Computer-assisted enumeration of finite geometries related to quantum contextuality

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Outline

Background

- Quantum computing basics
- Contextuality
- Mermin-Peres magic square
- Contextual geometries
- Multi-qubit doilies

Contributions

- Numbers of multi-qubit doilies
- Doily generation algorithm
- Multi-qubit classification

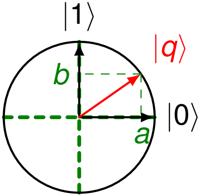
Conclusion

Quantum computing basics

Quantum bit (qubit)

ket notation $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

qubit $|q\rangle = a\,|0\rangle + b\,|1\rangle$ $a, b \in \mathbb{C}$ $|a|^2 + |b|^2 = 1$



$$|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

bra notation $\langle q| = (\bar{a}\ \bar{b})$

Single qubit measurement

Measurement of $|q\rangle = a|0\rangle + b|1\rangle$ in the basis $(|0\rangle, |1\rangle)$

$$|q\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \begin{array}{l} \xrightarrow{|a|^2} |0\rangle \rightsquigarrow \quad +1 \\ \xrightarrow[|b|^2]{} |1\rangle \rightsquigarrow \quad -1 \end{array}$$

encoded by the third Pauli matrix $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{array}{rcc} \text{eigenvalues} & 1 & -1 \\ \text{eigenvectors} & |0\rangle & |1\rangle \end{array}$$

Mean value

$$\langle q|Z|q\rangle = (\bar{a}\ \bar{b}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1.|a|^2 + (-1).|b|^2$$

Measurement in the basis $(|+\rangle, |-\rangle)$

$$|q\rangle = a|0\rangle + b|1\rangle$$

$$|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle \xrightarrow{|a+b|^2/2} |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \rightsquigarrow +1$$

$$|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle \xrightarrow{|a-b|^2/2} |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} \rightsquigarrow -1$$

encoded by the first Pauli matrix $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

eigenvalues	1	-1
eigenvectors	$|+\rangle$	$|-\rangle$

Mean value

$$\langle q|X|q\rangle = \bar{a}b + \bar{b}a = 1.|a+b|^2/2 + (-1).|a-b|^2/2$$

since $|a|^2 + |b|^2 = 1$

Pauli group

Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Y measures in the $\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \frac{|0\rangle - i|1\rangle}{\sqrt{2}}\right)$ basis

matrix product

.	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

Pauli group $\mathcal{P} = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}, .)$

commuting pair $X.I = I.X$

anticommuting pair $Y.Z = iX$ and $Z.Y = -iX$, so $Y.Z = -Z.Y$

Multi-qubit

tensor product $A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{pmatrix}$

notation $A_1 A_2 \cdots A_N$ for $A_1 \otimes A_2 \otimes \cdots \otimes A_N$
$|01\rangle$ for $|0\rangle \otimes |1\rangle$, etc

2-qubit $|q\rangle = q_{00}\,|00\rangle + q_{01}\,|01\rangle + q_{10}\,|10\rangle + q_{11}\,|11\rangle$

N-qubit $|q\rangle = q_{0..0}\,|0..0\rangle + \cdots + q_{1..1}\,|1..1\rangle \in \mathbb{C}^{2^N}$

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Generalized Pauli group

N-qubit Pauli operator	$G_1 G_2 \cdots G_N$, with $G_i \in \{I, X, Y, Z\}$
generalized Pauli group	$\mathcal{P}_N = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}^N, .)$
commuting pair	$YX.ZZ = (Y.Z)(X.Z) = (iX)(-iY) = XY$ $ZZ.YX = (Z.Y)(Z.X) = (-iX)(iY) = XY$
anticommuting pair	$XY.IZ = (X.I)(Y.Z) = iXX$ $IZ.XY = (I.X)(Z.Y) = -iXX$

Mutually commuting multi-qubit Pauli operators are
compatible observables

Contextuality

Kochen-Specker theorem

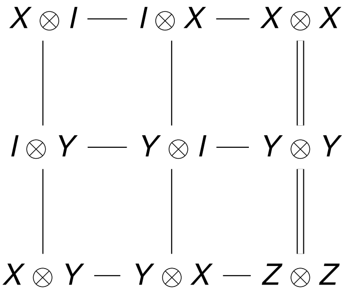
No non-contextual hidden-variable theory can reproduce the outcomes predicted by quantum physics.[1]

Without loss of generality, a non-contextual hidden-variable (NCHV) theory admits the existence of a function $v : \mathcal{P}_N \to \{-1, 1\}$ that determines (as $v(O)$) the result of any measurement with the multi-qubit Pauli observable O (among its two eigenvalues -1 and 1) independently of other measurements performed before or after this measurement.

Mermin-Peres square proves Kochen-Specker theorem by describing experiments with nine two-qubit Pauli observables which contradict the NCHV hypothesis.

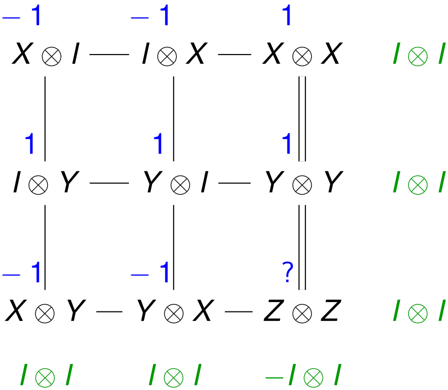
Contextuality

Mermin-Peres magic square[2]



Contextuality

Mermin-Peres magic square



Finite geometry with 9 points (two-qubit observables) and 6 lines (collinearity $\Leftrightarrow$ commutation), either positives ($M - \ldots - N$) or negatives ($M = \ldots = N$)

Contextual geometries

A *contextual geometry*[3] is a pair (O, C) such that

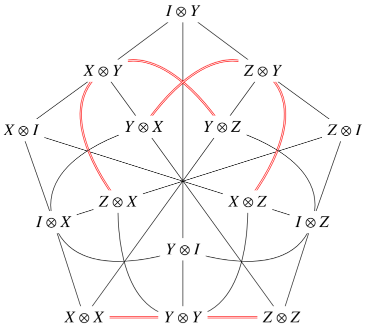
- O is a finite set of observables (*points*), i.e. Hermitian operators ($M = M^*$) of finite dimension;
- C is a finite set of subsets of O, called *contexts* (or *lines*) such that
  - each observable $M \in O$ satisfies $M^2 = Id$ (eigenvalues in $\{-1, 1\}$),
  - two observables M and N in the same context commute ($M.N = N.M$), and
  - the product of all observables in the same context is the identity matrix Id (*positive line*) or its opposite $-Id$ (*negative line*).

$\rightsquigarrow$ discover and classify contextual geometries (KS proofs)

Contextual geometries

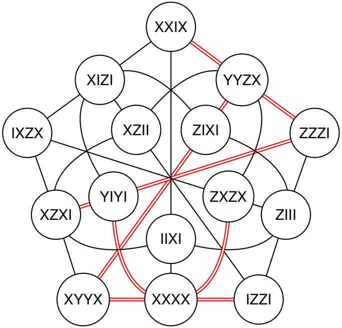
The two-qubit doily W_2

The *(two-qubit) doily* is the contextual geometry whose points are all the 2-qubit Pauli observables except $I \otimes I$



Multi-qubit doilies

A *N-qubit doily* is a geometry on *N*-qubit Pauli observables with the same point/line structure as the doily W_2



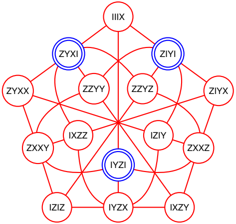
Example of 4-qubit doily

Multi-qubit doilies

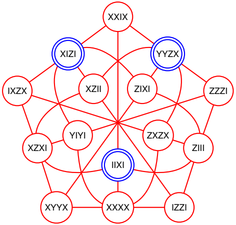
Linear and quadratic doilies

A doily is *linear* iff $A.B.C = Id$ for any *unicentric* (one common collinear point) *triad* (3 noncollinear points) $\{A, B, C\}$.

Otherwise, it is *quadratic*.



$ZYXI.IYZI.ZIYI = Id$



$XIZI.IIXI.YYZX \neq Id$

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Numbers of multi-qubit doilies

Closed formulas

Numbers $D(N)$ (resp. $D_l(N)$, $D_q(N)$) of (resp. linear, quadratic) N-qubit doilies

$$D(N) = D_l(N) + D_q(N)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-k+i}-1}{q^i-1}$$

$$D_l(N) = \begin{bmatrix} 2N \\ 4 \end{bmatrix}_2 - \begin{bmatrix} N \\ 4 \end{bmatrix}_2 \prod_{i=1}^{4}\left(2^{N+1-i}+1\right) - 7\begin{bmatrix} N \\ 3 \end{bmatrix}_2 2^{2N-6} \prod_{i=1}^{3}\left(2^{N+1-i}+1\right)/3$$

$$D_q(N) = 16\left(\begin{bmatrix} 2N \\ 5 \end{bmatrix}_2 - \begin{bmatrix} N \\ 5 \end{bmatrix}_2 \prod_{i=1}^{5}\left(2^{N+1-i}+1\right) - 15\begin{bmatrix} N \\ 4 \end{bmatrix}_2 2^{2N-8} \prod_{i=1}^{4}\left(2^{N+1-i}+1\right)/3\right)$$

N	$D_l(N)$	$D_q(N)$	$D(N)$
2	1	_	1
3	336	1 008	1 344
4	91 392	1 370 880	1 462 272
5	23 744 512	1 495 904 256	1 519 648 768
6	6 100 942 848	1 555 740 426 240	1 561 841 369 088
7	1 563 272 675 328	1 599 227 946 860 544	1 600 791 219 535 872
8	400 289 425 260 544	1 639 185 196 441 927 680	1 639 585 485 867 188 224
9	102 479 956 839 235 584	1 678 929 132 897 196 572 672	1 679 031 612 854 035 808 256
10	26 235 244 249 381 601 280	1 719 326 731 883 223 239 884 800	1 719 352 967 127 472 621 486 080

Doily generation algorithm

Goal: generate all N-qubit doilies for a given N, in order to classify them and check various properties about them

Binary encoding

$$I \leftrightarrow (0,0) \qquad X \leftrightarrow (0,1) \qquad Y \leftrightarrow (1,1) \qquad Z \leftrightarrow (1,0)$$

The N-qubit observable $G_1 G_2 \cdots G_N$ is encoded by the bitvector $(g_1 g_2 \dots g_{2N})_2$, with $G_j \leftrightarrow (g_j, g_{j+N})$ for $j \in \{1, 2, \dots, N\}$

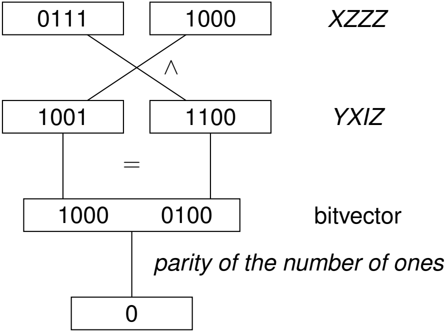
Operations

Product of observables $\rightsquigarrow$ exclusive or (the phase p is ignored)

$$ZZZZ.XYZI = 11110000_2 \oplus 01101100_2 = 10011100_2 = p.YXIZ$$

Symplectic product (for collinearity) $\rightsquigarrow$ conjunctions and parity check

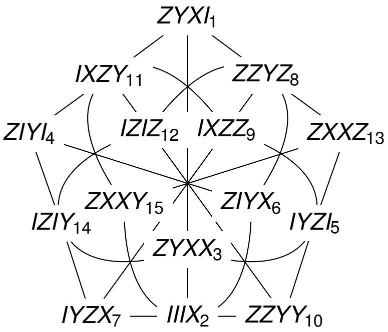
$$\langle a, b \rangle = a_1 b_{N+1} + a_{N+1} b_1 + a_2 b_{N+2} + a_{N+2} b_2 + \cdots + a_N b_{2N} + a_{2N} b_N$$



Representation of a multi-qubit doily

Each N-qubit doily is an injective labeling of the W_2 doily
⇝ we use the binary representation of two-qubit observables as array indices

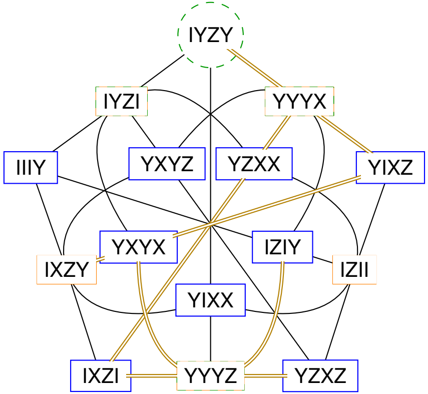
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
bv	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
W_2	II	IX	XI	XX	IZ	IY	XZ	XY	ZI	ZX	YI	YX	ZZ	ZY	YZ	YY
doily	∅	ZYXI	IIIX	ZYXX	ZIYI	IYZI	ZIYX	IYZX	ZZYZ	IXZZ	ZZYY	IXZY	IZIZ	ZXXZ	IZIY	ZXXY



Algorithm steps

main → find_ovoids → generate_center → generate_doily → classify_doily

- **main**: Main function initialization of the variables
- **find_ovoids**: Generation of every ovoid
- **generate_center**: For a predefined triad, compute every possible center
- **generate_doily**: Compute all the other observables of the doily
- **classify_doily**: Calculation of the properties and insertion in the results table



Doily generation algorithm

for each ovoid $O = \{o_1, o_2, o_3, o_4, o_5\}$ in W_N, with $o_1 < o_2 < o_3 < o_4 < o_5$ **do**
  $f(IX) \leftarrow o_1 \mid\mid f(IZ) \leftarrow o_2 \mid\mid f(XY) \leftarrow o_3 \mid\mid f(ZY) \leftarrow o_4 \mid\mid f(YY) \leftarrow o_5$
  for each center c of $\{o_1, o_2, o_3\}$ in W_N that anticommutes with o_4 and o_5 **do**
    $f(XI) \leftarrow c$
    for each line (p, q, r) in the order of the sequence S **do**
      $f(r) \leftarrow |f(p).f(q)|$
    end for
    if O is not the smallest ovoid of f **then** discard f **end if**
    ...   ▷ Classification of f
  end for
end for

S is $(XI, IX, XX), (XI, IZ, XZ), (XI, XY, IY), (ZY, XX, YZ), (ZY, XZ, YX),$
$(ZY, IY, ZI), (YY, XX, ZZ), (YY, XZ, ZX), (YY, IY, YI)$

Doily generation program

Implemented in *C* language for

- quick execution
- parallelization (with OpenMP), here when choosing the first observable o_1 of the ovoid ($4^N - 1$ processes)

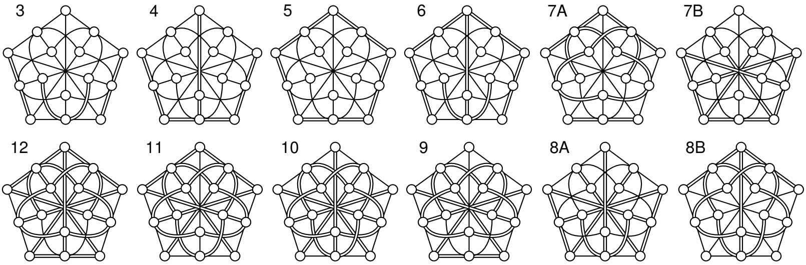
Execution time (Intel® Core™ i7-8665U CPU @ 1.90 GHz, 8 cores)

- 4 qubits: 1 462 272 doilies in 0.5 seconds, 1.4 Mb RAM
- 5 qubits: 1 519 648 768 doilies in 12 minutes, 1.8 Mb RAM

Multi-qubit classification

Classification criteria

- **Signature**: numbers of *I*s in the observables (*A* for $N-1$ *I*s, *B* for $N-2$ *I*s, *C* for $N-3$ *I*s, etc)
- **Nature** ν of the doily (*l*inear or *q*uadratic)
- **Configuration** of the negative lines



Classification results

95 types for 4 qubits

Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	3	0	12	q	216				648				648			
2	0	4	0	11	q				3888			3888					
3	0	5	0	10	q	972		1944		4860	1944			1944			
4	1	0	5	9	q	648								648			
5	3	0	3	9	l	144											
6	0	6	0	9	q		1296		5184								
7	0	1	6	8	q	972				3888						972	
8	1	1	5	8	q				7776								
9	2	1	4	8	q	1944		1944									
10	2	1	4	8	l	972					972						
11	0	7	0	8	q			1944		972							
12	0	2	6	7	q				15552			11664	19440				
13	1	2	5	7	q	7776		13608			15552			1944			
14	1	2	5	7	l	3888					7776						
15	2	2	4	7	q		11664						3888				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
95	6	9	0	0	l	6											

Classification results

447 types for 5 qubits

	Observables						Configuration of negative lines											
Type	A	B	C	D	E	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	0	1	5	9	q						58 320			19 440			
2	1	0	0	5	9	q				12 960								
3	0	1	1	4	9	q		58 320		233 280				233 280		116 640		19 440
4	0	2	1	3	9	q	68 040		174 960		116 640	466 560			262 440		116 640	
5	1	2	0	3	9	q				12 960								
6	0	3	1	2	9	q		116 640		421 200			116 640	174 960		116 640		
7	0	4	1	1	9	q			29 160		58 320	29 160						
8	0	5	1	0	9	q				9720								
9	0	0	0	7	8	q				19 440			58 320					
10	0	1	0	6	8	q			58 320		238 140	247 860			145 800			
11	0	0	2	5	8	q				291 600			233 280	233 280		58 320		
12	1	0	1	5	8	q			58 320			58 320						
13	0	2	0	5	8	q		116 640		291 600			174 960	233 280		116 640		
14	0	1	2	4	8	q	58 320		349 920		495 720	787 320			816 480		58 320	
15	1	1	1	4	8	q		58 320		233 280				58 320				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
446	0	15	0	0	0	q	360											
447	6	9	0	0	0	l	10											

Conclusion

Summary of (a significant part of) a recent publication[4]

Results available at `https://quantcert.github.io/doilies`

Perspectives

- Extend the algorithm and the C program to other contextual geometries
- Formal proofs of the discovered properties

Questions?

Fundings

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