Port-Hamiltonian modeling of a geometrically nonlinear hyperelastic beam ⋆

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Abstract: This paper is concerned with the port-Hamiltonian modeling of a Timoshenko beam subject geometric nonlinearities through von Kármán strains, material nonlinearity considering hyperelasticity with the assumption of neo-Hookean or Mooney-Rivlin material, in addition to the incompressible deformation constraint that corresponds to the preservation of volume. The model is suitable for representing the behavior of rubber like beams within the range of moderate deformations and rotations. Numerical simulations are carried out to illustrate the accuracy of the proposed model.

Keywords: Port-Hamiltonian systems; Modeling; Timoshenko beam; Nonlinear systems.

1. INTRODUCTION

Beam models find extensive applications in various fields of interest such as structural engineering and/or robotics. Nonlinear models are essential to represent the complex dynamic behavior of such systems subject to large deformations, or/and based on material with nonlinear behavior. The use of port-Hamiltonian formulations offers a powerful mathematical framework to model these complex physical systems in an efficient way, facilitating their analysis, control and design within a unified and structured approach.

The interconnection structure of port-Hamiltonian systems (PHS) serves as a framework for the effective modeling and control of complex nonlinear multi-physical systems (Duindam et al., 2009). The modeling and control of mechanical systems have been addressed in this framework in both linear and nonlinear cases. For example, in (Macchelli and Melchiorri, 2004) the PHS formulation of the Timoshenko beam is presented, which considers small deformations and linear elastic material by means of Hooke's law. Furthermore, in (Brugnoli, 2020) models for general linear elasticity, linear thermoelasticity, and classical plate models such as Kirchhoff-Love and Mindlin plates are presented within the PHS framework. Recently, in (Ponce et al., 2023) a methodology is presented for the port-Hamiltonian modeling of a wide range of linear mechanical systems that include bars in tension, circular bars in torsion, vibrating string, beams, plates, among others. All of these models are based on the infinitesimal strain theory and Hooke's law, which place these models in the category of geometrically and materially linear systems, respectively. On the other hand, several researchers have made notable contributions in the domain of geometrically nonlinear modeling, employing formulations based on von Kármán strains. (Brugnoli et al., 2021) presented an Euler-Bernoulli beam model, while (Voss et al., 2008) explored a piezo-actuated Euler-Bernoulli beam. In a similar vein, (Voss and Scherpen, 2014) introduced a piezo-actuated Timoshenko beam. (Trivedi et al., 2015) proposed a cart-mounted Timoshenko beam model, incorporating the additional constraint of inextensibility. Notably, (Brugnoli and Matignon, 2022) extended this idea by introducing a two-dimensional von Kármán plate model. In a broader context, (Thoma and Kotyczka, 2022) formulated elasticity incorporating geometric nonlinearity through the finite strain tensor (Green-Lagrange strain tensor), while maintaining material linearity through the assumption of hyperelasticity via a Saint Venant-Kirchhoff material. To the best of our knowledge, the only known work incorporating both sources of nonlinearities within an infinite-dimensional PHS model is the study by (Kinon et al., 2023), where the focus is made on modeling a geometrically exact string, considering hyperelasticity with a neo-Hookean material (in this context, geometrically exact implying no neglect of terms in the Green-Lagrange strain tensor).

In this paper we derive a port-Hamiltonian model for a geometrically nonlinear Timoshenko beam based on von Kármán strains, considering material nonlinearity due to hyperelasticity with neo-Hookean or Mooney-Rivlin mod-

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els for the solid, in addition to incompressible deformation constraint associated with conservation of volume during the deformation. These considerations make the model suitable for representing the dynamic behavior of rubber like beams in the range of moderate deformations and rotations, finding applications for example in the field of continuous robots. The paper is structured as follows: Section 2 presents the background regarding the modeling of nonlinear mechanical systems and some definitions about PHS. Section 3 presents the different stages in obtaining the model until reaching the port-Hamiltonian representation, and also are shown numerical simulations. Finally, section 4 gives some conclusions and discusses future work.

2. BACKGROUND

In this section we present a short overview of the physical laws that will be applied in the subsequent sections, and some usual definitions about infinite-dimensional PHS. For the mathematical description of the system we denote the reference domain by Ω, and its boundary is assumed to be divided in two subdomains as $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\partial\Omega_D \cap \partial\Omega_N = \{\emptyset\}$, where $\partial\Omega_D$, $\partial\Omega_N$ are the Dirichlet and Neumann boundary portions, and $\{\emptyset\}$ is the empty set. The material coordinates are denoted as $\mathtt{x} = \{\zeta_1, \zeta_2, \zeta_3\}$ and specify a point on Ω, and similarly $\mathtt{s}$ on $\partial\Omega$. To enhance the readability of the paper, reference to spatial and temporal dependencies will be often omitted.

2.1 Elasticity

The branch of physics that studies the relationship between external forces, internal stresses, and deformation is known as elasticity. A key variable in elasticity is the displacement field $\mathbf{u}(\mathtt{x}, t) \in \mathbb{R}^3$, which assigns to each point $\mathtt{x}$ in a body of volume $\forall \subset \mathbb{R}^3$ a displacement vector that specifies its current position in the deformed configuration regarding a reference configuration. For large deformations, is often used the Green-Lagrange strain tensor ($\underline{\varepsilon} \in \mathbb{R}^{3\times3}$) where its components are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial \zeta_j} + \frac{\partial \mathbf{u}_j}{\partial \zeta_i} + \sum_{k=1}^{3} \frac{\partial \mathbf{u}_k}{\partial \zeta_i}\frac{\partial \mathbf{u}_k}{\partial \zeta_j}\right) \tag{1}$$

with $i, j = (1, 2, 3)$. Since the Green-Lagrange strain tensor is symmetric, it can be represented as a six-components vector using the Voigt-Kelvin notation, then

$$\varepsilon = [\varepsilon_{11} \ \ \varepsilon_{22} \ \ \varepsilon_{33} \ \ 2\varepsilon_{12} \ \ 2\varepsilon_{13} \ \ 2\varepsilon_{23}]^\top. \tag{2}$$

When the body is deformed, it develops internal stresses, so there is a relationship between the strain and stress tensors. Denoting the strain energy density function (per unit volume) as $W(\underline{\varepsilon}) \in \mathbb{R}$, this relationship is given by

$$\underline{\sigma} = \frac{\partial W(\underline{\varepsilon})}{\partial \underline{\varepsilon}} \tag{3}$$

where $\underline{\sigma} \in \mathbb{R}^{3\times3}$ is the second Piola-Kirchhoff stress tensor. Similarly to the strain tensor, using the Voigt-Kelvin notation the stress tensor can be expressed as a vector $\sigma = [\sigma_{11} \ \ \sigma_{22} \ \ \sigma_{33} \ \ \sigma_{12} \ \ \sigma_{13} \ \ \sigma_{23}]^\top$. Usually the strain energy density function is expressed as $W(\underline{C})$, where $\underline{C}$ is the right Green-Cauchy tensor and is related to $\underline{\varepsilon}$ by

$$\underline{C} = 2\underline{\varepsilon} + I \tag{4}$$

with $I \in \mathbb{R}^{3\times3}$ the identity matrix. It is important to mention that the eigenvalues of $\underline{C}$ represent the square of the principal stretches, so the conservation of volume during the deformation can be expressed by the following condition

$$\det(\underline{C}) = 1. \tag{5}$$

One way to obtain the equations of motion is by using Hamilton's principle. This principle states that the true evolution of $\mathbf{u}(\mathtt{x}, t)$, between two specific states $\mathbf{u}(\mathtt{x}, t_1)$ and $\mathbf{u}(\mathtt{x}, t_2)$ at two specific times t_1 and t_2, is a stationary point of the action functional, that is

$$\delta \int_{t_1}^{t_2} (T - U + W_E)\, dt = 0 \tag{6}$$

subject to $\delta\mathbf{u}(\mathtt{x}, t_1) = \delta\mathbf{u}(\mathtt{x}, t_2) = 0$ for all $\mathtt{x}$, and $\delta\mathbf{u}(\mathtt{s}, t) = 0$ for $\mathtt{s} \in \partial\Omega_D$. In (6), T, U, and W_E denote the kinetic energy, the elastic potential energy, and the external work, respectively which are defined by

$$T = \frac{1}{2}\int_\forall \rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}}\, d\forall \tag{7}$$

$$U = \int_\forall W(\underline{\varepsilon})\, d\forall \tag{8}$$

$$W_E = \int_\forall f_\forall \cdot \mathbf{u}\, d\forall + \int_S f_S \cdot \mathbf{u}\, dS \tag{9}$$

where ρ is the constant density, $f_\forall, f_S \in \mathbb{R}^3$ represent body and surface forces, respectively, and S denotes the boundary surface of the volume $\forall$ where f_S is applied. For a more comprehensive review on these topics, refer to (Reddy, 2013, 2017).

2.2 Infinite-dimensional port-Hamiltonian systems

In simple terms, a conservative and autonomous infinite-dimensional PHS is defined by (van der Schaft and Maschke, 2002; Brugnoli, 2020)

$$\begin{aligned} \partial_t x &= \mathcal{J}\, \delta_x H \\ u_\partial &= \mathcal{B}_\partial \delta_x H \\ y_\partial &= \mathcal{C}_\partial \delta_x H \end{aligned} \tag{10}$$

where $\partial_t = \partial/\partial t$, x is the state and contains the energy variables, $\mathcal{J} = -\mathcal{J}^*$ is a formally skew-adjoint differential operator, $\delta_x H$ is the variational derivative of the Hamiltonian functional H with respect to x that defines the co-energy variables. $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ are boundary operators that provide the boundary input and output u_∂ and y_∂, respectively (Le Gorrec et al., 2004, 2005). To define a PHS in the Stokes-Dirac structure, the operators $\mathcal{J}, \mathcal{B}_\partial, \mathcal{C}_\partial$ must satisfy an integration by parts formula, see (Brugnoli, 2020, Assumption 1). The subsequent definition and lemma define a class of first-order differential operators and their formal adjoints, accompanied by an integration-by-parts formula allowing to define power conjugated boundary port variables. These formulations naturally extend certain findings outlined in (Voss and Scherpen, 2014; Brugnoli et al., 2021; Ponce et al., 2023).

Definition 1 Let $\mathtt{x} = \{\zeta_1, \ldots, \zeta_\ell\}$ be a set of orthogonal coordinate axes, $\Omega \subset \mathbb{R}^\ell$ an open set, $v(\mathtt{x}) \in \mathbb{R}^m$ and $w(\mathtt{x}) \in \mathbb{R}^n$ two vector functions. The first order differential operator $\mathcal{F}_\mathtt{x}$ and its formal adjoint $\mathcal{F}^*_\mathtt{x}$ are given by

$$\mathcal{F}_\mathtt{x}\, w(\mathtt{x}) = P_0(\mathtt{x})\, w(\mathtt{x}) + \textstyle\sum_{k=1}^{\ell} P_k(\mathtt{x})\, \partial_k w(\mathtt{x}) \tag{11}$$

$$\mathcal{F}^*_\mathtt{x} v(\mathtt{x}) = P_0(\mathtt{x})^\top v(\mathtt{x}) - \textstyle\sum_{k=1}^{\ell} \partial_k (P_k(\mathtt{x})^\top v(\mathtt{x})) \tag{12}$$

with $\partial_k = \partial/\partial\zeta_k$ and $P_0(\mathtt{x})$, $P_k(\mathtt{x}) \in \mathbb{R}^{m\times n}$.

Lemma 1 Consider that Definition 1 holds. Let be $\Omega \subset \mathbb{R}^\ell$ an ℓ-dimensional domain, its boundary $\partial\Omega$ and $\bar{\Omega} = \Omega \cup \partial\Omega$ the closure, such that $\mathtt{x} \in \Omega$ and $\mathtt{s} \in \partial\Omega$. Then for any $v(\mathtt{x}) \in \mathbb{R}^m$ and $w(\mathtt{x}) \in \mathbb{R}^n$ defined in $\bar{\Omega}$ we have that

$$\int_\Omega v(\mathtt{x})^\top \mathcal{F}_\mathtt{x} w(\mathtt{x}) - w(\mathtt{x})^\top \mathcal{F}_\mathtt{x}^* v(\mathtt{x}) d\mathtt{x} = \int_{\partial\Omega} w(\mathtt{s})^\top \mathcal{P}_\partial(\mathtt{s}) v(\mathtt{s}) d\mathtt{s} \tag{13}$$

with $\mathcal{P}_\partial(\mathtt{s}) \in \mathbb{R}^{n\times m}$ a boundary valued matrix given by

$$P_\partial(\mathtt{s}) = \sum_{k=1}^{\ell} P_k(\mathtt{s})^\top \hat{n}_k(\mathtt{s}) \tag{14}$$

where $\hat{n}_k(\mathtt{s})$ is the component of the outward unit normal vector to the boundary projected on the axis ζ_k.

3. MODELING

This section presents the modeling of the Timoshenko beam represented in Figure 1, where $u_0(\zeta_1, t)$ is the axial displacement, $w(\zeta_1, t)$ is the vertical displacement, and $\psi(\zeta_1, t)$ is the angle of rotation of the cross section. For this case, $\zeta_1 \in \Omega = (0, L) \subset \mathbb{R}$ with L the length of the beam in the reference configuration, $\mathtt{s} = \{0, L\}$ and is assumed that $\mathtt{s} = 0 \in \partial\Omega_D$ and $\mathtt{s} = L \in \partial\Omega_N$. This means that Dirichlet boundary conditions will be applied in $\zeta_1 = 0$, and Neumann boundary conditions in $\zeta_1 = L$. In addition, we denote the cross section area of the beam in the reference configuration as A, and we assume that it is symmetric with respect to their centroidal coordinates.

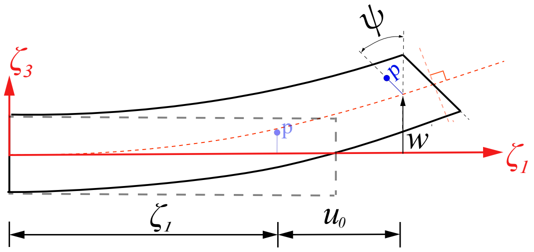

Fig. 1. Timoshenko beam scheme.

3.1 Kinematics

The kinematic assumption of the Timoshenko beam is that plane sections perpendicular to the neutral axis before deformation remain plane but not necessarily perpendicular to the neutral axis after deformation. The displacement field $\mathbf{u}(\mathtt{x}, t)$ of the Timoshenko beam, that mathematically represents its kinematic assumption, is given by

$$\mathbf{u}(\mathtt{x}, t) = \begin{bmatrix} 1 & -\zeta_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ \psi \\ w \end{bmatrix} = \bar{M}\, r \tag{15}$$

where $r = [u_0 \; \psi \; w]^\top \in \mathbb{R}^3$ is the generalized displacement field and gathers the primary unknowns of the problem, and $\bar{M}$ is a mapping matrix. From (1), the nonzero components of the Voigt-strain vector are given by

$$\begin{aligned} \varepsilon_{11} &= \partial_1 u_0 + \tfrac{1}{2}(\partial_1 w)^2 - \zeta_3 \partial_1 \psi + \tfrac{1}{2}(\partial_1 u_0 - \zeta_3 \partial_1 \psi)^2 \\ 2\varepsilon_{13} &= (\partial_1 w - \partial_1 \psi) + \psi(\partial_1 u_0 - \zeta_3 \partial_1 \psi) \\ \varepsilon_{33} &= \tfrac{1}{2}\psi^2 \end{aligned}$$

but using the hypothesis of von-Kármán strains, the nonlinear stretching terms are neglected and we obtain

$$\varepsilon_{11} = \partial_1 u_0 + \frac{1}{2}(\partial_1 w)^2 - \zeta_3 \partial_1 \psi \tag{16}$$

$$2\varepsilon_{13} = \partial_1 w - \partial_1 \psi \tag{17}$$

$$\varepsilon_{33} = \frac{1}{2}\psi^2. \tag{18}$$

Considering the above von-Kármán strains, from (4) the right Green-Cauchy tensor $\underline{C}$ is given by

$$\underline{C} = \begin{bmatrix} 2\varepsilon_{11}+1 & 0 & 2\varepsilon_{13} \\ 0 & 1 & 0 \\ 2\varepsilon_{13} & 0 & 2\varepsilon_{33}+1 \end{bmatrix}. \tag{19}$$

Then, the constraint of incompressible deformation in (5) is equivalent to

$$\det(\underline{C}) = (2\varepsilon_{11}+1)(2\varepsilon_{33}+1) - 4\varepsilon_{13}^2 = 1$$

from where it is obtained $\varepsilon_{33} = \frac{2\varepsilon_{13}^2 - \varepsilon_{11}}{2\varepsilon_{11}+1}$, but similar to the methodology discussed in (Azarniya et al., 2023), we approximate it by considering the initial terms of the Taylor series expansion around $\varepsilon_{11} = 0$, resulting in

$$\varepsilon_{33} \approx -\varepsilon_{11} + 2\varepsilon_{11}^2 + 2\varepsilon_{13}^2 - 4\varepsilon_{11}^3 - 4\varepsilon_{11}\varepsilon_{13}^2. \tag{20}$$

Now, analogously to (Voss and Scherpen, 2014), from (16) and (17) the generalized strain $\epsilon(\zeta_1, t) \in \mathbb{R}^3$ is defined as

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} \partial_1 u_0 + \frac{1}{2}(\partial_1 w)^2 \\ \partial_1 \psi \\ \partial_1 w - \psi \end{bmatrix} \tag{21}$$

and its time derivative can be written as

$$\dot{\epsilon} = \begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial_1 & 0 & \partial_1 w \partial_1 \\ 0 & \partial_1 & 0 \\ 0 & -1 & \partial_1 \end{bmatrix}}_{\mathcal{F}_\mathtt{x}} \underbrace{\begin{bmatrix} \dot{u}_0 \\ \dot{\psi} \\ \dot{w} \end{bmatrix}}_{\dot{r}} \tag{22}$$

where $\mathcal{F}_\mathtt{x}$ is a differential operator as in Definition 1 with associated matrices

$$P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 0 & \partial_1 w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

and formal adjoint according to (12) given by

$$\mathcal{F}_\mathtt{x}^* = -\begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 1 \\ \partial_1(\cdot\, \partial_1 w) & 0 & \partial_1 \end{bmatrix}. \tag{24}$$

3.2 Elastic potential energy

For the beam, hyperelastic behavior is first modeled using the neo-Hookean solid, chosen for its capacity to precisely and efficiently represent the nonlinear characteristics of elastomeric materials under large deformations. Additionally, its relative mathematical simplicity eases implementation and application in numerical and theoretical analyses.

In order to define the PHS, it is necessary to write the elastic potential energy in terms of the generalized strains (21). To start, the von-Kármán strains in (2) are equivalently written as

$$\begin{aligned} \varepsilon_{11} &= \epsilon_1 - \zeta_3 \epsilon_2 \\ \varepsilon_{13} &= \tfrac{1}{2}\epsilon_3 \\ \varepsilon_{33} &= \Psi_0(\epsilon) + \zeta_3 \Psi_1(\epsilon) + \zeta_3^2 \Psi_2(\epsilon) + \zeta_3^3 \Psi_3(\epsilon) \end{aligned}$$

where

$$\Psi_0(\epsilon) = 2\epsilon_1^2 - \epsilon_1 + \tfrac{1}{2}\epsilon_3^2 - \epsilon_1^2\epsilon_3^2 - 4\epsilon_1^3 \quad (25)$$
$$\Psi_1(\epsilon) = \epsilon_2 - 4\epsilon_1\epsilon_2 + 2\epsilon_1\epsilon_2\epsilon_3^2 + 12\epsilon_2\epsilon_1^2 \quad (26)$$
$$\Psi_2(\epsilon) = 2\epsilon_2^2 - \epsilon_2^2\epsilon_3^2 - 12\epsilon_1\epsilon_2^2 \quad (27)$$
$$\Psi_3(\epsilon) = 4\epsilon_2^3. \quad (28)$$

The strain energy density function of the compressible neo-Hookean solid is given by

$$W(\underline{C}) = \alpha_1(I_C - 3 - 2\ln(J)) + \alpha_2(J-1)^2 \quad (29)$$

where $I_C = tr(\underline{C})$ is the first invariant of $\underline{C}$, $J = \sqrt{\det(\underline{C})}$ is called the Jacobian, and the constants α_1, α_2 are properties of the material. Replacing (20) in (19) allows us to simplify (29) as

$$W(\underline{C}) = \alpha_1(I_C - 3) = 2\alpha_1(2\varepsilon_{11}^2 + 2\varepsilon_{13}^2 - 4\varepsilon_{11}^3 - 4\varepsilon_{11}\varepsilon_{13}^2)$$

which corresponds to the strain energy density function of the incompressible neo-Hookean beam. Then, writing a differential of volume as $d\forall = dAd\zeta_1$, by definition (8) we obtain

$$\begin{aligned} U &= \int_\Omega\int_A 2\alpha_1(2\varepsilon_{11}^2 + 2\varepsilon_{13}^2 - 4\varepsilon_{11}^3 - 4\varepsilon_{11}\varepsilon_{13}^2)\, dA\, d\zeta_1 \\ &= \int_\Omega\int_A 2\alpha_1[(\Psi_0(\epsilon)+\epsilon_1) + \zeta_3(\Psi_1(\epsilon) - \epsilon_2)\dots \\ &\qquad \dots + \zeta_3^2\Psi_2(\epsilon) + \zeta_3^3\Psi_3(\epsilon)]\, dA\, d\zeta_1 \\ &= \int_\Omega 2\alpha_1 A(\Psi_0(\epsilon)+\epsilon_1) + 2\alpha_1 I_2\Psi_2(\epsilon)\, d\zeta_1 \\ &= \int_\Omega \Psi_{NH}(\epsilon)\, d\zeta_1 \end{aligned} \quad (30)$$

where $\Psi_{NH}(\epsilon)$ is defined as the generalized strain energy density function of the neo-Hookean beam, and I_2 is the second moment of inertia of the cross section. Note that to go from line two to three in the previous expression it was applied $\int_A \zeta_3\, dA = \int_A \zeta_3^3\, dA = 0$, which is a consequence of the assumption of symmetry in the cross-section with respect to the centroidal coordinates.

Remark 1 The constitutive relation of the beam material is not restricted to neo-Hookean. For example, it is common for rubber like materials to be modeled using the Mooney-Rivlin model.

To illustrate how to use other constitutive relations, the case of the Mooney-Rivlin solid is then briefly presented below. Considering that in our case $J = 1$, the strain energy density function of the incompressible Mooney-Rivlin solid is given by

$$W(\underline{C}) = \beta_1(I_C - 3) + \beta_2(II_C - 3) \quad (31)$$

where $II_C = \frac{1}{2}\left(tr(\underline{C})^2 - tr(\underline{C}^2)\right)$ is the second invariant of $\underline{C}$, and the constants β_1, β_2 are related to the properties of the material. Similarly to what has been previously presented, according to (31) and after some computation, the elastic potential energy of the beam can be expressed as

$$\begin{aligned} U &= \int_\Omega 2\beta_1 A(\Psi_0(\epsilon)+\epsilon_1) + 2\beta_1 I_2\Psi_2(\epsilon) - 4\beta_2 I_4\epsilon_2\Psi_3(\epsilon)\dots \\ &\qquad \dots + 4\beta_2 I_2\left((\epsilon_1+1)\Psi_2(\epsilon) - \epsilon_2\Psi_1(\epsilon)\right)\dots \\ &\qquad \dots + \beta_2 A\left(4(\epsilon_1+1)\Psi_0(\epsilon) + 4\epsilon_1 - \epsilon_3\right)\, d\zeta_1 \\ &= \int_\Omega \Psi_{MR}(\epsilon)\, d\zeta_1 \end{aligned} \quad (32)$$

where $\Psi_{MR}(\epsilon)$ is defined as the generalized strain energy density function of the Mooney-Rivlin beam, and I_4 is the fourth moment of inertia of the cross section. In what follows, $\Psi(\epsilon)$ will represent either $\Psi_{NH}(\epsilon)$ or $\Psi_{MR}(\epsilon)$, depending on the hyperelastic model under consideration.

3.3 Kinetic energy and external work

From (7) and using the expression of the displacement field $\mathbf{u}(\mathbf{x}, t)$ in (15), the kinetic energy of the beam is given by

$$T = \frac{1}{2}\int_\Omega \dot r^\top \int_A \rho\bar M^\top \bar M\, dA\, \dot r\, d\zeta_1 = \frac{1}{2}\int_\Omega \dot r^\top \mathcal{M}\, \dot r\, d\zeta_1 \quad (33)$$

where $\mathcal{M}$ is defined as the distributed mass matrix, ans its expression is derived from

$$\mathcal{M} = \int_A \rho \begin{bmatrix} 1 & -\zeta_3 & 0 \\ -\zeta_3 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} dA = \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho A \end{bmatrix} \quad (34)$$

where again was applied $\int_A \zeta_3\, dA = \int_A \zeta_3^3\, dA = 0$. Then, defining the generalized momentum variable as $p = \mathcal{M}\dot r$, the kinetic energy of the beam can be expressed as

$$T = \frac{1}{2}\int_\Omega p^\top \mathcal{M}^{-1}\, p\, d\zeta_1. \quad (35)$$

Note that, as in the linear case, the kinetic energy is quadratic in the generalized momentum variables.

On the other hand, the first term of the external work in (9) involves the body forces $f_\forall$, which in the context of the beam, this is specifically regarded as the effect of its own weight, resulting in $f_\forall = [0\ 0\ -\rho g]^\top$ with g the acceleration of gravity. Consequently, the first term of (9) can be expressed as follows

$$\int_\forall f_\forall \cdot \mathbf{u}\, d\forall = \int_\Omega\int_A f_\forall^\top \bar M\, dA\, r\, d\zeta_1 = \int_\Omega b^\top r\, d\zeta_1 \quad (36)$$

where $b = [0\ 0\ -\rho g A]^\top \in \mathbb{R}^3$ is defined as the generalized body force. With respect to the second term of (9), S denotes the surface where f_S is applied (which in our case corresponds to the cross section at $\zeta_1 = L$ where Neumann boundary conditions are applied). Considering the above, the external work of the beam can be rewritten as

$$W_E = \int_\Omega b\cdot r\, d\zeta_1 + \int_{\partial\Omega_N} \tau_\partial \cdot r\, d\mathsf{s} \quad (37)$$

where $\tau_\partial \in \mathbb{R}^3$ is defined as the generalized boundary traction.

3.4 Port-Hamiltonian model

From the previous subsections it is now possible to propose a port-Hamiltonian representation of the Timoshenko beam featuring both geometric and material nonlinearities, considering also the constraint associated to the incompressible deformation.

Proposition 1 Let $x(\zeta_1, t) = [p(\zeta_1,t)^\top\ \epsilon(\zeta_1,t)^\top\ w(\zeta_1,t))]^\top$ be the state variable. Based on the displacement field (15), the elastic potential energy (30) or (32), the kinetic energy (35), and the external work (37), the beam dynamics defines an infinite-dimensional PHS of the form

$$
\underbrace{\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \\ \dot{w} \end{bmatrix}}_{\dot{x}} = \underbrace{\left[\begin{array}{ccc|ccc|c} 0 & 0 & 0 & \partial_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_1 & 1 & 0 \\ 0 & 0 & 0 & \partial_1(\cdot\,\partial_1 w) & 0 & \partial_1 & -1 \\ \hline \partial_1 & 0 & \partial_1 w \partial_1 & 0 & 0 & 0 & 0 \\ 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]}_{\mathcal{J}(x) = -\mathcal{J}(x)^*} \underbrace{\begin{bmatrix} e_{p_1} \\ e_{p_2} \\ e_{p_3} \\ e_{\epsilon_1} \\ e_{\epsilon_2} \\ e_{\epsilon_3} \\ e_w \end{bmatrix}}_{\delta_x H} \quad (38)
$$

$$
u_\partial(\mathbf{s}, t) = [v_\partial(0,t)^\top \; \tau_\partial(L,t)^\top]^\top
$$
$$
y_\partial(\mathbf{s}, t) = [y_v(0,t)^\top \; y_\tau(L,t)^\top]^\top
$$

Where

$$
H(p, \epsilon, w) = \int_0^L \frac{1}{2} p^\top \mathcal{M}^{-1} p + \Psi(\epsilon) + \rho g A w \, d\zeta_1 \quad (39)
$$

is the Hamiltonian and represents the sum of the kinetic energy, elastic potential energy and gravitational potential energy. The co-energy variables are defined as $e_p = \delta_p H = \dot{r}$, $e_\epsilon = \delta_\epsilon H = \partial\Psi/\partial\epsilon$ and $e_w = \delta_w H = \rho g A$. The boundary input u_∂ is composed of the generalized boundary velocity $v_\partial(0,t) = \dot{r}(0,t)$ and the generalized boundary tractions $\tau_\partial(L,t) = \mathcal{P}_\partial(L) e_\epsilon(L,t)$. The boundary output y_∂ contains the power conjugated variables $y_v(0,t) = \mathcal{P}_\partial(0) e_\epsilon(0,t)$ and $y_\tau(L,t) = \dot{r}(L,t)$. Then, the power exchange is given by $\dot{H} = u_\partial^\top y_\partial$.

Proof. The equations of motion are obtained by applying Hamilton's principle. First of all we have

$$
\delta T = \int_\Omega \delta \dot{r}^\top \mathcal{M} \dot{r} \, d\zeta_1
$$
$$
\delta W_E = \int_\Omega \delta r^\top b \, d\zeta_1 + \int_{\partial\Omega_N} \delta r^\top \tau_\partial \, d\mathbf{s}
$$
$$
\delta U = \delta \int_\Omega \Psi(\epsilon) \, d\zeta_1 = \int_\Omega \frac{\delta\Psi}{\delta\epsilon} \cdot \delta\epsilon \, d\zeta_1 = \int_\Omega e_\epsilon^\top \mathcal{F}_x \delta r \, d\zeta_1
$$

where in δU was used $\delta\Psi/\delta\epsilon = \partial\Psi/\partial\epsilon = e_\epsilon$ and $\delta\epsilon = \mathcal{F}_x \delta r$. Then, applying Lemma 1 to δU and considering that $\delta r(\mathbf{s}, t) = 0$ in $\partial\Omega_D$ since $\delta\mathbf{u}(\mathbf{s}, t) = 0$ in $\partial\Omega_D$ we obtain

$$
\delta U = \int_\Omega \delta r^\top \mathcal{F}_{\mathbf{x}}^* e_\epsilon \, d\zeta_1 + \int_{\partial\Omega_N} \delta r^\top \mathcal{P}_\partial e_\epsilon \, d\mathbf{s}.
$$

Before applying Hamilton's principle we integrate by parts δT with respect to time, then

$$
\int_{t_1}^{t_2} \delta T \, dt = -\int_{t_1}^{t_2} \int_\Omega \delta r^\top \mathcal{M} \ddot{r} \, d\zeta_1 \, dt + \int_\Omega \delta r^\top \mathcal{M} \dot{r} \Big|_{t_1}^{t_2} d\zeta_1.
$$

where the last term above is equal to zero because $\delta r(\zeta_1, t_1) = \delta r(\zeta_1, t_2) = 0$ since $\delta\mathbf{u}(\mathbf{x}, t_1) = \delta\mathbf{u}(\mathbf{x}, t_2) = 0$. Then, applying Hamilton's principle we obtain

$$
\int_{t_1}^{t_2} \left[\int_\Omega \delta r^\top (\mathcal{M}\ddot{r} + \mathcal{F}_{\mathbf{x}}^* e_\epsilon + b) \, d\zeta_1 + \ldots \right.
$$
$$
\left. \ldots \int_{\partial\Omega_N} \delta r^\top (\tau_\partial - P_\partial e_\epsilon) \, d\mathbf{s} \right] dt = 0.
$$

So applying the lemmas in Appendix A to each term in the above expression we obtain the following equation of motion and boundary conditions

$$
\text{for all } \zeta_1 \in \Omega: \quad \mathcal{M}\ddot{r} + \mathcal{F}_{\mathbf{x}}^* e_\epsilon + b = 0 \quad (40)
$$
$$
\text{for all } \mathbf{s} \in \partial\Omega_N: \quad \tau_\partial(\mathbf{s}, t) = \mathcal{P}_\partial(\mathbf{s}) e_\epsilon(\mathbf{s}, t). \quad (41)
$$

Then, (40) together with (22) and $\dot{w} = e_{p_3}$ can be written equivalently as in (38) with respect to the Hamiltonian defined in (39). The power exchanged with the environment is given by

$$
\dot{H} = \int_\Omega \delta_x H^\top \dot{x} \, d\zeta_1 = \int_\Omega e_\epsilon^\top \mathcal{F}_{\mathbf{x}} e_p - e_p^\top \mathcal{F}_{\mathbf{x}}^* e_\epsilon - e_{p_3} e_w + e_w e_{p_3} d\zeta_1
$$
$$
= \int_{\partial\Omega} e_p^\top \mathcal{P}_\partial e_\epsilon d\mathbf{s} = \int_{\partial\Omega_D} v_\partial^\top y_v \, d\mathbf{s} + \int_{\partial\Omega_N} y_\tau^\top \tau_\partial \, d\mathbf{s} = u_\partial^\top y_\partial
$$

where was applied Lemma 1, the boundary $\partial\Omega$ was partitioned into $\partial\Omega_D$ and $\partial\Omega_N$, and the boundary integrals were computed at their respective locations, specifically at $\mathbf{s} = 0$ in $\partial\Omega_D$ and $\mathbf{s} = L$ in $\partial\Omega_N$. ■

The model introduced in Proposition 1 encompasses geometric nonlinearity within the interconnection operator $\mathcal{J}$ and material nonlinearity in the co-energy variables e_ϵ, similarly to the hyperelastic string model proposed in (Kinon et al., 2023), where spatial and temporal discretization schemes are also presented. Consequently, finite-dimensional models suitable for controller design and analysis could be derived using these approaches, or using other conventional schemes for nonlinear mechanical systems. Note that the presented methodology facilitates a modular adjustment in the elastic potential energy, allowing the consideration of various hyperelastic materials without affecting the interconnection structure of the port-Hamiltonian system. Furthermore, the methodology is similar to the one presented in (Ponce et al., 2023) for the linear case. Therefore, further work on this topic could lead to a modeling methodology for a broader range of multidimensional mechanical systems that incorporate both geometric and material nonlinearities.

3.5 Numerical simulations

Adapting the spatial discretization scheme outlined in (Kinon et al., 2023) to our specific scenario, we obtained the following simulated results using the neo-Hookean beam model. The used physical parameters are $\alpha_1 = 45$ [GPa], $\rho = 7800$ [kg/m^3], $L = 30$ [cm], $A = 30$ [mm^2], $I_2 = 2.5$ [mm^4], $g = 9.8$ [m/s^2]. The boundary inputs are considered as $v_\partial(\mathbf{s}, t) = 0$ and

$$
\tau_\partial(\mathbf{s}, t) = [0 \;\; 0 \;\; 10 \sin(40\pi t)]^\top \quad (42)
$$

for $0 \le t \le 0.2$ [s]. With these considerations and the introduction of artificial damping, Fig. 2 illustrates the generalized displacements $u_0(\zeta_1, t)$ and $w(\zeta_1, t)$ evaluated at $\zeta_1 = L$, respectively. Fig. 3 presents the temporal evolution of the configuration. Both figures demonstrate the ability of the port-Hamiltonian model to capture the coupled behavior of the generalized displacements $u_0(\zeta_1, t)$ and $w(\zeta_1, t)$. This coupling arises from the integration of geometric and material nonlinearities, along with the constraint of incompressible deformation.

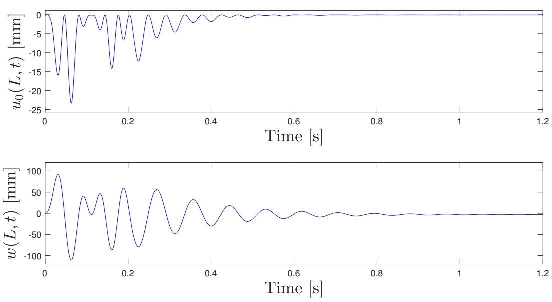

Fig. 2. Neo-Hookean beam: Tip dynamics.

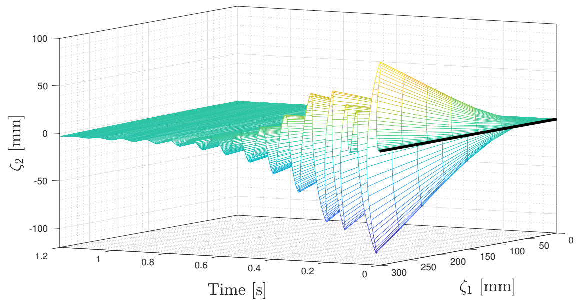

Fig. 3. Neo-Hookean beam: Configuration dynamics.

4. CONCLUSION

In this paper we derive a port-Hamiltonian model for an hyperelastic beam considering both geometric and material nonlinearities. The model is derived starting from its kinematic assumptions and constitutive relationships. The adopted methodology accounts for von-Kármán strains, enforces the incompressible deformation constraint internally, and provides flexibility in the choice of hyperelastic models without altering the interconnection structure of the derived model. Two specific cases are presented, the first case employing a neo-Hookean solid model and the second featuring a Mooney-Rivlin model. Furthermore, given the modularity of the presented methodology, we are currently working on a general method to obtain port-Hamiltonian representations for multidimensional mechanical systems that consider both types of geometric and material nonlinearities. Finally, the structure-preserving discretization and use of the resulting models for control purposes are also under consideration.

Appendix A. LEMMAS FROM VARIATIONAL CALCULUS

Lemma 2 (Gurtin, 1973, p.224.) Let $\mathcal{W}$ be an inner product space, and consider a C^0 field $h : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ with $\bar{\Omega}$ the closure $\bar{\Omega} = \Omega \cup \partial\Omega$. If the equation

$$\int_{t_1}^{t_2} \int_{\Omega} h(\mathtt{x}, t) \cdot \eta(\mathtt{x}, t)\, d\mathtt{x}\, dt = 0 \tag{A.1}$$

holds for every C^∞ field $\eta : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on $\partial\Omega$, then $h(\mathtt{x}, t) = 0$ on $\bar{\Omega} \times [t_1, t_2]$.

Lemma 3 (Gurtin, 1973, p.224.) Suppose that $\partial\Omega$ consists of complementary regular sub-surfaces $\partial\Omega_D$ and $\partial\Omega_N$. Let $\mathcal{W}$ be an inner product space, and consider a function $g : \partial\Omega_N \times [t_1, t_2] \to \mathcal{W}$ that is piecewise regular and continuous in time. If the equation

$$\int_{t_1}^{t_2} \int_{\partial\Omega_N} g(\mathtt{s}, t) \cdot \eta(\mathtt{s}, t)\, d\mathtt{s}\, dt = 0 \tag{A.2}$$

holds for every C^∞ field $\eta : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on $\partial\Omega_D$, then $g(\mathtt{s}, t) = 0$ on $\partial\Omega_N \times [t_1, t_2]$.

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