Computing the Degree of Contextuality of Various Finite Quantum Geometries

Quantum days at AMU, EPiQ WP2 workshop, CIRM, Marseille, 20 September 2023

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joint work with Alain Giorgetti[1] *, Metod Saniga*[2] *, Henri de Boutray*[3] *and Frédéric Holweck*[4,5]

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1. Background

2. Properties of multi-qubit doilies

3. Contextuality of quantum configurations with a SAT solver

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Single qubit measurement

Measurement of $|q\rangle = a\,|0\rangle + b\,|1\rangle$ in the basis $(|0\rangle\,, |1\rangle)$

$$\begin{tikzcd} & {|0\rangle} \arrow[r, squiggly] & +1 \\ {|q\rangle = \begin{pmatrix} a \\ b \end{pmatrix}} \arrow[ur, "|a|^2"] \arrow[dr, "|b|^2"'] & & \\ & {|1\rangle} \arrow[r, squiggly] & -1 \end{tikzcd}$$

encoded by the third Pauli matrix $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

eigenvalues	$+1$	-1
eigenvectors	$\|0\rangle$	$\|1\rangle$

Pauli group

Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

X measures in the $\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$ basis

Y measures in the $\left(\frac{|0\rangle+i|1\rangle}{\sqrt{2}}, \frac{|0\rangle-i|1\rangle}{\sqrt{2}}\right)$ basis

matrix product

.	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

Pauli group	$\mathcal{P} = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}, .)$
commuting pair	$X.I = I.X$
anticommuting pair	$Y.Z = iX$ and $Z.Y = -iX$, so $Y.Z = -Z.Y$

Generalized Pauli group

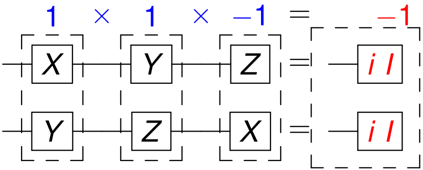
N-qubit Pauli operator	$G_1 G_2 \cdots G_N$, with $G_i \in \{I, X, Y, Z\}$
generalized Pauli group	$\mathcal{P}_N = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}^N, .)$
commuting pair	$YX.ZZ = (Y.Z)(X.Z) = (iX)(-iY) = XY$ $ZZ.YX = (Z.Y)(Z.X) = (-iX)(iY) = XY$
anticommuting pair	$XY.IZ = (X.I)(Y.Z) = iXX$ $IZ.XY = (I.X)(Z.Y) = -iXX$

Mutually commuting multi-qubit Pauli operators are *compatible observables*

Contextuality : The Kochen-Specker theorem

No non-contextual hidden-variable theory can reproduce the outcomes predicted by quantum physics[1]

Without loss of generality, a non-contextual hidden-variable (NCHV) theory admits the existence of a function $v : \mathcal{P}_N \to \{-1, 1\}$ that determines (as $v(M)$) the result of any measurement with the multi-qubit Pauli observable M (among its two eigenvalues -1 and 1) independently of former measurements, even when they are compatible (commuting)



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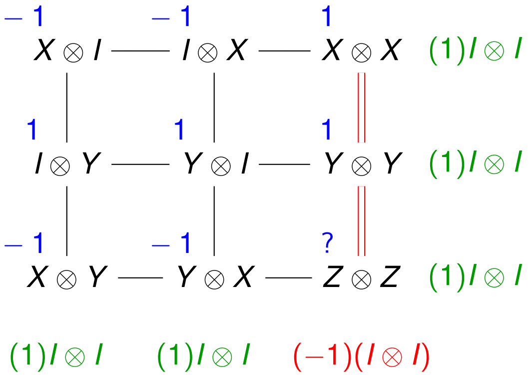
[1] Kochen, Simon and Specker, Ernst. “The Problem of Hidden Variables in Quantum Mechanics”. *Indiana Univ. Math. J.*. 1968.

The Mermin-Peres magic square

Finite geometry with 9 points and 6 lines

- Each point is an observable
- Each line is a measurement context



This geometry is *contextual*: no point valuation with -1 or $+1$ satisfies all context values

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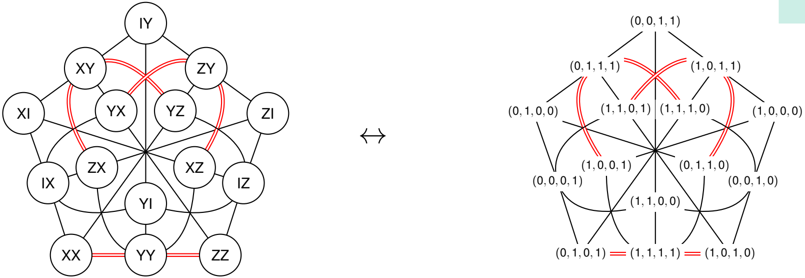
The two-qubit doily $W(3,2)$

The *(2-qubit) doily* is the contextual geometry of all the 2-qubit Pauli observables except $I \otimes I$

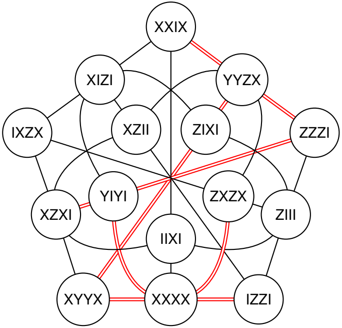


$$I \leftrightarrow (0,0) \quad X \leftrightarrow (0,1) \quad Y \leftrightarrow (1,1) \quad Z \leftrightarrow (1,0)$$

By using a bijection with the symplectic polar space $W(2N-1,2)$, two observables O and O' commute iff the symplectic product $\sigma(\tilde{O},\tilde{O}')$ of their images is 0

N-qubit doilies

N-qubit doily: Contextual geometry on *N* qubits with the same point/line structure as the 2-qubit doily $W(3,2)$



Example of 4-qubit doily

Classification results

	Observables					Configuration of negative lines											
Type	A	B	C	D	ν	3	4	5	6	7A	7B	8A	8B	9	10	11	12
1	0	3	0	12	q	216				648				648			
2	0	4	0	11	q				3888			3888					
3	0	5	0	10	q	972		1944		4860	1944			1944			
4	1	0	5	9	q	648								648			
5	3	0	3	9	l	144											
6	0	6	0	9	q		1296		5184								
7	0	1	6	8	q	972				3888						972	
8	1	1	5	8	q				7776								
9	2	1	4	8	q	1944		1944									
10	2	1	4	8	l	972					972						
11	0	7	0	8	q			1944		972							
12	0	2	6	7	q				15552			11664	19440				
13	1	2	5	7	q	7776		13608			15552			1944			
14	1	2	5	7	l	3888					7776						
15	2	2	4	7	q		11664						3888				
									⋮								
95	6	9	0	0	l	6											

Partial results for the number of 4-qubit doilies[1]

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[1] `https://quantcert.github.io/doilies/`

Doily generation program

All N-qubit doilies for a given N are generated in order to classify and check various properties about them[1][2]

- The *C* language is used for quick execution

Execution time (Intel® Core™ i7-8665U CPU @ 1.90GHz, 8 cores):

- **4 qubits**: 1 462 272 doilies in 0.5s and 1.4 Mo
- **5 qubits**: 1 519 648 768 doilies in 12min and 1.8 Mo

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[1] Muller, A., Saniga, M., Giorgetti, A., de Boutray, H., and Holweck, F. “Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five”. *Journal of Computational Science*. 2022b.

[2] Muller, A., Saniga, M., Giorgetti, A., de Boutray, H., and Holweck, F. “Computer-assisted enumeration and classification of multi-qubit doilies”. *Journées Informatique Quantique 2022 - JIQ’22*. 2022a.

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Contextual finite quantum geometries

$$\begin{tikzcd}
o_1 & o_2 & o_3 & \\
X \otimes I \arrow[r, dash] \arrow[d, dash, "o_4"'] & I \otimes X \arrow[r, dash] \arrow[d, dash, "o_5"'] & X \otimes X \arrow[d, equal, red, "o_6"'] & \ell_1 \\
I \otimes Y \arrow[r, dash] \arrow[d, dash, "o_7"'] & Y \otimes I \arrow[r, dash] \arrow[d, dash, "o_8"'] & Y \otimes Y \arrow[d, equal, red, "o_9"'] & \ell_2 \\
X \otimes Y \arrow[r, dash] & Y \otimes X \arrow[r, dash] & Z \otimes Z & \ell_3 \\
\ell_4 & \ell_5 & \ell_6 &
\end{tikzcd}$$

$$A = \begin{array}{c} \\ \end{array}\begin{array}{ccccccccc|c}
o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \ell_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \ell_3 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \ell_4 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \ell_5 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \ell_6
\end{array}
\qquad
E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\begin{array}{c}
I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ -(I \otimes I)
\end{array}$$

the product of observables on ℓ_i is $(-1)^{E_i}\, I$

$$+1 = (-1)^0, \; -1 = (-1)^1$$

The geometry is contextual iff $\nexists\, x.\ A\,x = E$

Contextuality degree

For every contextual geometry with an incidence matrix A, and for the valuation vector E related to the value of each line, we have

$$\nexists\, x.\ A\,x = E$$

The *contextuality degree*[1] is the minimal Hamming distance $d_H(A\,x, E)$ between E and a vector $A\,x$

$$d_H\left(\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}\right) = 2$$

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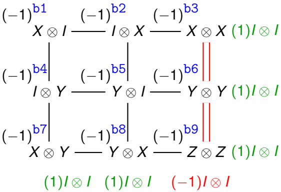
[1] de Boutray, H., Holweck, F., Giorgetti, A., Masson, P.-A., and Saniga, M. “Contextuality degree of quadrics in multi-qubit symplectic polar spaces”. *Journal of Physics A: Mathematical and Theoretical*. 2022.

Revealing contextuality in quantum configurations

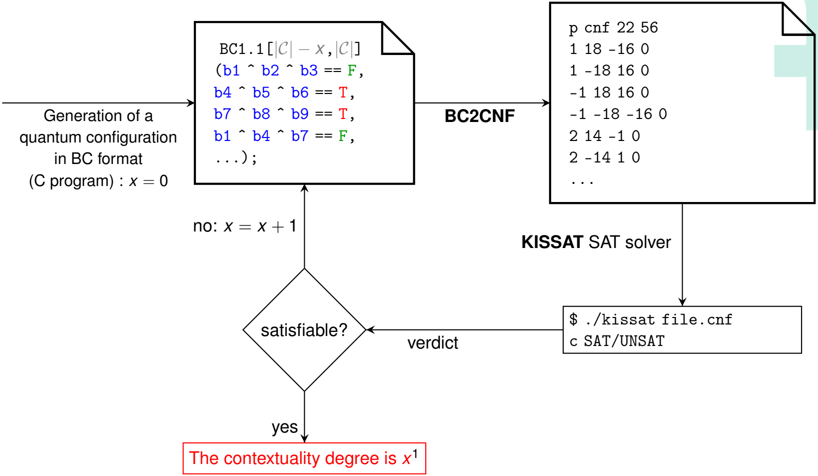


$$b_1 + b_2 + b_3 = 0$$
$$b_4 + b_5 + b_6 = 0$$
$$b_7 + b_8 + b_9 = 0$$
$$b_1 + b_4 + b_7 = 0$$
$$b_2 + b_5 + b_8 = 0$$
$$b_3 + b_6 + b_9 = 1$$

$\longrightarrow$ BC file generator for any quantum configuration

```
BC1.1
(b1 ^ b2 ^ b3 == F,
b4 ^ b5 ^ b6 == F,
b7 ^ b8 ^ b9 == F,
b1 ^ b4 ^ b7 == F,
b2 ^ b5 ^ b8 == F,
b3 ^ b6 ^ b9 == T );
```

Contextuality degree algorithm



Generation of a quantum configuration in BC format (C program) : $x = 0$

```
BC1.1[|𝒞| − x,|𝒞|]
(b1 ^ b2 ^ b3 == F,
b4 ^ b5 ^ b6 == T,
b7 ^ b8 ^ b9 == T,
b1 ^ b4 ^ b7 == F,
...);
```

BC2CNF

```
p cnf 22 56
1 18 -16 0
1 -18 16 0
-1 18 16 0
-1 -18 -16 0
2 14 -1 0
2 -14 1 0
...
```

KISSAT SAT solver

```
$ ./kissat file.cnf
c SAT/UNSAT
```

verdict

satisfiable?

no: $x = x + 1$

yes

The contextuality degree is x[1]

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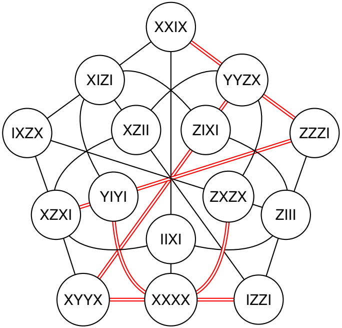
[1] Muller, A., Saniga, M., Giorgetti, A., Boutray, H. de, and Holweck, F. *New and improved bounds on the contextuality degree of multi-qubit configurations*. 2023c.

N-qubit doilies

N-qubit doily: Contextual geometry on N qubits with the same point/line structure as the W_2 doily



Example of 4-qubit doily

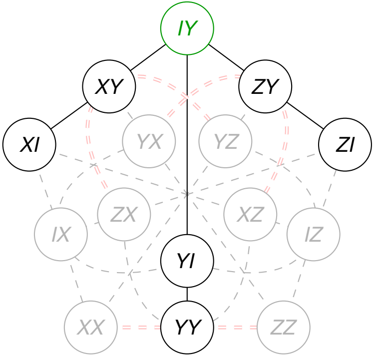
72 configurations checked in 1 second

Proposition: All N-qubit doilies are contextual, and their contextuality degree is 3 ($N \geq 2$)

Perpsets

The *perpset* P_r is the set of points that commute with a given point r:

$$P_r = \{p \in W_n, \text{p commutes with r}\}$$



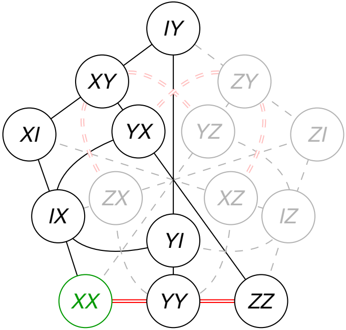
Contextuality checked on 21 834 configurations, 17 minutes

Proposition: All perpsets are non-contextual ($N \geq 2$)

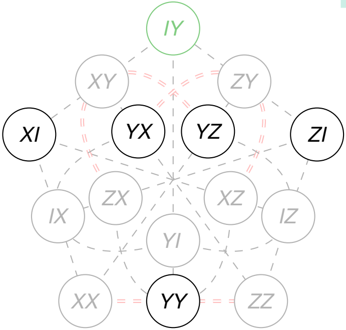
Quadrics

A *quadric* Q_p is a set of points annihilating a quadratic form $\mathcal{Q}_p(x) = \mathcal{Q}_0(x) + \langle x|p\rangle$.

Hyperbolic : $\mathcal{Q}_0(p) = 0$



Elliptic : $\mathcal{Q}_0(p) \neq 0$



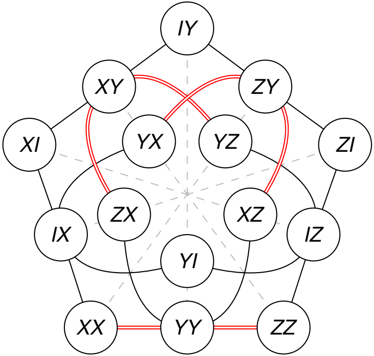
5 456 configurations checked in 33 minutes ($2 \leq N \leq 6$)

Conjecture: All elliptic and hyperbolic quadrics are contextual ($N \geq 2$), when the contexts are their lines

Two-spreads

A *spread* is a set of lines such that each point of the plane is on exactly one line of the spread

A *N-qubit two-spread* is a doily from which a spread is removed



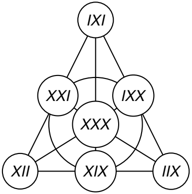
72 configurations checked in 1 second

Proposition: All two-spreads are contextual, and their contextuality degree is 1 ($N \geq 2$)

Totally isotropic subspaces

A *totally isotropic subspace* is a set of mutually commuting elements with a base of k points

$(k = 1, 2 \wedge N \leq 5, 3 \leq k \wedge N = 6, (k, N) = (6, 7))$:
14 configurations checked in less than 24 hours per configuration



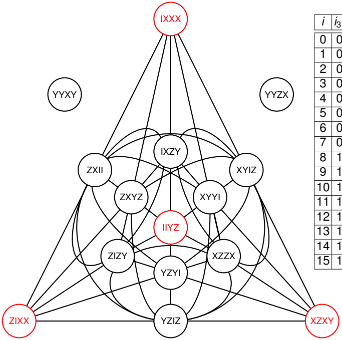
Example of a Fano plane $(k = 2, N = 3)$

The configuration whose contexts are:

- all the lines → contextual $(k = 1, N \geq 2)$
- *Conjecture: all the planes → non-contextual* $(k = 2, N \geq 3)$

Totally isotropic subspaces

- All the subspaces of some dimension $k \geq 3$ are non-contextual because they are all positive ($N > k$)



i	$i_3 i_2 i_1 i_0$	b^i	$\lvert b^i\rvert_3$	$\lvert b^{2i}\rvert_3.\lvert b^{2i+1}\rvert_3$	$\lvert b^i\rvert_2$	$\lvert b^{2i}\rvert_2.\lvert b^{2i+1}\rvert_2$	$\lvert b^i\rvert_1$	$\lvert b^{2i}\rvert_1.\lvert b^{2i+1}\rvert_1$	$\lvert b^i\rvert_0$	$\lvert b^{2i}\rvert_0.\lvert b^{2i+1}\rvert_0$
0	0000	$IIII$	I	$II = I$	I	$II = I$	I	$IY = Y$	I	$IZ = Z$
1	0001	$b_0 = IIYZ$	I		I		Y		Z	
2	0010	$b_1 = IXXX$	I	$II = I$	X	$XX = I$	Ⓧ	$XZ = -\mathrm{i}Y$	Ⓧ	$XY = \mathrm{i}Z$
3	0011	$b_1.b_0 = IXZY$	I		X		Z		Y	
4	0100	$b_2 = XZXY$	X	$XX = I$	Z	$ZZ = I$	X	$XZ = -\mathrm{i}Y$	Ⓨ	$YX = -\mathrm{i}Z$
5	0101	$b_2.b_0 = -XZZX$	X		Z		Z		X	
6	0110	$b_2.b_1 = XYIZ$	X	$XX = I$	Y	$YY = I$	I	$IY = Y$	Z	$ZI = Z$
7	0111	$b_2.b_1.b_0 = XYYI$	X		Y		Y		I	
8	1000	$b_3 = ZIXX$	Z	$ZZ = I$	I	$II = I$	X	$XZ = -\mathrm{i}Y$	X	$XY = \mathrm{i}Z$
9	1001	$b_3.b_0 = ZIZY$	Z		I		Z		Y	
10	1010	$b_3.b_1 = ZXII$	Z	$ZZ = I$	X	$XX = I$	I	$IY = Y$	I	$IZ = Z$
11	1011	$b_3.b_1.b_0 = ZXYZ$	Z		X		Y		Z	
12	1100	$b_3.b_2 = -YZIZ$	Y	$YY = I$	Z	$ZZ = I$	I	$IY = Y$	Z	$ZI = Z$
13	1101	$b_3.b_2.b_0 = -YZYI$	Y		Z		Y		I	
14	1110	$b_3.b_2.b_1 = YYXY$	Y	$YY = I$	Y	$YY = I$	X	$XZ = -\mathrm{i}Y$	Y	$YX = -\mathrm{i}Z$
15	1111	$b_3.b_2.b_1.b_0 = -YYZX$	Y		Y		Z		X	
			I^8		I^8		$Y^4.(-\mathrm{i}Y)^4 = I$		$Z^4.(\mathrm{i}Z)^2.(-\mathrm{i}Z)^2 = I$	
			trivial		*trivial*		*case* 1 ($r = 1$)		*case* 2 ($b^0 = b^1.b^2$)	

Illustrative example of the positivity of a subspace for $k = 3$ and $N = 4$

Conclusion

Achievements

- Program for generating quantum configurations
- Computed contextuality degree values leading to various conjectures and proved propositions

Perspectives

- Building more efficient algorithms to compute contextuality
- Finding more properties of quantum configurations
- Proving formally quantum properties

Questions?

Contributions

- A. Muller et al. (Oct. 2022b). “Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five”. In: *Journal of Computational Science* 64. ISSN: 1877-7503. DOI: `10.1016/j.jocs.2022.101853`
- A. Muller et al. (May 2023c). *New and improved bounds on the contextuality degree of multi-qubit configurations*. arXiv: `2305.10225`
- A. Muller et al. (2023b). “Décider la contextualité de configurations quantiques avec un solveur SAT”. In: *AFADL*, pp. 50–52
- A. Muller et al. (2023a). “Disclosing Quantum Contextuality: A Geometric Approach to N-Qubit Configurations”. In: *SICGT*, pp. 105–106

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