{2}|A''|^2 + \frac{1}{2}|B'|^2\right) dx < \kappa \quad \textit{and} \quad \int_I P(A,B)dx < \nu \tag{3.9}$$

imply that

$$\max_I \left(P(A,B) + |A'|\right) < \mu.$$

Proof. For all $x_1 < x_2$ in I, one has

$$A(x_2) - A(x_1) = A'(x_1)(x_2 - x_1) + \int_{x_1}^{x_2} (x_2 - s)A''(s)ds$$

and thus

$$\begin{aligned} |A(x_2) - A(x_1) - A'(x_1)(x_2 - x_1)| &\leqslant \left(\int_{x_1}^{x_2} (x_2 - s)^2 ds\right)^{1/2} \cdot \left(\int_{x_1}^{x_2} A''(s)^2 ds\right)^{1/2} \\ &\leqslant 3^{-1/2}|x_2 - x_1|^{3/2}\sqrt{2M_1}. \end{aligned}$$

This remains true if $x_2 \leqslant x_1$ in I.

As $\int_I P(A(x),B(x))dx \leqslant M_2$ and the integrand is nonnegative, there exists $x_1 \in I$ such that $P(A(x_1),B(x_1)) \leqslant M_2$. Thus

$$\|(A(x_1),B(x_1))\| \leqslant 1 + \sqrt{M_2/K}$$

thanks to (3.8). Let us check that $|A'(x_1)| \leqslant 8\left(\sqrt{2M_1/3} + 2 + 2\sqrt{M_2/K}\right)$.

After a possible translation in x, let us assume that $I = [-1/2, 1/2]$. Let us also assume to be in the case $x_1 \in [-1/2, 0]$ and $A'(x_1) \geqslant 0$. For $x \geqslant 1/4$, one has

$$A(x) \geqslant A(x_1) + A'(x_1)(1/4) - \sqrt{2M_1/3}.$$

If moreover $A'(x_1)(1/4) \geqslant 2\left(\sqrt{2M_1/3} + 2 + \sqrt{M_2/K}\right)$, then $A(x) \geqslant 1$ for $x \in [1/4, 1/2]$ and

$$\int_I P(A,B)dx \geqslant \int_{1/4}^{1/2} K(A-1)^2 dx \tag{3.10}$$

$$\begin{aligned}
&\geqslant \int_{1/4}^{1/2} K\left(-|A(x_1)| + A'(x_1)(1/4) - \sqrt{2M_1/3} - 1\right)^2 dx \\
&\geqslant \int_{1/4}^{1/2} KA'(x_1)^2(1/64)dx \geqslant KA'(x_1)^2(1/256).
\end{aligned}$$

Hence, if at the same time $A'(x_1)(1/4) \geqslant 2\left(\sqrt{2M_1/3} + 2 + \sqrt{M_2/K}\right)$ and $KA'(x_1)^2(1/256) > M_2$, we would get the contradiction $M_2 \geqslant \int_I P(A,B)dx > M_2$. This shows that

$$A'(x_1) \leqslant 8\left(\sqrt{2M_1/3} + 2 + 2\sqrt{M_2/K}\right)$$

if $I = [-1/2, 1/2]$, $x_1 \leqslant 0$ and $A'(x_1) \geqslant 0$. More generally, if I is any closed interval of length 1,

$$|A'(x_1)| \leqslant 8\left(\sqrt{2M_1/3} + 2 + 2\sqrt{M_2/K}\right).$$

Part (ii) is now proven ad absurdum by assuming the opposite. Let $\kappa > 0$ be given. For $I = [-1/2, 1/2]$ (after possible translations in x), there would exist $\mu > 0$ such that, for all integers $n \geqslant 1$, one could find $(A_n, B_n) \in H^2(I) \times H^1(I)$ such that

$$\int_I P(A_n, B_n)dx < 1/n \quad \text{and} \quad \max_I \left(P(A_n, B_n) + |A_n'|\right) \geqslant \mu.$$

From this, one also gets

$$\min_I \|(A_n, B_n)\| \leqslant 1 + \sqrt{1/(Kn)}$$

and

$$\min_I |A_n'| \leqslant 8\left(\sqrt{2\kappa/(3\epsilon)} + 2 + 2\sqrt{1/(Kn)}\right).$$

Hence the sequence $\{(A_n, B_n)\}$ is bounded in $H^2(I) \times H^1(I)$. Taking a subsequence instead if needed, it converges weakly in $H^2(I) \times H^1(I)$, and thus strongly in $C^1(I) \times C(I)$, to some $(A, B) \in H^2(I) \times H^1(I)$. One gets the contradiction

$$\int_I P(A,B)dx = 0 \quad \text{and} \quad \max_I \Big(P(A,B) + |A'|\Big) \geqslant \mu.$$

$\square$

Corollary 3.2. *If $(A, B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ satisfies $J_\epsilon(A, B) < \infty$, then*

$$\lim_{x\to\infty} P(A(x), B(x)) = \lim_{x\to-\infty} P(A(x), B(x)) = 0$$

and

$$\lim_{x\to\infty} A'(x) = \lim_{x\to-\infty} A'(x) = 0.$$

Therefore the two limits $\lim_{x\to\infty}(A(x), B(x))$ and $\lim_{x\to-\infty}(A(x), B(x))$ exist and belong to the set $\{(\pm1, 0), (0, \pm1)\}$.

3.3. Minimizing sequences

Let $\{(A_n, B_n)\} \subset X$ be a minimizing sequence of J_ϵ. Taking a subsequence if needed, it can be assumed to converge weakly in $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$, and strongly in $C^1_{loc}(\mathbb{R}) \times C_{loc}(\mathbb{R})$, to some $(A, B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \Big(\frac{\epsilon}{2}|A''|^2 + \frac{1}{2}|B'|^2 + P(A,B)\Big)dx \leqslant \inf_X J_\epsilon. \tag{3.11}$$

As $\inf_X J_\epsilon < \infty$, clearly $J_\epsilon(A, B) < \infty$. However property (3.3) (that appears in the definition of X) is in general not preserved by weak limits in $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$. Hence the weak limit (A, B) could a priori be in $\Big(H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})\Big)\backslash X$ and $J_\epsilon(A, B)$ could be strictly smaller than $\inf_X J_\epsilon$. Thus it is not yet possible to replace the inequality in (3.11) by an equality.

After possible translations in x, one can suppose that $B_n(0) = 1/2$ for all $n \in \mathbb{N}$, because $(A_n, B_n) \in X$, and thus $B(0) = 1/2$. It remains to show that, up to a subsequence,

$$\lim_{x\to-\infty}(A(x), B(x)) = (1, 0) \quad \text{and} \quad \lim_{x\to\infty}(A(x), B(x)) = (0, 1)$$

(that is, (3.3)). Observe that

$$\lim_{n\to\infty}\int_0^1 P(A_n,B_n)dx=\int_0^1 P(A,B)dx>0.$$

Consider the Hilbert space $H=L^2((0,1))$ and define $\{u_n\}_{n\geqslant 1}\subset l^2(\mathbb{Z},H)$ by $u_n=(u_{n,j})_{j\in\mathbb{Z}}$ with

$$u_{n,j}=P^{1/2}(A_n(\cdot+j),B_n(\cdot+j))\Big|_{(0,1)}.$$

We use the following compactness by concentration result that is a special case of the appendix in [4] and that is inspired by [1]:

Lemma 3.3. *Consider a sequence $\{u_n\}$ in $l^2(\mathbb{Z},H)$, where H is a Hilbert space. Writing $u_n=(u_{n,j})_{j\in\mathbb{Z}}$, where $u_{n,j}\in H$, suppose that*

(i) *$\{u_n\}$ is bounded in $l^2(\mathbb{Z},H)$,*
(ii) *$S=\{u_{n,j}:n\in\mathbb{N},j\in\mathbb{Z}\}$ is relatively compact in H,*
(iii) $\limsup_{n\to\infty}\|u_n\|_{l^\infty(\mathbb{Z},H)}>0$.

Let $T_w: l^2(\mathbb{Z},H)\to l^2(\mathbb{Z},H)$, $w\in\mathbb{Z}$, denotes the translation operator $T_w(u_j)=(u_{j-w})$. Then, for each $\delta>0$, the sequence $\{u_n\}$ admits a subsequence with the following properties. There exist a finite number k of non-zero vectors $u^1,\dots,u^k\in l^2(\mathbb{Z},H)$ and sequences $\{w_n^1\}$, …, $\{w_n^k\}\subset\mathbb{Z}$ such that

$$T_{-w_n^{k'}}\left(u_n-\sum_{\ell=1}^{k'-1}T_{w_n^\ell}u^\ell\right)\rightharpoonup u^{k'},$$

$$\|u^{k'}\|_{l^\infty(\mathbb{Z},H)}=\lim_{n\to\infty}\left\|u_n-\sum_{\ell=1}^{k'-1}T_{w_n^\ell}u^\ell\right\|_{l^\infty(\mathbb{Z},H)},$$

$$\lim_{n\to\infty}\|u_n\|^2_{l^2(\mathbb{Z},H)}=\sum_{\ell=1}^{k'}\|u^\ell\|^2_{l^2(\mathbb{Z},H)}+\lim_{n\to\infty}\left\|u_n-\sum_{\ell=1}^{k'}T_{w_n^\ell}u^\ell\right\|^2_{l^2(\mathbb{Z},H)}\tag{3.12}$$

for $k'=1,\dots,k$,

$$\limsup_{n\to\infty}\left\|u_n-\sum_{\ell=1}^{k}T_{w_n^\ell}u^\ell\right\|_{l^\infty(\mathbb{Z},H)}\leqslant\delta,\tag{3.13}$$

and

$$\lim_{n\to\infty} \left\| u_n - T_{w_n^1} u^1 \right\|_{l^\infty(\mathbb{Z},H)} = 0 \tag{3.14}$$

if $k = 1$. Here the weak convergence is understood in $l^2(\mathbb{Z}, H)$. Finally the sequences $\{w_n^1\}$, ..., $\{w_n^k\}$ satisfy

$$\lim_{n\to\infty} |w_n^{k''} - w_n^{k'}| \to \infty, \qquad 1 \leqslant k'' < k' \leqslant k \tag{3.15}$$

so that in particular

$$T_{-w_n^{k'}} u_n \rightharpoonup u^{k'}, \qquad k' = 1, \ldots, k. \tag{3.16}$$

Remark. Let us briefly compare with the usual version of the concentration-compactness method of P.L. Lions (see e.g. [16]) and its three standard possibilities: compactness, vanishing and dichotomy. Hypothesis (iii) can be interpreted as forbidding the 'vanishing' case. The case $k = 1$ would be analogous to the 'compactness' case, and the case $k \geqslant 2$ could be interpreted as 'dichotomy' occurring iteratively.

In our example, the sequence $\{u_n\}$ is bounded in $l^2(\mathbb{Z}, H)$, the set $\{u_{n,j} : j \in \mathbb{Z}, n \geqslant 1\}$ is relatively compact in H and

$$\liminf_{n\to\infty} \|u_{n,0}\|_H > 0.$$

Hence we can apply Lemma 3.3. Note that (3.12) implies

$$\sum_{\ell=1}^{k} \|u^\ell\|^2_{l^2(\mathbb{Z},H)} \leqslant \lim_{n\to\infty} \|u_n\|^2_{l^2(\mathbb{Z},H)}. \tag{3.17}$$

If $k \geqslant 2$, taking a subsequence if needed and relabelling $u^1, \ldots, u^k$, one can also assume that

$$w_n^1 < \ldots < w_n^k, \quad \forall n \in \mathbb{N}.$$

By (3.16), up to a subsequence, there exists, for $\ell \in \{1, \ldots, k\}$, $(A^\ell, B^\ell) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ such that

$$(A_n(\cdot + w_n^\ell), B_n(\cdot + w_n^\ell)) \to (A^\ell, B^\ell) \tag{3.18}$$

weakly in $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ and strongly in $C^1_{loc}(\mathbb{R}) \times C_{loc}(\mathbb{R})$, and

$$u_j^\ell = P^{1/2}(A^\ell(\cdot + j), B^\ell(\cdot + j)), \quad \forall j \in \mathbb{Z},$$

for $\ell \in \{1, \dots, k\}$. The equation (3.17) gives

$$\sum_{\ell=1}^{k} \int_{\mathbb{R}} P(A^\ell, B^\ell)dx \leqslant \lim_{n\to\infty} \int_{\mathbb{R}} P(A_n, B_n)dx.$$

Moreover, for all $1 \leqslant k' \leqslant k$,

$$\int_{\mathbb{R}} (B^{k'})' \left(B_n'(\cdot + w_n^{k'}) - \sum_{\ell=1}^{k'} (B^\ell)'(\cdot + w_n^{k'} - w_n^\ell) \right) dx$$

$$= \int_{\mathbb{R}} (B^{k'})' B_n'(\cdot + w_n^{k'})dx - \int_{\mathbb{R}} (B^{k'})'(B^{k'})'dx - \sum_{\ell=1}^{k'-1} \int_{\mathbb{R}} (B^{k'})'(B^\ell)'(\cdot + w_n^{k'} - w_n^\ell)dx \to 0,$$

by (3.15) and (3.18), which implies that

$$\lim_{n\to\infty} \left\| B_n' - \sum_{\ell=1}^{k'-1} (B^\ell)'(\cdot - w_n^\ell) \right\|_{L^2(\mathbb{R})}^2 = \lim_{n\to\infty} \left\| (B^{k'})'(\cdot - w_n^{k'}) + \left(B_n' - \sum_{\ell=1}^{k'} (B^\ell)'(\cdot - w_n^\ell) \right) \right\|_{L^2(\mathbb{R})}^2$$

$$= \lim_{n\to\infty} \left\| (B^{k'})' + \left(B_n'(\cdot + w_n^{k'}) - \sum_{\ell=1}^{k'} (B^\ell)'(\cdot + w_n^{k'} - w_n^\ell) \right) \right\|_{L^2(\mathbb{R})}^2$$

$$= \left\| (B^{k'})' \right\|_{L^2(\mathbb{R})}^2 + \lim_{n\to\infty} \left\| B_n' - \sum_{\ell=1}^{k'} (B^\ell)'(\cdot - w_n^\ell) \right\|_{L^2(\mathbb{R})}^2$$

and thus

$$\sum_{k'=1}^{k} \left\| (B^{k'})' \right\|_{L^2(\mathbb{R})}^2 \leqslant \lim_{n\to\infty} \|B_n'\|_{L^2(\mathbb{R})}^2 .$$

In the same way

$$\sum_{k'=1}^{k} \left\| (A^{k'})' \right\|_{L^2(\mathbb{R})}^2 \leqslant \lim_{n\to\infty} \|A_n'\|_{L^2(\mathbb{R})}^2$$

and

$$\sum_{k'=1}^{k} \left\| (A^{k'})'' \right\|_{L^2(\mathbb{R})}^2 \leqslant \lim_{n\to\infty} \|A_n''\|_{L^2(\mathbb{R})}^2 .$$

Hence

$$\sum_{\ell=1}^{k} J_\epsilon(A^\ell, B^\ell) \leqslant \lim_{n\to\infty} J_\epsilon(A_n, B_n) = \inf_X J_\epsilon \,. \tag{3.19}$$

From (3.13) and the fact that $\lim_{|j|\to\infty} \|u_j^\ell\|_H = 0$ for all $\ell \in \{1, \dots, k\}$, one gets

$$\sup \left\{ \|u_{n,j}\|_H : j \in \mathbb{Z}, |j - w_n^1| > p, \ \dots \ , |j - w_n^k| > p \right\}$$
$$\leqslant \left\| u_n - \sum_{\ell=1}^k T_{w_n^\ell} u^\ell \right\|_{l^\infty(\mathbb{Z},H)} + \sup \left\{ \sum_{\ell=1}^k \left\| u_j^\ell \right\|_H : j \in \mathbb{Z}, |j| > p \right\}$$

for each n and

$$\lim_{p\to\infty} \limsup_{n\to\infty} \sup \left\{ \|u_{n,j}\|_H : j \in \mathbb{Z}, |j - w_n^1| > p, \ \dots \ , |j - w_n^k| > p \right\} \leqslant \delta.$$

Given $\mu > 0$ and ϵ, one chooses $\kappa = 2 \inf_X J_\epsilon$ in (3.9) and then $\delta = \sqrt{\nu/2}$, with $\nu > 0$ as in (3.9), which gives

$$\lim_{p\to\infty} \limsup_{n\to\infty} \sup \left\{ P(A_n(x), B_n(x)), \ x \notin [w_n^j - p, w_n^j + p + 1], \ j = 1, \dots, k \right\} \leqslant \mu.$$

Let $\rho > 0$ be such that the open set $\{(a, b) \in \mathbb{R}^2 : P(a, b) < \rho\}$ is the union of four open sets $V_{(0,\pm1)}$ and $V_{(\pm1,0)}$ with disjoint adherence, containing the points $(0, \pm1)$ and $(\pm1, 0)$, respectively. One can also suppose that the line $\mathbb{R} \times \{1/2\}$ does not meet $\{(a, b) \in \mathbb{R}^2 : P(a, b) \leqslant \rho\}$.

If one chooses $\mu = \rho/2$, then p large enough, one gets for all n large enough, up to a subsequence,

$$P(A_n(x), B_n(x)) < \rho, \quad \forall x \in (-\infty, w_n^1 - p),$$
$$P(A_n(x), B_n(x)) < \rho, \quad \forall x \in (w_n^k + p + 1, \infty),$$

and, if $k \geqslant 2$,

$$P(A_n(x), B_n(x)) < \rho, \quad \forall x \in (w_n^{\ell-1} + p + 1, w_n^\ell - p) \neq \emptyset,$$

for all $\ell \in \{2, \dots, k\}$. In the two first cases, as well as in the last case for each $\ell \in \{2, \dots, k\}$, $(A_n(x), B_n(x))$ not only satisfies $P(A_n(x), B_n(x)) < \rho$, but $(A_n(x), B_n(x))$ even stays in $V_{(0,1)}$, $V_{(0,-1)}$, $V_{(1,0)}$ or $V_{(-1,0)}$ (this can change in each case and one uses the continuity of (A_n, B_n)). As $(A_n, B_n) \in X$, one has the following additional information:

$$(A_n(x), B_n(x)) \in V_{(1,0)}, \quad \forall x \in (-\infty, w_n^1 - p),$$

and

$$(A_n(x), B_n(x)) \in V_{(0,1)}, \quad \forall x \in (w_n^k + p + 1, \infty).$$

Hence there exists $\hat{\ell} \in \{1, \ldots, k\}$ such that

$$(A_n(x), B_n(x)) \in V_{(1,0)} \cup V_{(-1,0)}, \quad \forall x \in (w_n^{\hat{\ell}-1} + p + 1, w_n^{\hat{\ell}} - p),$$

and

$$(A_n(x), B_n(x)) \in V_{(0,1)} \cup V_{(0,-1)}, \quad \forall x \in (w_n^{\hat{\ell}} + p + 1, w_n^{\hat{\ell}+1} - p),$$

with the understanding that $w_n^0 = -\infty$ and $w_n^{k+1} = +\infty$. From (3.16), it follows that

$$\lim_{x \to -\infty} (A^{\hat{\ell}}(x), B^{\hat{\ell}}(x)) \in \{(\pm 1, 0)\} \quad \text{and} \quad \lim_{x \to \infty} (A^{\hat{\ell}}(x), B^{\hat{\ell}}(x)) \in \{(0, \pm 1)\}.$$

As, with the right choice of signs, one has $(\pm A^{\hat{\ell}}, \pm B^{\hat{\ell}}) \in X$ and

$$\inf_X J_\epsilon \leqslant J_\epsilon(\pm A^{\hat{\ell}}, \pm B^{\hat{\ell}}) = J_\epsilon(A^{\hat{\ell}}, B^{\hat{\ell}}),$$

one gets (see (3.19))

$$\inf_X J_\epsilon \leqslant \sum_{\ell=1}^{k} J_\epsilon(A^\ell, B^\ell) \leqslant \lim_{n \to \infty} J_\epsilon(A_n, B_n) = \inf_X J_\epsilon\,.$$

As $u^\ell \neq 0$ for all $\ell \in \{1, \ldots, k\}$, this is only possible if $k = 1$ and

$$J_\epsilon(\pm A^{\hat{\ell}}, \pm B^{\hat{\ell}}) = J_\epsilon(A^{\hat{\ell}}, B^{\hat{\ell}}) = \inf_X J_\epsilon.$$

Since $k = \hat{\ell} = 1$, one also has

$$\lim_{x \to -\infty} (A^1(x), B^1(x)) = (1, 0) \quad \text{and} \quad \lim_{x \to \infty} (A^1(x), B^1(x)) = (0, 1)$$

This shows that $(A^1, B^1) \in X$ minimizes J_ϵ. In addition, up to a translation in x, it is equal to $(A, B) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ introduced in (3.11), which therefore indeed belongs to X.

Finally, notice that $(A, |B|)$ is also a minimal pair and therefore one can assume that $B \geqslant 0$ on $\mathbb{R}$. However (A, B) tends to $(0, 1)$ as $x \to \infty$ in a way such that A oscillates around 0. This behavior is given by the linearization at $(A, B) = (0, 1)$, because this equilibrium is hyperbolic.

3.4. Limit $g \to 1$

It remains to prove the last property in Theorem 1. We show below that any heteroclinic orbit connecting the equilibria $(1,0)$ and $(0,1)$ which minimizes the functional J_ϵ in the space X remains in a neighborhood of the circle $A^2+B^2=1$ as $g \to 1^+$.

Let us first estimate $\min_X J_\epsilon$ as $g \to 1^+$. We introduce the "test function" (A_1, B_1) defined as follows:

$$(A_1, B_1)(x) = \left(\cos\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right),\ \sin\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right)\right), \quad x \in \mathbb{R}.$$

Then (A_1, B_1) belongs to the space X, and we have the formulas for the first and second order derivatives:

$$(A_1', B_1')(x) = \left(-\sin\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right), \cos\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right)\right)\frac{1}{2(x^2+1)},$$

$$\begin{aligned}(A_1'', B_1'')(x) = &\left(-\cos\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right), -\sin\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right)\right)\frac{1}{4(x^2+1)^2}\\ &-\left(-\sin\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right), \cos\left(\frac{\pi}{4} + \frac{\arctan(x)}{2}\right)\right)\frac{x}{(x^2+1)^2}.\end{aligned}$$

For a suitably chosen positive constant C, we obtain the estimates:

$$\begin{aligned}
|B_1(x) - 1| &\leqslant C\,\frac{1}{x}, \quad \forall\, x \leqslant -1,\\
0 < B_1(x) &\leqslant C\,\frac{1}{x}, \quad \forall\, x \geqslant 1,\\
0 < A_1(x) &\leqslant C\,\frac{1}{x}, \quad \forall\, x \leqslant -1,\\
|A_1(x) - 1| &\leqslant C\,\frac{1}{x}, \quad \forall\, x \geqslant 1.
\end{aligned}$$

For $\gamma > 0$, let $(A_\gamma, B_\gamma) \in X$ be defined by

$$(A_\gamma, B_\gamma)(x) = (A_1(\gamma x), B_1(\gamma x)), \quad \forall\, x \in \mathbb{R}.$$

Then

$$\min_X J_{\epsilon,g} \leqslant J_{\epsilon,g}(A_\gamma, B_\gamma) = \int_{\mathbb{R}} \left(\frac{\epsilon}{2}|A_\gamma''|^2 + \frac{1}{2}|B_\gamma'|^2 + \frac{g-1}{2}A_\gamma^2 B_\gamma^2\right)dx$$

$$= \int_{\mathbb{R}} \Big(\frac{\gamma^3 \epsilon}{2} |A_1''|^2 + \frac{\gamma}{2} |B_1'|^2 + \frac{\gamma^{-1}(g-1)}{2} A_1^2 B_1^2 \Big) dx.$$

By choosing $\gamma = (g-1)^{1/2}$, we get $\min_X J_{\epsilon,g} \to 0$ as $g \to 1^+$ (for fixed $\epsilon > 0$).

Let now (A_g, B_g) denote a minimizing heteroclinic orbit, where $\epsilon > 0$ is still fixed, but we insist on the dependence on the parameter $g > 1$. In the first part of Lemma 3.1, the constant $K > 0$ can be chosen independent of $g > 1$; this is due the explicit inequalities

$$P(A,B) \geqslant \frac{1}{4}(A^2 + B^2 - 1)^2 = \frac{1}{4}(||(A,B)|| + 1)^2 (||(A,B)|| - 1)^2$$

$$\geqslant \frac{1}{4}(||(A,B)|| - 1)^2, \qquad (A,B) \in \mathbb{R}^2,$$

and

$$P(A,B) \geqslant \frac{1}{4}(||(A,B)|| - 1)^2 \geqslant \frac{1}{4}(A-1)^2 \quad \text{if} \quad A \geqslant 1$$

(see respectively (3.8) and (3.10) used in the proof of Lemma 3.1). Hence, for some new constant $K > 0$ and each closed interval I of length 1, there exists $x_0 \in I$ such that

$$P(A_g(x_0), B_g(x_0)) \leqslant \min_X J_{\epsilon,g}, \quad \|(A_g(x_0), B_g(x_0))\| \leqslant 1 + \sqrt{\min_X J_{\epsilon,g}/K}$$

and

$$|A_g'(x_0)| \leqslant 1/K.$$

The constant $K > 0$ can be chosen independent of $g > 1$ with $g - 1$ as small as needed. Let us show that

$$\lim_{g \to 1^+} \sup_{x \in \mathbb{R}} \frac{1}{4}(A_g(x)^2 + B_g(x)^2 - 1)^2 = 0.$$

Suppose not. Then there exist $\chi > 0$ and a strictly decreasing sequence $g_n \to 1$ such that

$$\forall n \in \mathbb{N} \ \ \frac{1}{4}(A_{g_n}(0)^2 + B_{g_n}(0)^2 - 1)^2 \geqslant \chi > 0$$

(up to translations in x). Moreover $J_{\epsilon,g_n}(A_{g_n}, B_{g_n}) \to 0$ and thus $\{(A_{g_n}, B_{g_n})\} \subset X$ is bounded in $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$. See also the estimates just above on x_0.

Hence, up to a subsequence, it converges in $C^1_{loc}(\mathbb{R}) \times C_{loc}(\mathbb{R})$ to some $(A_1, B_1) \in H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$. Therefore

$$\int_{\mathbb{R}} P(A_1(x), B_1(x))dx \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}} P(A_{g_n}(x), B_{g_n}(x))dx$$

and thus $\int_{\mathbb{R}} P(A_1(x), B_1(x)) = 0$. Also

$$\frac{1}{4}(A_{g_n}(0)^2 + B_{g_n}(0)^2 - 1)^2 \to \frac{1}{4}(A_1(0)^2 + B_1(0)^2 - 1)^2$$

and thus $P(A_1(0), B_1(0)) \geqslant \frac{1}{4}(A_1(0)^2 + B_1(0)^2 - 1)^2 \geqslant \chi > 0$. This is in contradiction with the continuity of $P(A_1(x), B_1(x))$ at $x = 0$. We conclude that

$$\lim_{g\to 1^+} \sup_{x\in\mathbb{R}} \frac{1}{4}(A_g(x)^2 + B_g(x)^2 - 1)^2 = 0.$$

This completes the proof of Theorem 1.

Appendix A. A cubic normal form for 12-dimensional vector fields with symmetries

In this Appendix, we obtain a cubic normal form for 12-dimensional vector fields having the properties (2.16)-(2.17) and the symmetries (2.13)-(2.15).

Lemma A1. *Consider a system of ordinary differential equations of the form* (2.12) *in which the vector field F is of class C^m, for some $m \geqslant 4$, in a neighborhood $\mathcal{U}_1 \times \overline{\mathcal{U}_1} \times \mathcal{U}_2 \subset \mathbb{C}^6 \times \overline{\mathbb{C}^6} \times \mathbb{R}^2$ of the origin. Assume that the properties* (2.16)-(2.17) *hold and that F anti-commutes with $\boldsymbol{S}_1$ in* (2.13) *and commutes with $\boldsymbol{S}_2$ in* (2.14) *and $\boldsymbol{\tau}_a$ in* (2.15)*.*

There exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of 0 in $\mathbb{C}^6$ and $\mathbb{R}^2$, respectively, such that for any $(\widetilde{\mu}, \widetilde{k}) \in \mathcal{V}_2$, there is a polynomial $P(\cdot, \cdot, \widetilde{\mu}, \widetilde{k}) : \mathbb{C}^6 \times \overline{\mathbb{C}^6} \to \mathbb{C}^6$ of degree 3 in the variables $(Z, \overline{Z})$, such that for $Z \in \mathcal{V}_1$, the change of variable

$$X = Z + P(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k}),$$

transforms the equation (2.12) *into the normal form*

$$\frac{dZ}{dx} = L_c Z + N(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k}) + \rho(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k}), \tag{A.1}$$

with the following properties:

(i) *the map* ρ *belongs to* $C^m(\mathcal{V}_1 \times \overline{\mathcal{V}_1} \times \mathcal{V}_2, \mathbb{C}^6)$, *and*

$$\rho(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k}) = O(|(\widetilde{\mu}, \widetilde{k})|^2 \|Z\| + |(\widetilde{\mu}, \widetilde{k})| \, \|Z\|^3 + \|Z\|^4);$$

(ii) *both* $N(\cdot, \cdot, \widetilde{\mu}, \widetilde{k})$ *and* $\rho(\cdot, \cdot, \widetilde{\mu}, \widetilde{k})$ *anti-commute with* $\boldsymbol{S}_1$ *and commute with* $\boldsymbol{S}_2$ *and* $\boldsymbol{\tau}_a$, *for any* $(\widetilde{\mu}, \widetilde{k}) \in \mathcal{V}_2$;

(iii) *the six components* $(N_0, N_1, N_2, N_3, M_0, M_1)$ *of* N *are of the form*

$$\begin{aligned}
N_0 &= iA_0P_0,\\
N_1 &= iA_1P_0 + A_0P_1 + b_7u_7,\\
N_2 &= iA_2P_0 + A_1P_1 + iA_0P_2 + b_7v_7 + c_8u_8 + c_9u_9,\\
N_3 &= iA_3P_0 + A_2P_1 + iA_1P_2 + A_0P_3 + b_7w_7 + c_8v_8 + c_9v_9\\
&\quad + d_7u_7 + d_{10}u_{10} + d_{11}u_{11},\\
M_0 &= iB_0Q_0 + \alpha_{12}u_{12},\\
M_1 &= iB_1Q_0 + B_0Q_1 + \alpha_{12}v_{12} + i\beta_{12}u_{12} + i\beta_{13}u_{13},
\end{aligned}$$

with

$$\begin{aligned}
P_0 &= a_2u_2 + a_4u_4,\\
P_1 &= b_0\widetilde{\mu} + b_0'\widetilde{k} + b_1u_1 + b_3u_3 + b_5u_5 + b_6u_6,\\
P_2 &= c_2u_2 + c_4u_4,\\
P_3 &= d_0\widetilde{\mu} + d_0'\widetilde{k} + d_1u_1 + d_3u_3 + d_5u_5 + d_6u_6,\\
Q_0 &= \alpha_0\widetilde{\mu} + \alpha_0'\widetilde{k} + \alpha_1u_1 + \alpha_3u_3 + \alpha_5u_5 + \alpha_6u_6,\\
Q_1 &= \beta_0\widetilde{\mu} + \beta_0'\widetilde{k} + \beta_1u_1 + \beta_3u_3 + \beta_5u_5 + \beta_6u_6,
\end{aligned}$$

where $(A_0, A_1, A_2, A_3, B_0, B_1)$ *are the six components of* Z, *the coefficients* a_j, b_j, c_j, d_j, α_j, *and* β_j *are all real, and*

$$u_1 = A_0\overline{A_0}, \quad u_2 = i(A_0\overline{A_1} - \overline{A_0}A_1),$$
$$u_3 = A_0\overline{A_2} + \overline{A_0}A_2 - A_1\overline{A_1},$$
$$u_4 = i(A_0\overline{A_3} - \overline{A_0}A_3 - A_1\overline{A_2} + \overline{A_1}A_2),$$
$$u_5 = B_0\overline{B_0}, \quad u_6 = i(B_0\overline{B_1} - \overline{B_0}B_1),$$

$$u_7 = \overline{A_0}(A_1^2 - 2A_0A_2), \quad v_7 = \overline{A_1}(A_1^2 - 2A_0A_2),$$
$$w_7 = \overline{A_2}(A_1^2 - 2A_0A_2), \quad u_8 = A_0v_3 - A_1u_3, \quad v_8 = A_1v_3 - 2A_2u_3,$$
$$v_3 = \tfrac{1}{2}(3A_0\overline{A_3} + 3\overline{A_0}A_3 - A_1\overline{A_2} - \overline{A_1}A_2),$$
$$u_9 = \tfrac{1}{2}A_0(B_0\overline{B_1} + \overline{B_0}B_1) - A_1u_5, \quad v_9 = \tfrac{1}{2}A_0B_1\overline{B_1} - A_2u_5,$$

$$u_{10} = 3iA_3u_2 + 2A_2(A_0\overline{A_2} - \overline{A_0}A_2) - A_1(A_1\overline{A_2} - \overline{A_1}A_2),$$
$$u_{11} = \tfrac{1}{2}A_0B_1\overline{B_1} + A_2u_5 - \tfrac{1}{2}A_1(B_0\overline{B_1} + \overline{B_0}B_1),$$
$$u_{12} = \tfrac{1}{2}B_0(A_0\overline{A_1} + \overline{A_0}A_1) - B_1u_1,$$
$$v_{12} = B_0\overline{A_1}A_1 - \tfrac{1}{2}B_1(A_0\overline{A_1} + \overline{A_0}A_1), \quad u_{13} = B_0v_3 - B_1u_3.$$

Proof. We closely follow the arguments in the proofs from [7, Lemma 6.1] and [8, Theorem 2]. The first part of the lemma and the properties (i) and (ii) are obtained exactly in the same way from general normal form theorems. It remains to determine the properties (iii) of the six components of the polynomial N. As in the proofs mentioned above, the problem is reduced to the one of determining the homogeneous monomials

$$A_0^{p_0}\overline{A_0}^{q_0}A_1^{p_1}\overline{A_1}^{q_1}A_2^{p_2}\overline{A_2}^{q_2}A_3^{p_3}\overline{A_3}^{q_3}B_0^{r_0}\overline{B_0}^{s_0}B_1^{r_1}\overline{B_1}^{s_1}, \tag{A.2}$$

which appear in each of the six components of N. Restricting to monomials of degrees 1, 2, and 3, the nonnegative exponents from (A.2) satisfy

$$(p_0 + p_1 + p_2 + p_3) + (q_0 + q_1 + q_2 + q_3) + (r_0 + r_1) + (s_0 + s_1) = m, \tag{A.3}$$

with $m \in \{1, 2, 3\}$. Further conditions on these exponents are obtained as explained in the proofs from [7, Lemma 6.1] and [8, Theorem 2] from the commutativity properties in (ii) and the normal form property

$$D_Z N(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k})L_0^*Z + D_{\overline{Z}}N(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k})\overline{L_0^*Z} = L_0^*N(Z, \overline{Z}, \widetilde{\mu}, \widetilde{k}), \tag{A.4}$$

which must hold for all $(Z, \widetilde{\mu}, \widetilde{k}) \in \mathbb{C}^6 \times \mathcal{V}_2$. Differences with the proofs mentioned above being only at the computational level, we skip the details of the remaining arguments. □

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