Revealing contextuality of quantum configurations with a SAT solver

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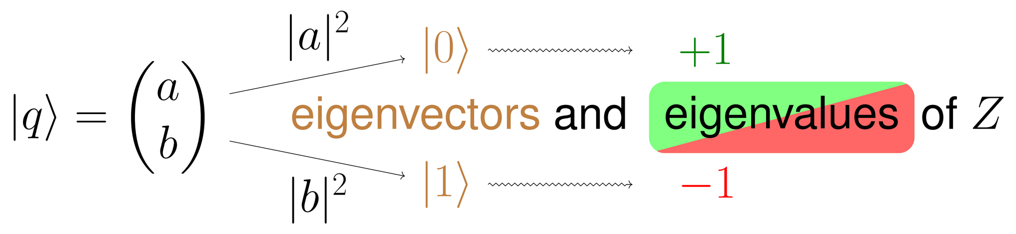
Quantum state measure

ket notation $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $|q\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

qubit $|q\rangle = a\,|0\rangle + b\,|1\rangle$ $a, b \in \mathbb{C}$ $|a|^2 + |b|^2 = 1$

Measure with the Pauli matrix $Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$



Observables

Pauli matrices (1-qubit *observables*): $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ $Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

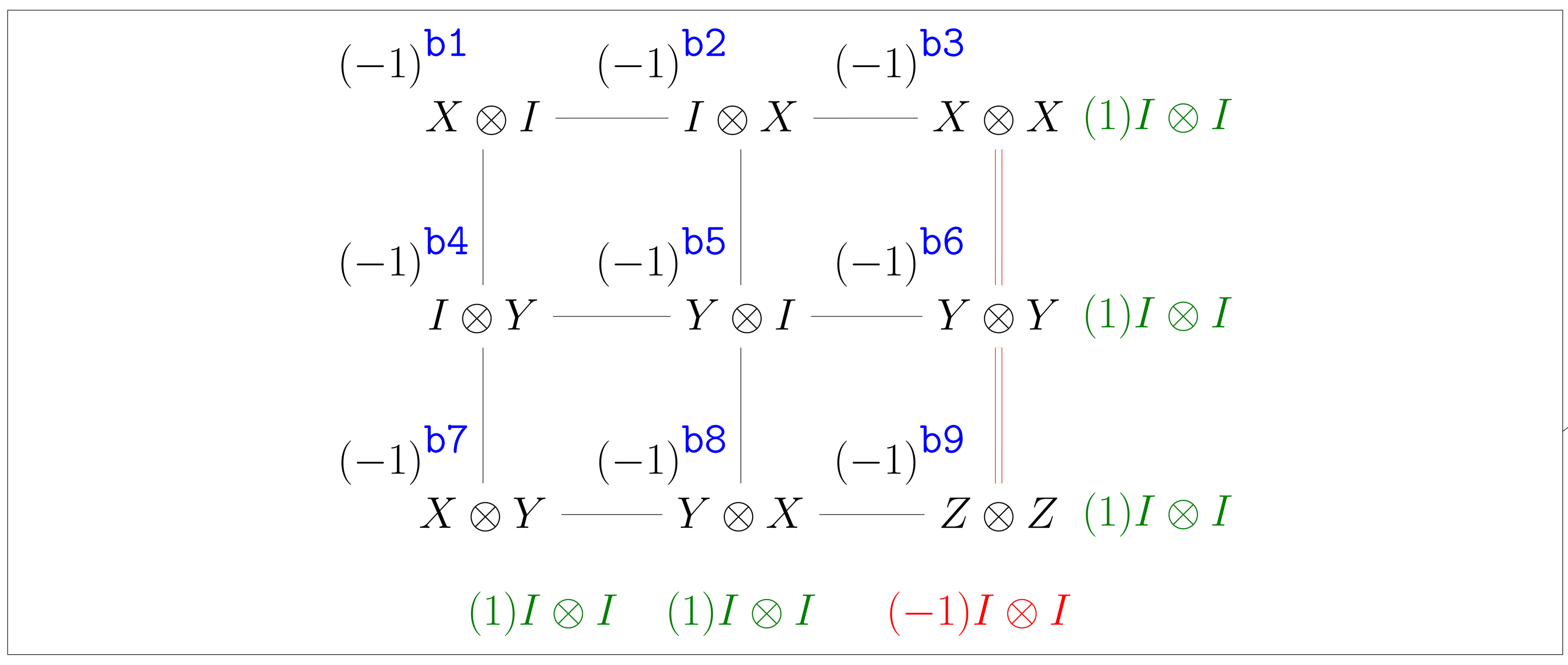
matrix product:

.	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

N-qubit Pauli operator (N-qubit *observable*): $G_1 \otimes G_2 \otimes \cdots \otimes G_N$, with $G_i \in \{I, X, Y, Z\}$
generalized Pauli group: $\mathcal{P}_N = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}^N, .)$

Contextuality

A *context* is a finite subset c of mutually commuting N-qubit observables (eigenvalues in $\{-1, 1\}$, i.e. $(-1)^b$ for a Boolean variable $b \in \{0, 1\}$) whose matrix product $\prod_{o \in c} o$ is $\pm I \otimes \ldots \otimes I$. A *quantum configuration* [1] is a finite set of contexts.



Example: **Mermin-Peres quantum configuration** [2, 3], with 9 two-qubit observables and 6 contexts, either positive ($o_1 - o_2 - o_3$) or negative ($o_1 = o_2 = o_3$), for instance $(X \otimes X).(Y \otimes Y).(Z \otimes Z) = (X.Y.Z) \otimes (X.Y.Z) = i.I \otimes i.I = -I \otimes I$.

The Mermin-Peres configuration is *contextual*: no value for $(\mathtt{b1}, \ldots, \mathtt{b9}) \in \{0, 1\}^9$ is consistent with the eigenvalue ± 1 of the matrix products of each context.

Process

quantum configuration generator [4, 5] (C program)

```
BC1.1
(b1 ^ b2 ^ b3 == F,
b4 ^ b5 ^ b6 == F,
b7 ^ b8 ^ b9 == F,
b1 ^ b4 ^ b7 == F,
b2 ^ b5 ^ b8 == F,
b3 ^ b6 ^ b9 == T );
```

BC2CNF [6]

```
p cnf 22 56
1 18 -16 0
1 -18 16 0
-1 18 16 0
-1 -18 -16 0
2 14 -1 0
2 -14 1 0
...
```

22 variables, 56 clauses

KISSAT SAT solver [7]

```
$ ./kissat file.cnf
c UNSATISFIABLE
```

verdict

Results [4, 5, 8]

Contextuality checked* **for several configurations**	**Proofs and** ***conjectures*****, for an arbitrary of qubits** N
N-qubit doilies ($2 \le N \le 5$), 12 configurations, less than 1 second	All multi-qubit doilies are contextual, and their *contextuality degree* (minimal number of unsatisfied constraints) is 3 ($N \ge 2$)
N-qubit 2-spreads ($2 \le N \le 5$), 72 configurations, 1 second	All 2-spreads are contextual, and their contextuality degree is 1 ($N \ge 2$)
elliptic and hyperbolic quadrics ($2 \le N \le 6$), $5\,456$ configurations, 33 minutes	*Conjecture: All elliptic and hyperbolic quadrics are contextual ($N \ge 2$), when the contexts are their lines*
N-qubit perpsets ($2 \le N \le 7$), $21\,834$ configurations, 17 minutes	All perpsets are non-contextual ($N \ge 2$)
totally isotropic subspaces of dimension $1 \le k < N$ of the symplectic space $W(2N-1, 2)$ ($k = 1, 2 \wedge N \le 5, 3 \le k \wedge N = 6, (k, N) = (6, 7)$), 14 configurations, less than 24 hours per configuration	The configuration whose contexts are all the lines is contextual ($k = 1, N \ge 2$) *Conjecture: The configuration whose contexts are all the planes is non-contextual ($k = 2, N \ge 3$)* The configuration whose contexts are all the subspaces of some dimension $k \ge 3$ is non-contextual ($N > k$)

* computed with a PC equipped with an Intel(R) Core(TM) i7-12700H and 16 GB RAM

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