Disclosing Quantum Contextuality of Several Multi-Qubit Finite Configurations

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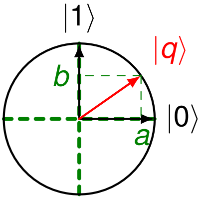
1. Quantum computing basics
2. Properties of multi-qubit doilies
3. Contextuality of quantum configurations

Quantum computing basics

Quantum bit (qubit)

ket notation $\quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

qubit $\quad |q\rangle = a\,|0\rangle + b\,|1\rangle \quad a, b \in \mathbb{C} \quad |a|^2 + |b|^2 = 1$



Single qubit measurement

Measurement of $|q\rangle = a\,|0\rangle + b\,|1\rangle$ in the basis $(|0\rangle\,,|1\rangle)$

$$
\begin{tikzcd}
 & {|0\rangle} \arrow[r, squiggly] & {\color{green}+1} \\
{|q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}} \arrow[ur, "|a|^2"] \arrow[dr, "|b|^2"'] & & \\
 & {|1\rangle} \arrow[r, squiggly] & {\color{red}-1}
\end{tikzcd}
$$

encoded by the third Pauli matrix $Z = \begin{bmatrix} {\color{green}1} & 0 \\ 0 & {\color{red}-1} \end{bmatrix}$

$$
\begin{array}{rcc}
\text{eigenvalues} & {\color{green}+1} & {\color{red}-1} \\
\text{eigenvectors} & |0\rangle & |1\rangle
\end{array}
$$

Pauli group

Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

X measures in the $\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$ basis

Y measures in the $\left(\frac{|0\rangle+i|1\rangle}{\sqrt{2}}, \frac{|0\rangle-i|1\rangle}{\sqrt{2}}\right)$ basis

matrix product

.	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

Pauli group $\mathcal{P} = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}, .)$

commuting pair $X.I = I.X$

anticommuting pair $Y.Z = iX$ and $Z.Y = -iX$, so $Y.Z = -Z.Y$

Multi-qubit

tensor product $A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{pmatrix}$

notation
$A_1 A_2 \cdots A_N$ for $A_1 \otimes A_2 \otimes \cdots \otimes A_N$
$|01\rangle$ for $|0\rangle \otimes |1\rangle$, etc
I^N for $\underbrace{I \otimes \ldots \otimes I}_{N}$

2-qubit $|q\rangle = q_{00}|00\rangle + q_{01}|01\rangle + q_{10}|10\rangle + q_{11}|11\rangle$

N-qubit $|q\rangle = q_{0..0}\underbrace{|0..0\rangle}_{N} + \cdots + q_{1..1}\underbrace{|1..1\rangle}_{N} \in \mathbb{C}^{2^N}$

Generalized Pauli group

N-qubit Pauli operator $\quad G_1 G_2 \cdots G_N$, with $G_i \in \{I, X, Y, Z\}$

generalized Pauli group $\quad \mathcal{P}_N = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}^N, .)$

commuting pair
$$YX.ZZ = (Y.Z)(X.Z) = (iX)(-iY) = XY$$
$$ZZ.YX = (Z.Y)(Z.X) = (-iX)(iY) = XY$$

anticommuting pair
$$XY.IZ = (X.I)(Y.Z) = iXX$$
$$IZ.XY = (I.X)(Z.Y) = -iXX$$

Mutually commuting multi-qubit Pauli operators are
compatible observables

The Mermin-Peres magic square

Finite geometry with 9 points and 6 lines

- Each point is an observable
- Each line is a measurement context

$$
\begin{array}{ccccccc}
-1 & & -1 & & 1 & & \\
X \otimes I & — & I \otimes X & — & X \otimes X & & I \otimes I \\
| & & | & & \| & & \\
1 & & 1 & & 1 & & \\
I \otimes Y & — & Y \otimes I & — & Y \otimes Y & & I \otimes I \\
| & & | & & \| & & \\
-1 & & -1 & & ? & & \\
X \otimes Y & — & Y \otimes X & — & Z \otimes Z & & I \otimes I \\
 & & & & & & \\
I \otimes I & & I \otimes I & & -(I \otimes I) & &
\end{array}
$$

This geometry is *contextual*: no point valuation with -1 or $+1$ satisfies all context values

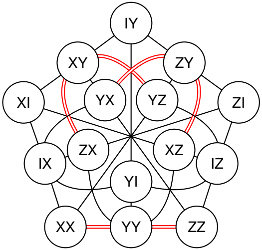
Quantum geometries

Definition of a *quantum geometry* (O, C):

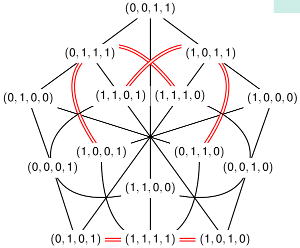
- O is a finite set of observables (points): hermitian operators ($M = M^\dagger$) of finite dimension
- C is a finite set of subsets of O called *contexts* (lines) such that
  - each observable $M \in O$ satisfies $M^2 = I^N$ (eigenvalues in $\{-1, 1\}$),
  - two observables M and N in a context commute ($M.N = N.M$), and
  - the product of all observables in a context is the identity matrix I^N or its opposite $-I^N$

The two-qubit doily, named $W(3,2)$

The doily is the contextual geometry of all the 2-qubit Pauli observables except $I \otimes I$



$\leftrightarrow$

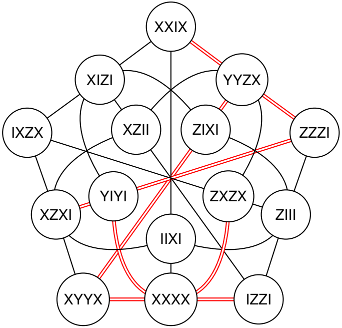


$$I \leftrightarrow (0,0) \quad X \leftrightarrow (0,1) \quad Y \leftrightarrow (1,1) \quad Z \leftrightarrow (1,0)$$

By using a bijection with the symplectic polar space $W(2N-1,2)$, two observables O and O' commute iff the symplectic product $\sigma(\tilde{O}, \tilde{O}')$ of their images is 0

N-qubit doilies

N-qubit doily: Contextual geometry on N qubits with the same point-line geometry as the doily $W(3,2)$

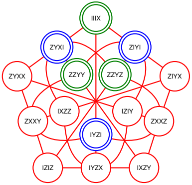


Example of 4-qubit doily

N-qubit doily classification

- **Signature**: number of I per observable (A: $N-1$ identities I per observable, B: $N-2$, C: $N-3$, ...)
- **Nature** ν of a doily

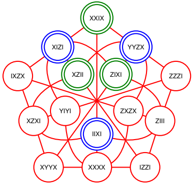
spans a $PG(3,2)$: *linear*



$ZYXI.IYZI.ZIYI = I^4$

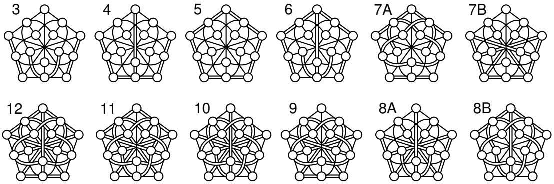
for any tricentric triad (3 observables commuting with 3 common observables)

spans a $PG(4,2)$: *quadratic*



$XIZI.IIXI.YYZX \neq I^4$

- **Configuration** of the negative lines



Classification results

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	3	0	12	q	216				648				648			
2	0	4	0	11	q				3888			3888					
3	0	5	0	10	q	972		1944		4860	1944			1944			
4	1	0	5	9	q	648								648			
5	3	0	3	9	l	144											
6	0	6	0	9	q		1296		5184								
7	0	1	6	8	q	972				3888						972	
8	1	1	5	8	q				7776								
9	2	1	4	8	q	1944		1944									
10	2	1	4	8	l	972					972						
11	0	7	0	8	q			1944		972							
12	0	2	6	7	q				15552			11664	19440				
13	1	2	5	7	q	7776		13608			15552			1944			
14	1	2	5	7	l	3888					7776						
15	2	2	4	7	q		11664						3888				
⋮																	
95	6	9	0	0	l	6											

Partial results for the number of 4-qubit doilies[1]

[1] https://quantcert.github.io/doilies/

Doily generation program

All N-qubit doilies for a given N are generated in order to classify them and check various properties about them[2]

- *C* language used for quick execution

Execution time (Intel® Core™ i7-8665U CPU @ 1.90GHz, 8 cores)

- **4 qubits**: 1 462 272 doilies in 0.5s and 1.4 Mb of RAM
- **5 qubits**: 1 519 648 768 doilies in 12min and 1.8 Mb

[2] Muller, A., Saniga, M., Giorgetti, A., de Boutray, H., and Holweck, F. “Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five”. Journal of Computational Science. 2022.

Contextual finite quantum geometries

$$
\begin{array}{ccccccc}
o_1 & & o_2 & & o_3 & & \\
X \otimes I & — & I \otimes X & – & X \otimes X & & \ell_1 \\
o_4 \mid & & o_5 \mid & & o_6 \parallel & & \\
I \otimes Y & — & Y \otimes I & – & Y \otimes Y & & \ell_2 \\
o_7 \mid & & o_8 \mid & & o_9 \parallel & & \\
X \otimes Y & - & Y \otimes X & – & Z \otimes Z & & \ell_3 \\
\ell_4 & & \ell_5 & & \ell_6 & &
\end{array}
$$

$$
A = \begin{array}{c}
\begin{array}{ccccccccc} o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}
\begin{array}{c} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \end{array}
\qquad
E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\begin{array}{c} I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ -(I \otimes I) \end{array}
$$

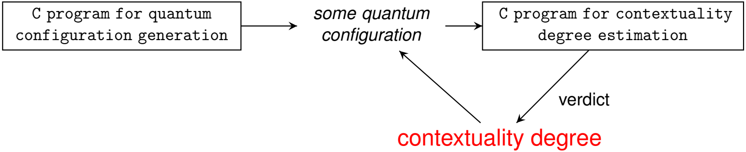
the product of observables on ℓ_i is $(-1)^{E_i}\, I$

The geometry is contextual iff $\nexists\, x.\; Ax = E$

Revealing contextuality in quantum configurations

The *contextuality degree*[3] is the minimal Hamming distance (i.e. the minimal number of unsatisfied constraints) between E and a vector Ax

Computed by a C program using a SAT solver



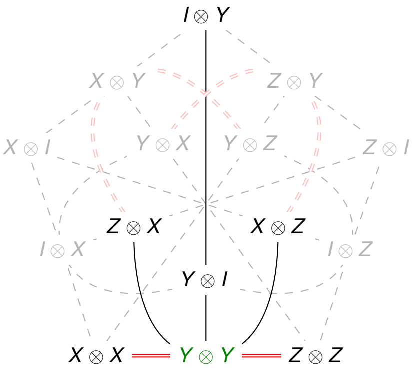
Proposition: The contextuality degree of all multi-qubit doilies is 3

[3] de Boutray, H., Holweck, F., Giorgetti, A., Masson, P.-A., and Saniga, M. “Contextuality degree of quadrics in multi-qubit symplectic polar spaces”. Journal of Physics A: Mathematical and Theoretical. 2022.

Perpsets

The *perpset* P_r is the set of points that commute with a given point r:

$$P_r = \{p \in W(2N-1,2) \mid p \text{ commutes with } r\}$$



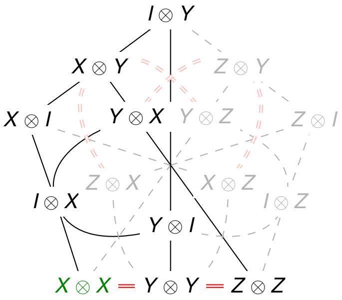
Contextuality checked on 21 834 configurations, 17 minutes

Proposition: All perpsets are non-contextual ($N \geq 2$)

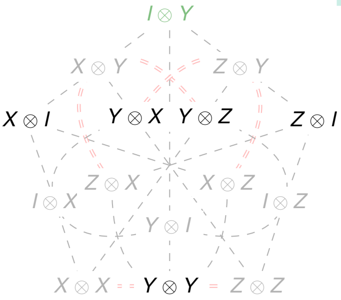
Quadrics

A *quadric* Q_p is a set of points annihilating a quadratic form
$\mathcal{Q}_p(x) = \mathcal{Q}_0(x) + \sigma(x, p)$

Hyperbolic: $\mathcal{Q}_0(p) = 0$



Elliptic: $\mathcal{Q}_0(p) \neq 0$

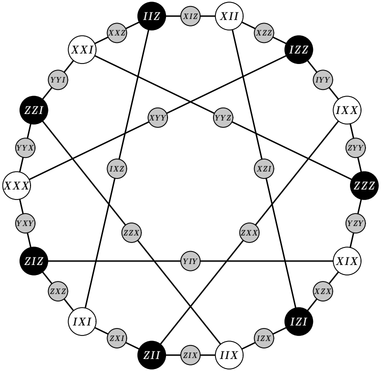


5 456 configurations checked in 33 minutes ($2 \leq N \leq 6$)

Conjecture: All elliptic ($N \geq 3$) and hyperbolic ($N \geq 2$) quadrics are contextual, when the contexts are their lines

3-qubit hyperbolic quadrics

The contextuality degree of all 3-qubit hyperbolic quadrics is 21, and the 21 invalid contexts form a Heawood graph

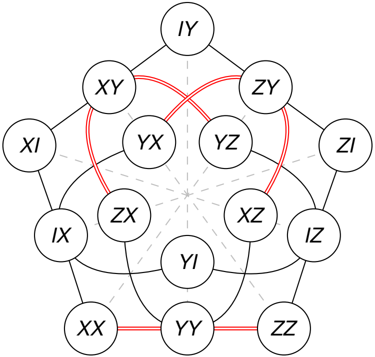


Example of set of invalid contexts in a 3-qubit hyperbolic quadric

Two-spreads

A *spread* is a set of lines such that each point of the plane is on exactly one line of the spread

N-qubit two-spread: Doily from which a spread is removed



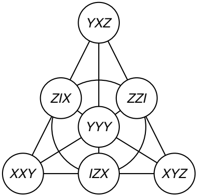
72 configurations checked in 1 second

Proposition: All 2-spreads are contextual, and their contextuality degree is 1 ($N \geq 2$)

Totally isotropic subspaces

A *totally isotropic subspace* is a set of mutually commuting elements with a base of k points

(for $k = 1, 2$ and $N \leq 5$, $3 \leq k$ and $N = 6$, $(k, N) = (6, 7)$) :
14 configurations checked in less than 24 hours per configuration



Example of a Fano plane ($k = 2$)

The configuration whose contexts are:

- all the lines $\rightarrow$ contextual ($k = 1, N \geq 2$)
- *Conjecture: all the planes* $\rightarrow$ *non-contextual* ($k = 2, N \geq 3$)
- all the subspaces of some dimension $k \geq 3 \rightarrow$ non-contextual ($N > k$)

Conclusion

Review

- Computed contextuality degree values leading to various conjectures and proved propositions
- Work leading to a recent publication[4]

Perspectives

- Building more efficient algorithms to compute contextuality degrees
- Finding more properties of quantum configurations
- Proving formally quantum properties

[4] Muller, A., Saniga, M., Giorgetti, A., de Boutray, H., and Holweck, F. *New and improved bounds on the contextuality degree of multi-qubit configurations*. arXiv. 2023.

Questions?

Fundings

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