Estimation of Vibration Delay Rates for a Nonlinear Flexible Beam with Nonlinear Boundary Damping

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Abstract: In this paper, the decay rate estimation of solution and energy function for a nonlinear flexible beam with nonlinear boundary damping are established. The nonlinear boundary feedback criterion, which covers a large class of nonlinear functions, is based on a negative feedback of the transverse velocity at the right boundary of flexible beam. Several decay rates for the solution and energy are provided corresponding to various growth restriction on the nonlinear boundary feedback near the origin. To verify the effectiveness of the results, numerical simulations are shown by the finite element method.

Keywords: Axially moving; Nonlinear beam; Nonlinear boundary damping; Energy decay rate.

1. INTRODUCTION

Flexible materials are widely used in engineering practice due to their light weight, low energy consumption and other advantages, such as marine mooring lines He (2014), flexible manipulators Hu (2008); Pereira (2010), marine risers for oil transportation Do (2008), and crane cables He (2013). However, flexible materials often modeled as beam or string equations are more prone to vibration, which will reduce work efficiency and damage product quality. Boundary control with its own unique advantages is one of the most effective methods to suppress the vibration of flexible systems, for instance Morgul (1992); Geniele (1997); Ge (1998); Krstic et al. (2008); Krstic (2008).

Considering the significant bending stiffness of flexible materials, the vibration system of this structure should be modeled as a beam equations Wu and Wang (2014); Kelleche and Tatar (2017). When the amplitude of the flexible beam is large, the disturbed strain shows strong nonlinearity, which brings about analysis difficulties for stability and well posed results. Ignoring the longitudinal vibration of the beam, the disturbed strain relationship can be approximated by the Kirchhoff function under the quasi-static assumption Arosio (1993). In this context, boundary stabilization of Kirchhoff beam equations had been addressed in Guo-Guo (2007), Kobayashi (2009) and Cheng-Wu-Guo (2021a).

In this paper, considering the disturbing strains deduced by the nonlinear geometric relations Ding and Chen (2009, 2011), we investigate the attenuation estimation of transverse vibration for the nonlinear flexible beam described by the following partial differential equation (PDE)

$$\begin{cases} \rho A w_{tt} + EI w_{xxxx} = [(EA + \dfrac{P-EA}{\sqrt{1+w_x^2}})w_x]_x, \\ EI w_{xxx}(L,t) - (EA + \dfrac{P-EA}{\sqrt{1+w_x^2(L,t)}})w_x(L,t) \\ \qquad = \mathscr{F}(w_t(L,t)), \\ w(0,t) = w_x(0,t) = w_{xx}(L,t) = 0, \\ w(x,0) = g_1(x),\ w_t(x,0) = g_2(x), \end{cases} \tag{1}$$

for all $x \in (0, L)$ and $t > 0$, where $w(x,t)$ denotes the transversal deflection of beam at time t and at the position x, L is the length of beam, ρ is the mass per unit area, A is the cross-sectional area of the beam, E is the Young modulus, P is the initial tension, I is the moment of inertia, $\mathscr{F}$ represents a nonlinear damping, g_1 and g_2 are the initial displacement and the initial velocity of the beam system, respectively. From a physical point of view, the initial tension is usually much smaller than the tensile stiffness ($P \le EA$).

When the deformation of the beam is limited but the amplitude is small, the nonlinear tension $EA + \frac{P-EA}{\sqrt{1+w_x^2}}$ of (1) can be reduced to $P - \frac{P-EA}{2} w_x^2$ in view of $\frac{1}{\sqrt{1+\mu^2}} \approx 1 - \frac{\mu^2}{2}$ as $|\mu| \ll 1$. In this case, the approximate equation of the

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nonlinear beam (1) has been examined in Yang and Hong (2002); Ding and Chen (2010); Kelleche and Tatar (2017). It is worth mentioning that, by using nonlinear boundary controls, the exponential stability of an axially moving beam with same nonlinear tension as (1) is established by the integral-type multiplier method in Cheng-Wu-Guo (2021b).

The purpose of this paper is to establish the estimation of vibration attenuation of the nonlinear flexible beam (1) with nonlinear boundary damping $\mathscr{F}$. In the present paper, we identify a explicit criteria to choose a class of nonlinear damping functions, for which we obtain a explicit energy decay formula. When various growth constraints of nonlinear damping $\mathscr{F}$ at infinity and near the origin are provided, the explicit solutions and energy decay rates for the nonlinear flexible beam (1) are guaranteed by a dissipative ordinary differential equation (ODE). By invoking the method introduced by Lasiecka (1993), we conclude that if we impose some additional restrictions on the growth of nonlinear damping near the origin, we find that when the damping term increases linearly near the origin, the energy function decays exponentially, and when the damping increases as a superlinear or sublinear behavior near the origin, the energy function decays as a polynomial.

The remainder of this paper is planned as follows. In Sect. 2, decay estimates of energy and solution for the nonlinear flexible beam (1) are provided. The proof of main results are given in Sect. 3. In Sect. 4, some simulation results are shown to illustrate the theoretical results. A brief conclusion follows in Section 5.

2. MAIN RESULT

In order to estimate the delay rate of energy function for the nonlinear flexible beam (1), we need the following assumptions on $\mathscr{F}$:

Assumption 1. $\mathscr{F}$ is a non-decreasing continuous function on $\mathbb{R}$ with $\mathscr{F}(0) = 0$ such that

$$\begin{aligned} &0 < \mathscr{F}(s)s, \qquad \forall s \neq 0, \\ &k_1 \leq \frac{\mathscr{F}(s)}{s} \leq k_2, \quad \forall\ |s| \geq N, \end{aligned} \tag{2}$$

for the given constants $0 < k_1 \leq k_2$ and $N > 0$.

Remark 1. It is easy to see that, we do not require explicit change behavior of $\mathscr{F}$ near zero in Assumption 1. Therefore, this condition is more generally compared with lope-restricted condition Cheng-Wu-Guo (2021a) and slope-sector condition Cheng-Wu-Guo (2021b).

Let

$$\begin{aligned} \mathcal{E}(t) := &\frac{\rho A}{2}\int_0^L w_t^2 dx + (P - EA)\int_0^L \sqrt{1 + w_x^2} dx \\ &+ \frac{EI}{2}\int_0^L w_{xx}^2 dx + \frac{EA}{2}\int_0^L w_x^2 dx, \end{aligned} \tag{3}$$

stand for the energy corresponding to the beam system (1). In what follows, we mainly focus on the stability analysis of closed-loop systems (1), since the proof of well-posedness of the problem (1) is analogous to Cheng-Wu-Guo (2021a,b), where the Faedo-Galerkin method is used to complete the two estimates of solutions.

Now, we present the idea based on convexity arguments from Lasiecka (1993), which will play a crucial role in establishing the stability for the nonlinear flexible beam (1). In this context, it is important to point out other important works in the previous results that considered optimal energy decay estimates, such as Alabau-Boussouira (2005, 2010); Alabau-Boussouira-Ammari (2011); Martinez (1999).

Assume that a function $U(s)$ is concave and strictly increasing for $s \geq 0$, with $U(0) = 0$, satisfying

$$U(s\mathscr{F}(s)) \geq s^2 + [\mathscr{F}(s)]^2, \ \forall\ |s| \leq N, \tag{4}$$

for the constant $N > 0$ given in (2). In light of the property of $\mathscr{F}(s)$, such a function $U(s)$ can always be constructed, for detail, see Lasiecka (1993). Set

$$\hat{U}(s) := U(\frac{s}{T}),\ \forall\ s \geq 0, \tag{5}$$

where T is a constant to be determined later. For $\xi > 0$, it is easy to check that $\xi\mathcal{I} + \hat{U}$ is invertible and strictly increasing, where $\mathcal{I}$ is the identity mapping. Define a map

$$\mathcal{P}(s) := (\xi\mathcal{I} + \hat{U})^{-1}(\hat{\xi}s) \tag{6}$$

for a constant $\hat{\xi} > 0$, which is a strictly increasing, positive and continuous function with $\mathcal{P}(0) = 0$. Define $\mathcal{Q}(s) := s - (\mathcal{I} + \mathcal{P})^{-1}(s)$ for $s \geq 0$. Then $\mathcal{Q}(s)$ is also a positive, strictly increasing and continuous function. Recalling an ODE system given by

$$\begin{cases} \dfrac{d}{dt}\mathscr{S}(t) + \mathcal{Q}(\mathscr{S}(t)) = 0, \quad t > 0, \\ \mathscr{S}(0) = s_0, \end{cases} \tag{7}$$

if $\mathcal{P}(t) > 0$ defined in (6) for any $t > 0$, one has $\lim_{t\to\infty}\mathscr{S}(t) = 0$, as discussed in Lasiecka (1993) (see also Cavalcanti (2007)). From the above preliminary work, we state our stability result.

Theorem 2. Let $w(x,t)$ and $\mathcal{E}(t)$ be the solution and energy function of the nonlinear flexible beam (1). Assume that the assumption (2) on $\mathscr{F}$ is satisfied. Then there exist a positive constant $T_0 > 0$ such that

$$\mathcal{E}(t) \leq \mathscr{S}(\frac{t}{T_0} - 1) \tag{8}$$

for all $t > T_0$ with $\lim_{t\to\infty}\mathscr{S}(t) = 0$, where $\mathscr{S}(t)$ is the solution of ODE system (7) with $s_0 = \mathcal{E}(0)$. Moreover,

$$|w(x,t)|^2 \leq \frac{2L^2}{EI}\mathscr{S}(\frac{t}{T_0} - 1) \tag{9}$$

for any $x \in [0, L]$ and all $t \geq 0$.

It should be noted that the asymptotic stability of solutions and energy function to the nonlinear flexible beam (1) can only be guaranteed from the above theorem. Furthermore, if the nonlinear damping function $\mathscr{F}$ satisfies additional specific growth conditions at the origin, a explicit energy decay rate is deduced by applying Theorem 2, as discussed in Cavalcanti (2014). Hence, we present the following further result.

Corollary 3. Under the assumptions of Theorem 2, assume that there exists two positive constants k_1, k_2, such that

$$k_1|s|^p \leq |\mathscr{F}(s)| \leq k_2|s|^{1/p}, \ \forall\ |s| \leq 1, \tag{10}$$

where $p \in [1, \infty)$. Then the energy $\mathcal{E}(t)$ and the solution $w(x,t)$ of nonlinear flexible beam (1) satisfies

$$|w(x,t)|^2 \leq \frac{2L^2}{EI}\mathcal{E}(t) \leq Ct^{\frac{2}{1-p}}, \text{ if } p > 1, \tag{11}$$

and

$$|w(x,t)|^2 \leq \frac{2L^2}{EI}\mathcal{E}(t) \leq Ce^{-\mu t}, \text{ if } p = 1, \tag{12}$$

where μ, C are positive constants, for any $x \in [0, L]$ and all $t \geq 0$.

3. PROOF OF MAIN RESULTS

In order to prove the main result, an auxiliary lemma needs to be shown.

Lemma 4. Let w be the solution of nonlinear flexible beam (1), then for any $T > S > 0$

$$\mathcal{E}(T) = \mathcal{E}(S) - \int_S^T \mathscr{F}(w_t(L,t))w_t(L,t)dt. \tag{13}$$

Proof. Due to (3), the derivative rule and integration by parts show,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = \rho A\int_0^L w_t w_{tt}dx + EI\int_0^L w_{xx}w_{xxt}dx \\
&+ \int_0^L (EA + \frac{P-EA}{\sqrt{1+w_x^2}})w_x w_{xt}dx \\
&= \int_0^L \left\{\rho A w_{tt} - [(EA + \frac{P-EA}{\sqrt{1+w_x^2(x,t)}})w_x]_x\right\}w_t dx \\
&-[EIw_{xxx}(L,t) - (EA + \frac{P-EA}{\sqrt{1+w_x^2(L,t)}})w_x(L,t)]w_t(L,t) \\
&+EIw_{xx}(L,t)w_{xt}(L,t) + EI\int_0^L w_{xxxx}w_t dx \\
&= -\mathscr{F}(w_t(L,t))w_t(L,t),
\end{aligned} \tag{14}$$

where the boundary conditions of (1) are applied. Therefore, integrating on both sides of (14) from T to S, our desired result follows.

Remark 5. Since $\mathscr{F}(w_t(L,t))w_t(L,t) \geq 0$, it easy to see that the energy function $\mathcal{E}(t)$ is non-increasing and $\mathcal{E}(t) \leq \mathcal{E}(0)$ for all $t \geq 0$.

Proof of Theorem 2.

Taking the inner product with xw_x on both sides of the first equation in (1) shows

$$\begin{aligned}
&\rho A\langle xw_x, w_{tt}\rangle + EI\langle xw_x, w_{xxxx}\rangle \\
&= \langle xw_x, \big[(EA + \frac{P-EA}{\sqrt{1+w_x^2(x,t)}})w_x\big]_x\rangle.
\end{aligned} \tag{15}$$

Thanks to the basic derivative rules and integration by parts, one gets

$$\begin{aligned}
\langle xw_x, w_{tt}\rangle &= \int_0^L [xw_x w_t]_t dx - \int_0^L xw_{xt}w_t dx \\
&= \int_0^L [xw_x w_t]_t dx - \frac{L}{2}w_t^2(L,t) + \frac{1}{2}\int_0^L w_t^2 dx,
\end{aligned} \tag{16}$$

and

$$\langle xw_x, w_{xxxx}\rangle = \frac{3}{2}\int_0^L w_{xx}^2 dx + Lw_x(L,t)w_{xxx}(L,t). \tag{17}$$

Likewise, it follows that

$$\begin{aligned}
\Delta :&= \langle xw_x, \big[(EA + \frac{P-EA}{\sqrt{1+w_x^2}})w_x\big]_x\rangle \\
&= Lw_x^2(L,t)(EA + \frac{P-EA}{\sqrt{1+w_x^2(L,t)}}) \\
&- \int_0^L (EA + \frac{P-EA}{\sqrt{1+w_x^2}})[w_x^2 + xw_x w_{xx}]dx.
\end{aligned} \tag{18}$$

Then we can also deduce

$$\begin{aligned}
&\int_0^L (EA + \frac{P-EA}{\sqrt{1+w_x^2}})xw_x w_{xx}dx \\
&= \int_0^L \left[x(\frac{EA}{2}w_x^2 + (P-EA)\sqrt{1+w_x^2})\right]_x dx \\
&- \int_0^L [\frac{EA}{2}w_x^2 + (P-EA)\sqrt{1+w_x^2}]dx \\
&= L[\frac{EA}{2}w_x^2(L,t) + (P-EA)\sqrt{1+w_x^2(L,t)}] \\
&- \int_0^L [\frac{EA}{2}w_x^2 + (P-EA)\sqrt{1+w_x^2}]dx.
\end{aligned} \tag{19}$$

The fact $0 < P \leq EA$ leads to

$$s(EA + \frac{P-EA}{\sqrt{1+s}}) \geq EAs + 2(P-EA)\sqrt{1+s},$$

for all $s \geq 0$. Hence, inserting (19) into (18), this implies

$$\begin{aligned}
\Delta \leq &-\frac{1}{2}\int_0^L [EAw_x^2 + 2(P-EA)\sqrt{1+w_x^2}]dx \\
&- L[\frac{EA}{2}w_x^2(L,t) + (P-EA)\sqrt{1+w_x^2(L,t)}] \\
&+ Lw_x^2(L,t)(EA + \frac{P-EA}{\sqrt{1+w_x^2(L,t)}}).
\end{aligned} \tag{20}$$

Recalling the definition of energy function $\mathcal{E}(t)$, and substituting (16), (17) and (20) into (15) leads to

$$\begin{aligned}\mathcal{E}(t) \leq & -\rho A \int_0^L [x w_x w_t]_t dx + \frac{\rho A L}{2} w_t^2(L,t) \\ & - L[\frac{EA}{2} w_x^2(L,t) + (P-EA)\sqrt{1+w_x^2(L,t)}] \\ & - LEI w_x(L,t) w_{xxx}(L,t) \\ & + [L w_x^2(L,t)(EA + \frac{P-EA}{\sqrt{1+w_x^2(L,t)}})]. \end{aligned} \tag{21}$$

Taking the boundary condition in (1) into account, the estimate above becomes

$$\begin{aligned}\mathcal{E}(t) \leq & - \rho A \int_0^L [x w_x w_t]_t dx + \frac{\rho A L}{2} w_t^2(L,t) \\ & - \frac{L}{2}[EA w_x^2(L,t) + 2(P-EA)\sqrt{1+w_x^2(L,t)}] \\ & - L\mathscr{F}(w_t(L,t)) w_x(L,t). \end{aligned} \tag{22}$$

According to $EA + \frac{P-EA}{\sqrt{1+s}} \geq P$ $(\forall s \geq 0)$ and Young's inequality, it holds that

$$\begin{aligned} & -\frac{L}{2}[EA w_x^2(L,t) + 2(P-EA)\sqrt{1+w_x^2(L,t)}] \\ & -L\mathscr{F}(w_t(L,t)) w_x(L,t) \\ & \leq -\frac{\rho A P L}{2} w_x^2(L,t) + \frac{L}{4\varepsilon}\mathscr{F}^2(w_t(L,t)) + L\varepsilon w_x^2(L,t), \end{aligned} \tag{23}$$

where the fact that

$$\begin{aligned} & [EA w_x^2(L,t) + 2(P-EA)\sqrt{1+w_x^2(L,t)}] \\ & = \int_0^{w_x^2(L,t)} \left(EA + \frac{P-EA}{\sqrt{1+s}} \right) ds \\ & \geq P w_x^2(L,t) \end{aligned} \tag{24}$$

is applied. In view of the arbitrariness of parameters $\varepsilon > 0$, we can choose $\varepsilon = \frac{\rho A P}{2}$. Then insert (23) into (22) to get

$$\begin{aligned}\mathcal{E}(t) \leq & -\rho A \int_0^L [x w_x w_t]_t dx + \frac{\rho A L}{2} w_t^2(L,t) \\ & + \frac{L}{2\rho A P}\mathscr{F}^2(w_t(L,t)), \end{aligned} \tag{25}$$

which yields

$$\mathcal{E}(t) \leq -\rho A \int_0^L [x w_x w_t]_t dx + \hat{C}[w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))], \tag{26}$$

where $\hat{C} = \max\{\frac{L^2}{4P\rho A}, \frac{L}{2}\}$. It is easy to see that

$$\rho A \int_0^L x w_x w_t dx \leq \frac{\rho A L}{2} \int_0^L (w_x^2 + w_t^2) dx \leq \lambda \mathcal{E}(t)$$

with $\lambda = \max\{L, \frac{\rho A P}{L}\}$. This with Remark 5 implies

$$\int_0^T \int_0^L [x w_x w_t]_t dx dt \leq \lambda(\mathcal{E}(0) + \mathcal{E}(T)) \leq 2\lambda \mathcal{E}(0). \tag{27}$$

Integrate simultaneously both sides of (26) from 0 to $T (0 < T)$, and invoke (27) to obtain

$$\int_0^T \mathcal{E}(t) dt \leq 2\lambda \mathcal{E}(0) + \hat{C} \int_0^T [w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))] dt, \tag{28}$$

where $\hat{C}$ is the constant given by (26). Letting $T > T_0$, we immediately see

$$\int_0^T \mathcal{E}(t) dt \leq C_1 \mathcal{E}(0) + \hat{C} \int_0^T [w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))] dt. \tag{29}$$

Due to (13), it follows from (29) that for $T > \hat{T} := \max\{T_0, C_1\}$,

$$\mathcal{E}(T) \leq C_T \int_0^T [w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))] dt, \tag{30}$$

where $C_T > 0$ is a constant depending on T. Denote $\Sigma_N := \{t \in [0,T]; |w_t(L,t)| \leq N\}$, with the constant $N \geq 1$ given by (4). From (2), it is easy to deduce that

$$\begin{aligned} & \int_{[0,T]\setminus\Sigma_N} [w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))] dt \\ & \qquad \leq C_2 \int_{[0,T]\setminus\Sigma_N} w_t(L,t)\mathscr{F}(w_t(L,t)) dt, \end{aligned} \tag{31}$$

where $C_2 = \frac{1+k_2^2}{k_1^2}$. On the other hand, from (4) we obtain

$$\begin{aligned} & \int_{\Sigma_N} [w_t^2(L,t) + \mathscr{F}^2(w_t(L,t))] dt \\ & \qquad \leq \int_{\Sigma_N} U(w_t(L,t)\mathscr{F}(w_t(L,t))) dt. \end{aligned} \tag{32}$$

Based on Jensen's inequality, it holds that

$$\begin{aligned} & \int_{\Sigma_N} U(w_t(L,t)\mathscr{F}(w_t(L,t))) dt \\ & \leq T \cdot U\left(\int_0^T \frac{w_t(L,t)\mathscr{F}(w_t(L,t))}{T} dt \right) \\ & \leq T \cdot \hat{U}\left(\int_0^T w_t(L,t)\mathscr{F}(w_t(L,t)) dt \right). \end{aligned} \tag{33}$$

From (30), we can find

$$\mathcal{E}(T) \leq C_T T \hat{U} \left(\int_0^T w_t(L,t)\mathscr{F}(w_t(L,t)) dt \right)$$

$$+ C_T C_2 \int_0^T w_t(L,t)\mathscr{F}(w_t(L,t))dt. \qquad (34)$$

Set $\hat{\xi} = \frac{1}{C_T T}$ and $\xi = \frac{C_2}{T}$ in (6), we can derive from (34) the following inequality

$$\mathcal{P}(\mathcal{E}(T)) + \mathcal{E}(T) \leq \mathcal{E}(0), \qquad (35)$$

where $\mathcal{P}(s)$ is defined as (6). Using this result repeatedly, we can show

$$\mathcal{P}(\mathcal{E}((n+1)T)) + \mathcal{E}((n+1)T) \leq \mathcal{E}(nT), n = 0, 1, \cdots. \qquad (36)$$

According to the Lemma 3.3 in Lasiecka (1993) with $s_n = \mathcal{E}(nT)$, $s_0 = \mathcal{E}(0)$, $n = 1, 2, \cdots$, then we can find $\mathcal{E}(nT) \leq \mathscr{S}(n)$, where $\mathscr{S}$ is the solution of the ODE system (7). For $t \geq T$, let $t = nT + \eta$, with $0 \leq \eta < T$ and $n = 0, 1, 2, \cdots$. Thus, we can derive

$$\mathcal{E}(t) \leq \mathcal{E}(nT) \leq \mathscr{S}(n) = \mathscr{S}(\frac{t-\eta}{T}) \leq \mathscr{S}(\frac{t}{T} - 1). \qquad (37)$$

The definition of $\mathcal{E}(t)$ with Poincaré inequality implies $|w(x,t)|^2 \leq L^2 \|w_{xx}\|^2 \leq \frac{2L^2}{EI}\mathcal{E}(t)$ for any $x \in [0, L]$ and all $t \geq 0$. The proof of Theorem 2 is complete.

Proof of Corollary 3.
The key point of the proof is to construct such a function $U(s)$ satisfying property (4). In view of (10), for any $|s| < 1$ we have $k_1|s|^p \leq |\mathscr{F}(s)|$ and $|\mathscr{F}(s)|^p \leq k_2^p|s|$. Then we can set $U(s) = (k_1^{\frac{-2}{p+1}} + k_2^{\frac{2p}{p+1}})s^{\frac{2}{p+1}}$. It easy to check that U fulfills the condition (4). Thus, define the map $\mathcal{P}(s) = (\xi\mathcal{I} + \hat{U})^{-1}(\hat{\xi}s)$, i.e., $\xi\mathcal{P}(s) + C_{k_1,k_2}\mathcal{P}(s)^{\frac{2}{p+1}} = \hat{\xi}s$ where C_{k_1,k_2} is a suitable constant depending on k_1 and k_2. Recalling the map $\mathcal{Q}(s) = s - (\mathcal{I} + \mathcal{P})^{-1}(s)$ and when s is very small, one gets $\mathcal{P}(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ and $\mathcal{Q}(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ with some constant $\mathcal{C} > 0$. Consequently, our desired estimates (11) and (12) are demonstrated by solving (7) with $\mathcal{Q}(s)$ as above and invoking Theorem 2.

4. NUMERICAL SIMULATION

In this section, two simulation examples are present for the nonlinear flexible beam (1) to show the effectiveness of the proposed results. The simulation is carried out by using the finite element method, where the the quadratic Lagrange basis of the finite element equidistant meshes is applied.

To show the numerical results, the parameters of nonlinear flexible beam (1) are assigned as follows. Consider a flexible beam with Young modulus of elasticity $E = 2.3 \times 10^{11} Pa$, density $\rho = 7900 kg/m^3$, $L = 1$, $A = 0.0045 m^2$, the initial tension $P = 7850 N$ and $I = 7.0846 \times 10^{-6} m^4$. The initial displacement and velocity of the flexible beam adopted in simulation are $g_1(x) = 0.4\sin(6x)$ and $g_2(x) = 0.5\cos(4x)$. Two damping functions satisfying the restrictive conditions (2) and (10) near infinity and zero are present in simulation as follows:

$$F_1(s) = 2s, \qquad (38)$$

$$F_2(s) = \begin{cases} 2s - 8, & s \leq -1, \\ 10s^3, & -1 < s < 1, \\ 3s + 7, & 1 \leq s. \end{cases} \qquad (39)$$

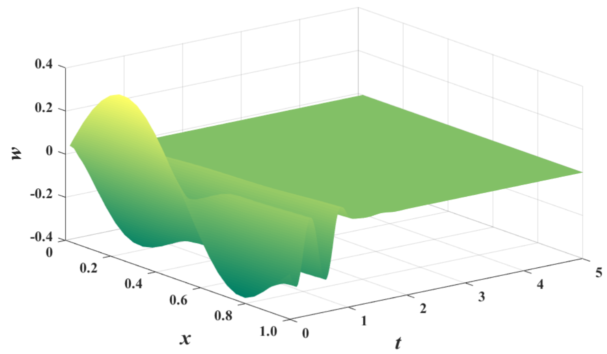

Fig. 1. Transverse displacements of the nonlinear flexible beam (1) with the nonlinear feedback function (38).

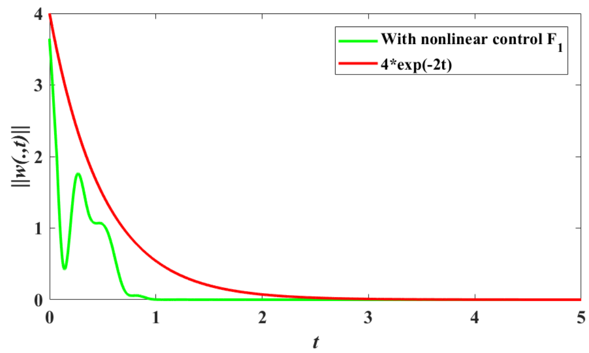

Fig. 2. Norm of the nonlinear flexible beam (1) with the nonlinear feedback function (38).

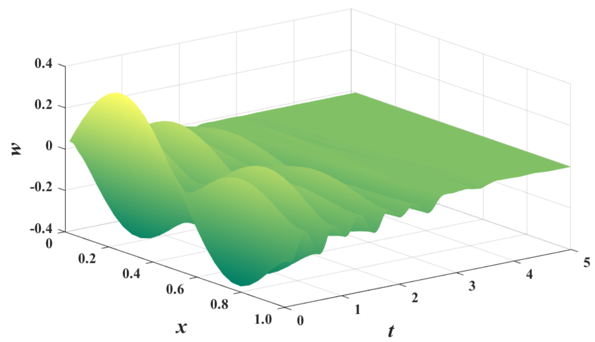

Fig. 3. Transverse displacements of the nonlinear flexible beam (1) with the nonlinear feedback function (39).

The transverse vibration $w(x,t)$ of the nonlinear flexible beam (1) with the nonlinear damping (39) and the linear damping (38) are illustrated in Fig. 1 and Fig. 3, which indicates that the transverse vibration of the nonlinear flexible beam has been suppressed. It is easy to see that the corresponding norm $\|w(\cdot,t)\|$ of the nonlinear flexible beam (1) with the nonlinear damping (39) and the linear damping (38) are depicted in Fig. 2 and Fig. 4, respectively, which coincides with the theoretical result Corollary 3.

5. CONCLUSION

In this paper, the amplitude of vibration and energy decay rates of the nonlinear flexible beam (1) are estimated. The asymptotic stability of solutions of the nonlinear flexible

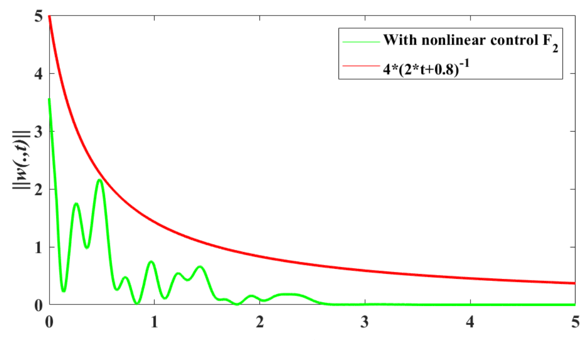

Fig. 4. Norm of the nonlinear flexible beam (1) with the nonlinear feedback function (39).

beam is guaranteed by a dissipative ODE system. Under the explicit growth of nonlinear boundary damping near zero, the explicit decay rates of energy and solutions for the nonlinear flexible beam can be obtained. If a time-delay damping is applied at the boundary, the boundary stabilization of the nonlinear flexible beam is still an open problem and will be the focus of future work.

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