Stabilization and Decay Rate Estimation for Axially Moving Kirchhoff-type Beam with Rotational Inertia under Nonlinear Boundary Feedback Controls ⋆

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Abstract

In this paper, we consider the stabilization of an axially moving Kirchhoff-type beam with rotational inertia under controls of force and torque at one boundary. The proposed negative feedbacks of the transverse velocity and angular velocity applied at the control end covers a large class of nonlinear feedback functions. The well-posedness of the resulting closed-loop system is established by means of the nonlinear semigroup theory, where the solution is shown to be depending continuously on the initial value. The asymptotic stability of the closed-loop system is guaranteed by resolving a dissipative ordinary differential equation. The decay rates of the vibration for some special nonlinear feedback functions can be estimated by the dissipative ordinary differential equation provided that growth restrictions on these nonlinear feedbacks near the origin are required. Three types of examples including exponential, polynomial and polynomial-logarithmic decay forms are deduced, and the numerical simulations are presented to verify the proposed control approach.

Key words: Axially moving; Kirchhoff-type beam; boundary control; nonlinear semigroup theory.

1 Introduction

Vibration control for axially moving beam systems has attracted much attention over the decades due to many engineering applications such as aerial cable tram ways, oil pipelines, magnetic tapes, paper sheet processes, fiber winding power, band saws and transmission belts, which can be found respectively in papers (Öz-Pakdemirli, 1999; Ghayesh-Khadem, 2008; Hong et al., 2022) and the references therein. Among many different control strategies of passive and active controls, the boundary control is much efficient and practically feasible active control strategy to suppress the vibration of the systems described by partial differential equations (PDEs), as such, it takes advantages of fewer sensors and easy implementations (Alabau-Komornik, 1999; Alabau , 1997; Prieur-Trélat, 2019; Karafyllis-Krstic, 2019).

Actually, boundary controllability and stabilization for non-moving and moving beams such as fixed Euler-Bernoulli beam (Wu-Wang, 2014), linear shear beam (Krstic et al., 2008), axially moving Euler-Bernoulli beams (Choi et al., 2004), linear thermoelastic beam (Hansen-Zhang, 1997) and axially moving Timoshenko beam (Mokhtari-Mirdamadi, 2018), name just a few, have been widely investigated particularly in recent years from both perspectives of mathematics and control engineering.

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Under a quasi-static stretch assumption given in Wickert (1992), a vibration model of elastic system can be reduced to the axially moving Kirchhoff beam as follows:

$$\rho\mathcal{A}_c(w_{tt}(x,t)+2vw_{xt}(x,t)+v^2w_{xx}(x,t)) \\ = M(\|w_x(\cdot,t)\|^2)w_{xx}(x,t)-\mathcal{E}\hat{\mathcal{I}}w_{xxxx}(x,t), \tag{1}$$

for all $x\in(0,L)$ and $t>0$, where $w(x,t)$ stands for the transversal deflection of beam at the position x and time t, ρ is the mass per unit area, $\mathcal{A}_c$ is the cross-sectional area of the beam, v is the moving speed of beam, $\mathcal{E}$ is the Young modulus, $\hat{\mathcal{I}}$ is the moment of inertia, L is the length of the beam, and

$$M(\|w_x(\cdot,t)\|^2)=\tilde{P}+\mathcal{E}\mathcal{A}_c\int_0^L w_x^2(x,t)dx$$

represents the nonlinear tension with the initial axial tension $\tilde{P}$ and the Kirchhoff correction $\int_0^L w_x^2(x,t)dx$. Hereafter, for notational simplicity, we miss the obvious variable components by using the abbreviated notations $w:=w(x,t)$, $w_x:=\frac{\partial w}{\partial x}$, and $w_t:=\frac{\partial w}{\partial t}$.

When $M(\cdot)\equiv$ Const. in (1), i.e., by ignoring the tension change caused by the vibration of beam in the deflection process, the system is simplified to be a moving Euler-Bernoulli beam. The exponential delay rate of the moving Euler-Bernoulli beam with the linear boundary control was considered in Choi et al. (2004). When the nonlinear feedback satisfies the slope-restricted condition, absolute stability for axially moving Kirchhoff-beam (1) with $v>0$ was established by the integral-type multiplier method in our previous work (Cheng-Wu-Guo, 2021). The boundary stabilization of other nonlinear moving beam where the axial strain $1/L\int_0^L w_x^2(x,t)dx$ is replaced by $w_x^2(x,t)$ at position x has been examined in Kelleche-Tatar (2017).

When $M(\cdot)\in C^1(0,\infty)$ is an abstract non negative function, some adaptive boundary output feedback controls for the non-moving Kirchhoff-type beam (1) with $v=0$ was investigated in Kobayashi et al. (2009). However, in all the aforementioned papers, the influence of the rotational inertia was not taken into account for nonlinear beam equations. On the other hand, the rotation of the moving beam occurs often in the process of vibration, and the effect of the rotational inertia was considered in Ghayesh-Khadem (2008); Wang (2018) where the dynamic behavior of the systems with simply supported boundary conditions was discussed by means of the multiple scales method. To the best of our knowledge, there is no published result on well-posedness and boundary stabilization of axially moving Kirchhoff-type beams with rotational inertia. Taking the influence of the torsional deformation into account, we consider, in this paper, the decay rate of the solution and energy to the Kirchhoff-type beam with rotational inertia described by

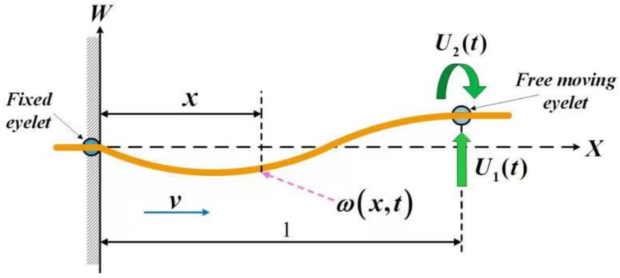


Fig. 1. Schematic representation of a moving beam under two boundary controls.

$$\rho\mathcal{A}_c(w_{tt}+2vw_{xt}+v^2w_{xx})+\mathcal{E}\hat{\mathcal{I}}w_{xxxx}-\rho\hat{\mathcal{I}}w_{xxtt} \\ = M(\|w_x(\cdot,t)\|^2)w_{xx}, \tag{2}$$

where $M(\cdot)$ is a continuous differentiable function: ($M\in C^1(0,\infty)$) with $M(s)\geq\tilde{P}$ for any $s\geq 0$ and $\rho\hat{\mathcal{I}}w_{xxtt}$ represents the rotational inertia of the beam (Hegarty-Taylor, 2012). Introduce non-dimensional variables

$$t^*=t\sqrt{\frac{\tilde{P}}{\rho\mathcal{A}_cL^2}},\ v^*=v\sqrt{\frac{\rho\mathcal{A}_c}{\tilde{P}}},\ \gamma=\frac{\hat{\mathcal{I}}}{\mathcal{A}_cL^2}, \\ x^*=\frac{x}{L},\ w^*=\frac{w}{L},\ \beta=\frac{\mathcal{E}\hat{\mathcal{I}}}{L^2\tilde{P}},\ M^*=\frac{M}{\tilde{P}}. \tag{3}$$

Ignoring the sign "*" for brevity, the system (2) can be rewritten as

$$w_{tt}+2vw_{xt}+v^2w_{xx}+\beta w_{xxxx}-\gamma w_{xxtt} \\ = M(\|w_x(\cdot,t)\|^2)w_{xx}, \tag{4}$$

for all $x\in(0,1)$ and $t>0$. The major concern of this paper is about stabilization and decay rate of system (2) under two boundary controls:

$$\begin{cases}(M(\|w_x(\cdot,t)\|^2)-v^2)w_x(1,t)-\beta w_{xxx}(1,t)\\ \quad +\gamma w_{xtt}(1,t)-vw_t(1,t)=U_1(t),\\ \beta w_{xx}(1,t)=U_2(t),\\ w(0,t)=w_x(0,t)=0,\\ w(x,0)=f_1(x),\ w_t(x,0)=f_2(x),\end{cases} \tag{5}$$

where $f_1(\cdot)$ is the initial displacement of beam, $f_2(\cdot)$ is the initial velocity of the beam, and $U_1(\cdot)$ and $U_2(\cdot)$ are boundary control force and torque applied at the right boundary of the beam respectively, as illustrated in Fig. 1. We consider the nonlinear boundary feedback controls as follows:

$$\begin{cases}U_1(t)=-F_1(w_t(1,t)),\\ U_2(t)=-F_2(w_{xt}(1,t)),\end{cases} \tag{6}$$

where $w_t(1,\cdot)$ and $w_{xt}(1,\cdot)$ denote respectively the transverse velocity and angular velocity at the right boundary of the beam, $F_i(\cdot)$, $i=1,2$, are non-decreasing continuous functions, that is, $(F_i(s_1)-F_i(s_2))(s_1-s_2)\geq 0$ for all $s_1,s_2\in\mathbb{R}$, satisfying

$$F_i(0)=0,\ 0<F_i(s)s,\ \forall s\neq 0, \tag{7}$$

and

$$k_1\leq \frac{F_i(s)}{s}\leq k_2,\ \forall |s|\geq N, \tag{8}$$

where $0<k_1\leq k_2$ and $N>0$ are given constants. When the axial tension $M(\cdot)=M(x,t)$, $v=0$ and $\beta=1$ in (4), both linear boundary feedbacks make an cantilevered beam exponentially stable proved in Hegarty-Taylor (2012) by means of the integral multiplier method. If the influence of rotational inertia is not considered, the preliminary energy decay estimate were discussed in our previous work (Cheng-Wu-Wu, 2022). The decay rate of wave equations was first discussed in Lasiecka (1993) and further investigated by Liu-Zuazua (1999) with an ODE approach. A construction method of explicit energy decay rates were presented in Martinez (1999). Note that none of the above work involves the optimal decay rate which was partially considered in Alabau-Boussouira (2005). The paper Alabau-Boussouira (2010) completely addressed the optimality of the explicit energy decay by choosing, with convexity arguments, an optimal weight function. However, the explicit decay rate method and the optimal decay rate method are difficult to be applied to address the stabilization problem of the nonlinear beam system (2) with multiple controls. The main reason behind is that these methods require appropriate multipliers.

Firstly, from mathematical point of view, the presence of the rotational inertia introduces major technical difficulties for the analysis of the well posedness and stability of the system. This is attributed to the fact that an appropriate energy perturbation or integral multiplier is difficult to be found for establishment of the stability of systems, particularly for nonlinear beam systems. To overcome this difficulty, we develop a new method which introduces an adjustment term to determine the range of integral multiplier parameters to give the boundary stabilization and the decay rate simultaneously for this kind of nonlinear beam systems with rotational inertia. Since the nonlinear feedbacks $F_i(\cdot)$ in (7) are not required to be strictly monotonic increasing on $\mathbb{R}$, it is difficult to get the high-order estimate of the approximate solution and hence the Faedo-Galerkin approximation method is not working for well-posed analysis. Fortunately, the well-posedness of the closed-loop system (4)-(6) can be established by the nonlinear semigroup theory, where the solution depends continuously on the initial value. This is essentially inspired by the works of Komornik (1994) and Brezis (1973). Secondly, since no growth assumptions near zero are imposed on the nonlinear feedback functions $F_i(\cdot)$ in (7) and the nonlinearity of the system, the commonly used methods for analyzing the stability of systems like frequency domain method (Lee-Jang, 2007), energy perturbation method (Kobayashi et al., 2009), the integral-type multiplier method (Hegarty-Taylor, 2012), the approximately observable method (Ramirez-Zwart-Le Gorrec, 2017) are failure. Instead, inspired by an idea presented in Lasiecka (1993), we generalize the method there to deal with the asymptotic stability of the closed-loop system (4)-(6) by the solution of a dissipative ODE under nonlinear feedback controls. Furthermore, when various growth constraints of the nonlinear feedback $F_i(\cdot)$ near the origin are available, the decay rate estimation of both the solution and energy for the nonlinear beam (4) can be calculated by solving the associated dissipative ODE. When the nonlinear feedbacks $F_i(\cdot)$ near the origin are non-decreasing continuous, a new construction method for estimating the decay rate of the system is provided. More specifically, it is concluded that when the nonlinear feedback functions increases linearly near the origin, the solution and energy decay exponentially, which is consistent with Hegarty-Taylor (2012) by linear feedbacks. When the nonlinear feedback functions increase as a power function near the origin, the solution and energy function decay polynomially, and when the feedback function growth is weaker near zero like $e^{-\frac{1}{s^2}}$, the solution and energy are proved to be stable in the polynomial and logarithmic senses. These results are verified by numerical simulations for three different groups of examples. The estimation method is quite general to be used to deal with other PDEs under nonlinear feedbacks.

The rest of the paper is organized as follows. In the next section, Section 2, we state the well posedness, stability, and the decay estimates for both solution and energy of the closed-loop system. Three types of examples including exponential, polynomial and polynomial-logarithmic decay forms are also presented. Section 3 is devoted to the proof of the well posedness and asymptotic stability of the closed-loop system. In Section 4, some numerical simulations are carried out to illustrate the theoretical results, followed up by a brief conclusion in Section 5.

2 Main results

The closed-loop form of the system (4)-(5) under feedbacks (6) is as follows:

$$\begin{cases} w_{tt}+2vw_{xt}+v^2w_{xx}+\beta w_{xxxx}-\gamma w_{xxtt}\\ \quad =M(\|w_x(\cdot,t)\|^2)w_{xx},\\ (M(\|w_x(\cdot,t)\|^2)-v^2)w_x(1,t)-\beta w_{xxx}(1,t)\\ \quad +\gamma w_{xtt}(1,t)-vw_t(1,t)=-F_1(w_t(1,t)),\\ \beta w_{xx}(1,t)=-F_2(w_{xt}(1,t)),\\ w(0,t)=w_x(0,t)=0,\\ w(x,0)=f_1(x),\ w_t(x,0)=f_2(x), \end{cases} \tag{9}$$

where $\beta > v^2/2$ and $\gamma > 0$. The energy of the system (9) is defined physically as

$$\begin{aligned} E(t) = & \frac{1}{2}\int_0^1 w_t^2 dx + \frac{1}{2}\widetilde{M}(\|w_x\|^2) \\ & + \frac{\beta}{2}\int_0^1 w_{xx}^2 dx + \frac{\gamma}{2}\int_0^1 w_{xt}^2 dx, \end{aligned} \tag{10}$$

where $\widetilde{M}(s) = \int_0^s [M(\theta) - v^2]d\theta$. From physical point of view, the velocity of an axially moving beam does not surpass a critical value ($v < \sqrt{\tilde{P}/\rho\mathcal{A}_c}$) for the beam (2). As such, it is easy to deduce that $M(\theta) - v^2 \geq 1 - v^2 > 0$ for any $\theta \geq 0$ (Cheng-Wu-Guo, 2021). From this point of view and $v^2 \ll 1$, it is reasonable to assume that $\beta > v^2/2$ in the dimensionless form (9).

2.1 Well-posedness of closed-loop system

In this subsection, we discuss the well-posedness of the closed-loop system (9) by nonlinear semigroup theory (Komornik, 1994; Barbu, 1976). The procedure follows from Lagnese-Leugering (1991). Let $(\cdot,\cdot)_{L^2}$ represent the inner product in $L^2 := L^2(0,1)$ with the norm $\|\cdot\|$. The closed subspaces H_E^1 and H_E^2 of Hilbert spaces $H^1(0,1)$ and $H^2(0,1)$ are defined respectively by

$$\begin{aligned} H_E^1 &:= \{w \in H^1(0,1) : w(0) = 0\}, \\ H_E^2 &:= \{w \in H^2(0,1) : w(0) = w_x(0) = 0\}, \end{aligned}$$

with the corresponding inner products given by

$$\begin{aligned} (w,y)_{H_E^1} &= \gamma\int_0^1 w_x y_x dx + \int_0^1 wy dx, \ \forall w,y \in H_E^1, \\ (w,y)_{H_E^2} &= \beta\int_0^1 w_{xx} y_{xx} dx, \ \forall w,y \in H_E^2, \end{aligned} \tag{11}$$

with the inner product induced norms $\|\cdot\|_i, i = 1,2$. Let H_E^{-2} and H_E^{-1} be the dual spaces of H_E^2 and H_E^1 by considering $L^2(0,1)$ as a pivot space. Then we have Gelfand's inclusions: $H_E^2 \subset H_E^1 \subset L^2(0,1) = (L^2(0,1))^* \subset H_E^{-1} \subset H_E^{-2}$. Set $H_E^3 = H^3(0,1) \cap H_E^2$.

Multiplying by $y \in H_E^2$ on both sides of the first equation of (9) and integrating over $x \in (0,1)$ yields the variational form of system (9) as

$$\begin{aligned} & \int_0^1 w_{tt} y dx + [M(\|w_x(t)\|^2) - v^2]\int_0^1 y_x w_x dx \\ & +2v\int_0^1 w_{xt} y dx \ + \ \beta\int_0^1 w_{xx} y_{xx} dx \\ & +\gamma\int_0^1 w_{xtt} y_x dx + F_2(w_{xt}(1,t)) y_x(1) \\ & -[v w_t(1,t) - F_1(w_t(1,t))]y(1) = 0, \ \forall y \in H_E^2, \end{aligned} \tag{12}$$

where the boundary conditions in (9) were applied. Introduce linear operators $B_1 : H_E^2 \to H_E^{-2}$ and $A, \mathcal{D} : H_E^1 \to H_E^{-1}$ as follows:

$$\begin{aligned} \langle Aw, y\rangle_{-1,1} &= 2v\int_0^1 w_x y dx, \ \forall w,y \in H_E^1, \\ \langle B_1 w, y\rangle_{-2,2} &= (w,y)_{H_E^2}, \ \forall w,y \in H_E^2, \\ \langle \mathcal{D} w, y\rangle_{-1,1} &= (w,y)_{H_E^1}, \ \forall w,y \in H_E^1, \end{aligned} \tag{13}$$

and nonlinear operators $G_2 : H_E^2 \to H_E^{-2}$ and $G_1, B_2 : H_E^1 \to H_E^{-1}$ as

$$\begin{aligned} \langle B_2 w, y\rangle_{-1,1} &= \int_0^1 [M(\|w_x\|^2) - v^2] w_x y_x dx, \ \forall w,y \in H_E^1, \\ \langle G_1 w, y\rangle_{-1,1} &= v w(1) y(1) - F_1(w(1)) y(1), \ \forall w,y \in H_E^1, \\ \langle G_2 w, y\rangle_{-2,2} &= -F_2(w_x(1)) y_x(1), \quad \forall w,y \in H_E^2. \end{aligned}$$

It is easy to find that $\mathcal{D} : H_E^1 \to H_E^{-1}$ is an isomorphic mapping. In this context, the variational form (12) is equivalent to the following equation

$$\langle \mathcal{D} w_{tt} + A w_t + B_1 w + B_2 w - G_1 w_t - G_2 w_t, y\rangle_{-2,2} = 0$$

for all $y \in H_E^2$. The existence of the weak solution of (9) is therefore equivalent to

$$\mathcal{D} w_{tt} + A w_t + B_1 w + B_2 w - G_1 w_t - G_2 w_t = 0 \text{ in } H_E^{-2}. \tag{14}$$

However, by the nonlinear semigroup theory, we could show that there exists a solution for (14) in the state space following as Lagnese-Leugering (1991):

$$H = H_E^2 \times H_E^1 \tag{15}$$

with inner product $\langle (w_1,u_1)^\top, (w_2,u_2)^\top\rangle_H = (w_1,w_2)_{H_E^2} + (u_1,u_2)_{H_E^1}$ for $(w_1,u_1)^\top, (w_2,u_2)^\top \in H$ and the norm $\|(w,u)^\top\|_H^2 = \|w\|_{H_E^2}^2 + \|u\|_{H_E^1}^2$ for $(w,u)^\top \in H$. To this purpose, let $X := (w, w_t)^\top$. Then

$$\begin{aligned} \dot{X} = & \begin{pmatrix} w_t \\ w_{tt} \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{D}^{-1} B_2 w \end{pmatrix} \\ & + \begin{pmatrix} w_t \\ \mathcal{D}^{-1}[-A w_t - B_1 w + G_1 w_t + G_2 w_t] \end{pmatrix}. \end{aligned} \tag{16}$$

Thus, (14) can be formulated into an evolution equation following:

$$\dot{X} + \mathcal{A} X = \mathcal{B} X, \quad X_0 = (f_1, f_2)^\top, \tag{17}$$

where the operators $\mathcal{A} : \mathscr{D}_{\mathcal{A}} \subset H \to H$ and $\mathcal{B} : H \to H$ are defined by

$$\mathcal{A}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} -u \\ \mathcal{D}^{-1}[B_1 w + A u - G_1 u - G_2 u] \end{pmatrix}, \tag{18}$$

for any $(w,u)^\top \in \mathscr{D}_{\mathcal{A}}$ with

$$\mathscr{D}_{\mathcal{A}} = \left\{(w,u)^\top \in H_E^2 \times H_E^2 | B_1 w + Au - G_1 u - G_2 u \in H_E^{-1}\right\},$$

and

$$\mathcal{B}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{D}^{-1}B_2 w \end{pmatrix}, \ \forall (w,u)^\top \in H. \tag{19}$$

Proposition 2.1 following expresses precisely the $\mathscr{D}_{\mathcal{A}}$.

Proposition 2.1 *The set $\mathscr{D}_{\mathcal{A}}$ consists of pairs $(w,u)^\top \in H_E^3 \times H_E^2$ satisfying the boundary condition $\beta w_{xx}(1) = -F_2(u_x(1))$.*

Proof. From the definition (18), let $(w,u)^\top \in \mathscr{D}_{\mathcal{A}}$ and $\mathcal{A}((w,u)^\top) = (x,z) \in H$. Then we have $-u = x \in H_E^2$ and

$$\mathcal{D}^{-1}[B_1 w + Au - G_1 u - G_2 u] = z.$$

Since $z \in H_E^1$ and $\mathcal{D} : H_E^1 \to H_E^{-1}$ is an isomorphism, so one has

$$B_1 w + Au - G_1 u - G_2 u = \mathcal{D}z \text{ in } H_E^{-1} \subset H_E^{-2}.$$

Due to the definition of $B_1, A, G_1, G_2, \mathcal{D}$, it follows that (see, e.g., Lemma 2.2 of Wang-Xu-Yung (2004))

$$\beta(w_{xx}, y_{xx})_{L^2} + F_2(u_x(1))y_x(1) = \langle f, y\rangle_{-1,1}, \forall y \in H_E^2, \tag{20}$$

with $f = \mathcal{D}z - Au + G_1 u \in H_E^{-1}$. We claim that $w \in H^3(0,1)$. Actually, let $\widetilde{w} = \mu_1 x^2 + \mu_2 x^3$ with $\mu_2 = (vu(1) - F_1(u(1)))/6\beta$ and $\mu_1 = [F_2(u_x(1)) - vu(1) + F_1(u(1))]/2\beta$. Then $\beta(\widetilde{w}_{xx}, y_{xx})_{L^2} = F_2(u_x(1))y_x(1) - vu(1)y(1) + F_1(u(1))y(1)$. Therefore, (20) is equivalent to $B_1(w - \widetilde{w}) = \mathcal{D}z - Au \in H_E^{-1}$. Next, apply interpolation to show that $\widehat{w} := w - \widetilde{w} \in H^3(0,1)$. The elliptic operator theory gives that the problem $B_1\widehat{w} = \hat{f}$ has a unique solution $\widehat{w} \in H_E^2$ if $\hat{f} \in H_E^{-2}$, which leads to that the map $\hat{f} \to \widehat{w} : H_E^{-2} \to H_E^2$ is linear and continuous. Also, if $\hat{f} \in L^2$, the problem $B_1\widehat{w} = \hat{f}$ with boundary value conditions $\widehat{w}(0) = \widehat{w}_x(0) = \widehat{w}_{xx}(1) = \widehat{w}_{xxx}(1) = 0$ has a unique solution $\widehat{w} \in H^4(0,1) \cap H_E^2$ and the map $\hat{f} \to \widehat{w} : L^2 \to H^4(0,1)$ is linear and continuous. Consequently, by interpolation, if $\hat{f} \in H_E^{-1}$, then $\widehat{w} \in H^3(0,1) \cap H_E^2$, which, together with $w = \widehat{w} + \widetilde{w}$, shows that $w \in H_E^3$.

Letting $y \in H_0^2 := \{y \in H_E^2 : y_x(1) = 0\}$ in (20) with $w \in H_E^3$ yields

$$-\beta(w_{xxx}, y_x)_{L^2} = \langle f, y\rangle_{-1,1}. \tag{21}$$

By integration by parts, it follows from (20) that

$$-\beta(w_{xxx}, y_x)_{L^2} + [F_2(u_x(1)) + \beta w_{xx}(1)]y_x(1) = \langle f, y\rangle_{-1,1},$$

for any $y \in H_E^2$, which with (21) together leads to $\beta w_{xx}(1) = -F_2(u_x(1))$.

□

Now, we use the classical definition for strong solutions to (9) via nonlinear semigroup theory where the initial data belongs to the domain of the generator, and the weak (generalized) solutions to (9) are the limits of strong solutions for almost everywhere in t (Barbu (1976); Brezis (1973); Chueshov (2002)).

Theorem 2.1 *Assume Assumption (7) and $X_0 = (f_1, f_2)^\top \in \mathscr{D}_{\mathcal{A}}$ satisfying the compatibility condition $\beta f_{1xx}(1) = -F_2(f_{2x}(1))$. Then there exists a unique strong solution $X = (w, w_t)^\top \in W^{1,\infty}((0,T], \mathscr{D}_{\mathcal{A}})$ for the evolution equation (17), i.e., the closed-loop system (9) admits a unique strong solution $w(\cdot,\cdot)$ such that*

$$w \in W^{1,\infty}((0,T]; H_E^3) \cap W^{2,\infty}((0,T]; H_E^2), \ \ \forall\, T > 0.$$

Moreover, if $X_0 = (f_1, f_2)^\top \in H$, then there exists a unique weak solution $X = (w, w_t)^\top \in C((0,T], H)$ for the evolution equation (17), i.e., $w \in C((0,T]; H_E^2) \cap C^1((0,T]; H_E^1)$ for the closed-loop system (9) for all $T > 0$, which depends continuously on the initial value $(f_1, f_2)^\top$ in H.

2.2 Stability of closed-loop system

In this subsection, we present the stability of the closed-loop system. Assume that two maps $W_i(s), i = 1,2$, are concave and strictly increasing in $s \in \mathbb{R}$, with $W_i(0) = 0$ satisfying

$$W_i(sF_i(s)) \ge s^2 + [F_i(s)]^2, \ \forall\, |s| \le N, \tag{22}$$

for some $N > 0$, where the nonlinear feedback functions $F_i(\cdot), i = 1,2$, are given as in (9). According to the property of $F_i(\cdot)$ $(i = 1,2)$, the functions $W_i(\cdot)$, $i = 1,2$, can be easily found to satisfy (22), as shown by some examples later. For each $i = 1,2$, set $\hat{W}_i(s) = W_i\left(\frac{s}{T}\right)$, $\forall s \in \mathbb{R}$, where $T > 0$ is a constant to be determined later. For $\delta > 0$,

$$\delta\mathcal{I} + \frac{1}{2}[\hat{W}_1 + \hat{W}_2] : \mathbb{R} \to \mathbb{R} \tag{23}$$

is invertible and strictly increasing, where $\mathcal{I}$ is the identity mapping. Define a map

$$P(s) = 2\left(\delta\mathcal{I} + \frac{1}{2}[\hat{W}_1 + \hat{W}_2]\right)^{-1}(\hat{\delta}s), \ \forall s \in \mathbb{R}, \tag{24}$$

for a constant $\hat{\delta} > 0$, which is strictly increasing and continuous with $P(0) = 0$. Define $Q(s) = s - (\mathcal{I} + P)^{-1}(s)$ for $s \in \mathbb{R}$. Recall the ODE system of the following:

$$\begin{cases} \dot{S}(t) + Q(S(t)) = 0, \ \ t > 0, \\ S(0) = S_0. \end{cases} \tag{25}$$

If the function defined by (24) satisfies $P(s) > 0$ for all $s > 0$, then it must have $\lim_{t\to\infty} S(t) = 0$ as discussed in Lasiecka (1993). Lemma 2.1 following shows that the energy $E(t)$ is nonincreasing.

Lemma 2.1 *Let $w(\cdot,\cdot)$ and $E(\cdot)$ defined by (10) be the strong solution and energy of the closed-loop system (9). Then for any $T > S > 0$,*

$$\begin{aligned} E(T) &= E(S) - \int_S^T F_1(w_t(1,t))w_t(1,t)dt \\ &\quad - \int_S^T F_2(w_{xt}(1,t))w_{xt}(1,t)dt. \end{aligned} \tag{26}$$

Moreover, $E(t) \le E(0)$ for all $t \ge 0$.

From the above preliminaries, we come up the stability.

Theorem 2.2 *Let $w(\cdot,\cdot)$ and $E(\cdot)$ be the strong solution and energy defined by (10) of the closed-loop system (9). Assume assumptions (7) and (8). Then there exist a positive constant $T_0 > 0$ such that*

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le \frac{2}{\beta}S\left(\frac{t}{T_0} - 1\right) \tag{27}$$

for all $x \in [0,1]$ and $t \ge T_0$, and $\lim_{t\to\infty} S(t) = 0$, where $S(\cdot)$ is the solution of ODE (25) with $S_0 = E(0)$ and the constant β is given as in (9).

Remark 2.1 *By approximating ODE (25), the decay rates were estimated in Horn-Lasiecka (1996) for Kirchhoff plates, Toundykov (2007) for nonlinear wave equation, Horn (2000) for anisotropic elasticity system and Horn-Lasiecka (1995, 1994a) for the von Kármán plates. It is worth mentioning that the explicit decay rates were considered in (Liu-Zuazua, 1999; Martinez, 1999) for wave equations, (Alabau-Boussouira, 2005) for general abstract equations, (Alabau-Boussouira, 2006) for plate equations and (Alabau-Boussouira, 2007) for Timoshenko beams. The optimal decay rates closely related to optimal weight function was established in Alabau-Boussouira (2010); Alabau-Boussouira-Ammari (2011). However, the above mentioned works have not provided relevant results for estimating the decay rate of PDEs under multiple controls. The proposed method in this paper can be applied to other PDEs with two nonlinear controls.*

Corollary 2.1 *Under assumptions of Theorem 2.2, for $p \in [1,\infty)$, $N = 1$ and assume that there exist constants $k_1, k_2 > 0$ such that*

$$k_1|s|^p \le |F_i(s)| \le k_2|s|^{1/p}, \ \forall |s| \le 1. \tag{28}$$

Then the energy $E(\cdot)$ and the solution $w(\cdot,\cdot)$ of the closed-loop system (9) satisfy

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le Ct^{\frac{2}{1-p}} \ \text{for}\ p > 1, \tag{29}$$

and

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le Ce^{-\mu t} \ \text{for}\ p = 1, \tag{30}$$

for all $x \in [0,1]$ and $t \ge 0$, where μ and C are two positive constants.

The main approach of the proof for Theorem 2.2 in the next section is to construct a concave function $W_i(s)$ $(i = 1,2)$ to satisfy (22). Due to (28), there holds $k_1|s|^p \le |F_i(s)|$ and $|F_i(s)|^p \le k_2^p|s|$ $(i = 1,2)$ for any $|s| \le 1$. Hence we can take

$$W_1(s) = W_2(s) = \left(k_1^{\frac{-2}{p+1}} + k_2^{\frac{2p}{p+1}}\right) s^{\frac{2}{p+1}}.$$

In this way, the map $P(s) = (\delta\mathcal{I} + \hat{W})^{-1}(\hat{\delta}s)$, i.e.,

$$\delta P(s) + C_{k_1,k_2}P(s)^{\frac{2}{p+1}} = \hat{\delta}s,$$

where C_{k_1,k_2} is a suitable constant related to k_1 and k_2. Recalling the map $Q(s) = s - (\mathcal{I} + P)^{-1}(s)$ and for sufficiently small s, we have $P(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ and $Q(s) \sim \mathcal{C}s^{\frac{p+1}{2}}$ for some constant $\mathcal{C} > 0$. As a result, the estimates (29) and (30) follow from solution (25) with $Q(s)$ as above by invoking Theorem 2.2.

Remark 2.2 *To estimate the decay rate, a key point is to construct the concave function $W(\cdot)$ because the change behavior of the function $Q(\cdot)$ near zero point is asymptotically equivalent to the convex function $W^{-1}(\cdot)$. From the definition of $Q(\cdot)$ and (22), it suffices to restrict the analysis to positive values of s. Therefore, for two nonlinear control cases, we can find $Q(\cdot) \sim (\frac{1}{2}W_1 + \frac{1}{2}W_2)^{-1} \sim \frac{1}{2}(W_1^{-1} + W_2^{-1})$ near zero, which implies that the estimation of the delay rates is calculated by the dissipative ODE*

$$\begin{cases} \dot{S} + \frac{1}{2}(W_1^{-1}(S) + W_2^{-1}(S)) = 0, & t > 0, \\ S(0) = S_0, \end{cases} \tag{31}$$

where $W_i(\cdot), i = 1,2$, are constructed corresponding to the nonlinear feedbacks $F_i(\cdot)$.

Remark 2.3 *If $F_i(\cdot), i = 1,2$, are strictly monotone near zero, the construction of the concave function W_i can be specified from (Liu-Zuazua, 1999; Martinez, 1999; Alabau-Boussouira, 2005). Actually, from the convexity arguments, we can let $W_i(s) = \sqrt{s}F_i(\sqrt{s})$ for s at the origin to satisfy the growth condition $W_i(sF_i(s)) \ge s^2 + F_i^2(s)$ once the feedback functions $F_i(\cdot)$ decay towards to zero faster than the linear ones. Moreover, we can let $W_i(s) = \sqrt{s}F_i^{-1}(\sqrt{s})$ near zero satisfy the growth condition $W_i(sF_i(s)) \ge s^2 + F_i^2(s)$ if the feedback functions $F_i(\cdot)$ decay towards zero slower than the linear ones.*

Inspired by examples of decay rates (Liu-Zuazua, 1999; Martinez, 1999; Alabau-Boussouira, 2005), when the nonlinear feedback functions decay beyond the power function near the zero point like $s^3e^{-\frac{1}{s}}$, which does not satisfy the condition of Corollary 2.1, we have Corollaries 2.2 and 2.3 below.

Corollary 2.2 *Under assumptions of Theorem 2.2, let*

$$F_1(s) = s^3e^{-\frac{1}{s^2}} \ \text{and}\ F_2(s) = s^3 \tag{32}$$

for s at the origin. Then the energy $E(\cdot)$ and the solution $w(\cdot,\cdot)$ of the closed-loop system (9) satisfy

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le \frac{C}{\ln[(1+e^{\frac{1}{E(0)}})e^{\frac{t}{2}}-1]}, \qquad (33)$$

for all $x \in [0,1]$ and $t \ge 0$, where C is a positive constant.

According to Remark 2.3, we can take $W_1^{-1}(s) = s^2 e^{-\frac{1}{s}}$ and $W_2^{-1}(s) = s^2$, which keep convexity near zero. Due to (31), the result (33) follows by resolving the following ODE

$$\begin{cases} \dot{S} + \frac{1}{2}(S^2 e^{-\frac{1}{S}} + S^2) = 0, \ t > 0, \\ S(0) = E(0). \end{cases} \qquad (34)$$

When both nonlinear feedback functions decay beyond the power function near the zero, a class of polynomial-logarithmic form decay can be deduced as claimed by Corollary 2.3 following.

Corollary 2.3 *Let the assumptions of Theorem 2.2 be fulfilled. Assume*

$$F_1(s) = F_2(s) = s^{2p+1} e^{-\frac{1}{s^{2p}}} \qquad (35)$$

for s at the origin and $p > 0$. For $W_1^{-1}(s) = W_2^{-1}(s) = F_1(\sqrt{s})\sqrt{s} = s^{p+1}e^{-\frac{1}{s^p}}$ and (31), the energy $E(\cdot)$ and solution $w(\cdot,\cdot)$ of closed-loop system (9) satisfy

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le \left(\frac{C}{\ln(pt + C_e)}\right)^{\frac{1}{p}}, \qquad (36)$$

where C is a positive constant and $C_e = e^{\frac{1}{(E(0))^p}}$, for any $x \in [0,1]$ and all $t \ge 0$.

Remark 2.4 *Let $\eth_i(s), i = 1,2$ be strictly monotonically increasing odd functions near zero with $\eth_i(0) = 0$. We only discuss, without loss of generality, the case where s is greater than 0. Assume that $F_i(s), i = 1,2$ are non-decreasing continuous (locally saturated) on $0 < s < N$, such as*

$$F_i(s) = \begin{cases} \eth_i(s), & 0 < s \le \mu, \\ \eth_i(\mu), & \mu < s < N, \end{cases} \qquad (37)$$

with $N = 1$. Now we present construction methods by taking (37) as an example for $W_i^{-1}(s)$ as follows. When $\eth_i(s)$ decays faster on $0 < s \le \mu$ than linear function, we take $W_i^{-1}(s) := s^2 F_i(s^{\frac{1}{2}}), i = 1,2$, which satisfy the convexity condition. It is easy to see that $s^2 = W_i(s^4 F_i(s)) \le W_i(sF_i(s))$ for $0 < s \le N$, which shows that (22) holds. When $\eth_i(s)$ decay slower than linear function, we can take $W_i^{-1}(s) := s^{\frac{1}{2}}\eth_i^{-1}(s^2), i = 1,2$ satisfying the convexity condition. It is easy to deduce that $s^2 = W_i(s\eth_i^{-1}(s^4)) \le W_i(s\eth_i^{-1}(s))$ with $0 < s \le \mu$, which leads to $\eth_i^2(s) \le W_i(s\eth_i(s))$ for $0 < s \le \mu$. This implies that $\eth_i^2(\mu) \le W_i(\mu\eth_i(\mu)) \le W_i(s\eth_i(\mu))$ for $\mu < s \le N$. Therefore $F_i^2(s) \le W_i(sF_i(s))$ on $0 < s \le N$, which leads to (22).

Corollary 2.4 *Let $\eth_i(s) = s^p$, $i = 1,2$ with $p > 1$ in (37) near the origin. Setting*

$$W_1^{-1}(s) = W_2^{-1}(s) = \begin{cases} s^{\frac{p+4}{2}}, & 0 < s \le \mu^2, \\ \mu^p s^2, & \mu^2 < s < 1, \end{cases} \qquad (38)$$

it follows from (31) that

$$\mathcal{S}(t) = \begin{cases} \left[[E(0)]^{-\frac{p+2}{2}} + \frac{p+2}{2}t\right]^{-\frac{2}{p+2}}, & 0 < t \le C_\mu, \\ \left[\hat{C}_\mu + \mu^p t\right]^{-1}, & t > C_\mu, \end{cases}$$

where C_μ is a certain constant and $\hat{C}_\mu = [[E(0)]^{-\frac{p+2}{2}} + \frac{p+2}{2}C_\mu]^{\frac{2}{p+2}} - \mu^p C_\mu$, which implies that the energy $E(\cdot)$ and solution $w(\cdot,\cdot)$ of the closed-loop system (9) satisfy

$$|w(x,t)|^2 \le \frac{2}{\beta}E(t) \le C\left[E(0)^{-\frac{p+2}{2}} + \frac{p+2}{2}t\right]^{-\frac{2}{p+2}},$$

for all $t > 0$ and $x \in [0,1]$.

3 Proof of main results

In this section, we present the proof of the main results stated in Section 2.

3.1 Proof of well-posedness

Three Lemmas 3.1-3.2 and 2.1 following play significant roles in the proof of Theorem 2.1.

Lemma 3.1 *The operator $\mathcal{A}$ defined by (18) is a maximal monotone operator in H.*

Proof. For any $X_1 = (w,u)^\top, X_2 = (\hat{w},\hat{u})^\top \in \mathscr{D}_{\mathcal{A}} \subset H$, it follows from (13) and (18) that

$$\begin{aligned} &\langle \mathcal{A}X_1 - \mathcal{A}X_2, X_1 - X_2\rangle_H \\ &= -(u-\hat{u}, w-\hat{w})_{H_E^2} + (\mathcal{D}^{-1}(f-\hat{f}), u-\hat{u})_{H_E^1} \\ &= -(u-\hat{u}, w-\hat{w})_{H_E^2} + \langle \mathcal{D}\mathcal{D}^{-1}(f-\hat{f}), u-\hat{u}\rangle_{-1,1} \\ &= -(u-\hat{u}, w-\hat{w})_{H_E^2} + \langle f-\hat{f}, u-\hat{u}\rangle_{-1,1}, \end{aligned} \qquad (39)$$

where $f := B_1 w + Au - G_1 u - G_2 u$ and $\hat{f} = B_1\hat{w} + A\hat{u} - G_1\hat{u} - G_2\hat{u}$. This, together with (20), leads to

$$\begin{aligned} &\langle \mathcal{A}X_1 - \mathcal{A}X_2, X_1 - X_2\rangle_H \\ &= 2v\int_0^1 (u_x - \hat{u}_x)(u-\hat{u})dx - v(u(1)-\hat{u}(1))^2 \\ &\quad + [F_1(u(1)) - F_1(\hat{u}(1))](u(1)-\hat{u}(1)) \\ &\quad + [F_2(u_x(1)) - F_2(\hat{u}_x(1))](u_x(1)-\hat{u}_x(1)). \end{aligned} \qquad (40)$$

Since F_i, $i = 1,2$, are non-decreasing continuous over $\mathbb{R}$, there holds

$$[F_i(s_1) - F_i(s_2)](s_1 - s_2) \ge 0, \quad \forall s_1, s_2 \in \mathbb{R}. \qquad (41)$$

Since $u_1(0) = u_2(0) = 0$, it has

$$2v \int_0^1 (u_x - \hat{u}_x)(u - \hat{u})dx = v(u(1) - \hat{u}(1))^2. \quad (42)$$

Plug (41) and (42) into (40) to obtain

$$\langle \mathcal{A}X_1 - \mathcal{A}X_2, X_1 - X_2 \rangle_H \geq 0, \quad (43)$$

which shows that $\mathcal{A}$ is monotonic.

It remains to show that $\mathcal{I} + \mathcal{A} : \mathscr{D}_{\mathcal{A}} \to H$ is surjective where $\mathcal{I}$ is the identity mapping on H. To this purpose, it suffices to show that $I + \mathcal{D}^{-1}(B_1 + A - G_1 - G_2) : H_E^2 \to H_E^1$ is onto, where I is the identity mapping on H_E^2. Indeed, if $I+\mathcal{D}^{-1}(B_1+A-G_1-G_2)$ is onto, it means that for arbitrary $(x_1, x_2)^\top \in H$, there exists $u \in H_E^2$ satisfying $(\mathcal{D} + B_1 + A - G_1 - G_2)u = \mathcal{D}x_2 - B_1x_1$. Let $w = x_1 + u$. Then $(w, u) \in H_E^2 \times H_E^2$. This implies that

$$\begin{aligned} &\mathcal{D}^{-1}(B_1w + Au - G_1u - G_2u) \\ &= \mathcal{D}^{-1}(B_1u + B_1x_1 + Au - G_1u - G_2u) \\ &= x_2 - u \in H_E^1, \end{aligned} \quad (44)$$

which leads to $(w, u)^\top \in \mathscr{D}_{\mathcal{A}}$ and $(\mathcal{I} + \mathcal{A})(w, u)^\top = (x_1, x_2)^\top$. Now we demonstrate the surjectivity of $\mathcal{D}u + B_1u + Au - G_1u - G_2u$. Fix arbitrarily $x_2 \in H_E^{-2}$ and define the functional $\Psi(u) : H_E^2 \to \mathbb{R}$ by

$$\begin{aligned} \Psi(u) =& \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\langle Au, u\rangle_{-1,1} - \mathscr{G}_1(u) \\ &- \mathscr{G}_2(u) - \langle x_2, u\rangle_{-2,2}, \ \forall u \in H_E^2, \end{aligned} \quad (45)$$

where the maps $\mathscr{G}_i : H_E^2 \to \mathbb{R}, i = 1, 2$ are defined by

$$\begin{aligned} \mathscr{G}_1(u) &= \tfrac{1}{2}vu^2(1) - \textstyle\int_0^{u(1)} F_1(\tau)d\tau, \\ \mathscr{G}_2(u) &= -\textstyle\int_0^{u_x(1)} F_2(\tau)d\tau. \end{aligned} \quad (46)$$

Once again, since $F_i(\cdot), i = 1, 2$, are non-decreasing continuous, it is easy to check that the maps $\Psi, \mathscr{G}_i, i = 1, 2$, are well-defined, continuously differentiable and

$$\langle \Psi'(u), v\rangle_{-2,2} = \langle (\mathcal{D} + B_1 + A - G_1 - G_2)u - x_2, v\rangle_{-2,2},$$

for all $u, v \in H_E^2$. Furthermore, by the definition of the operator A, we can find

$$\langle Au, u\rangle_{-1,1} = 2v \int_0^1 u_x u dx = vu^2(1).$$

By condition (7), it follows that

$$\begin{aligned} &\frac{1}{2}\langle Au, u\rangle_{-1,1} - \mathscr{G}_1(u) - \mathscr{G}_2(u) \\ &= \int_0^{u(1)} F_1(\tau)d\tau + \int_0^{u_x(1)} F_2(\tau)d\tau \geq 0. \end{aligned} \quad (47)$$

Substituting (47) into (45), it is easy to verify that

$$\Psi(u) \geq C\big[\frac{1}{2}\|u\|_2 - \|x_2\|_{-2}\big]\|u\|_2, \quad (48)$$

for some constant $C > 0$. Hence $\Psi(u) \to +\infty$ as $\|u\|_2 \to +\infty$, which implies that the $\inf \Psi$ can be attained at some point $u \in H_E^2$ by (Mawhin, 2013, Theorem 1.1, p. 4). Therefore $\Psi'(u) = 0$ which means that $(\mathcal{D}+B_1+A-G_1-G_2)u = x_2$. This proves the maximal monotonicity of $\mathcal{A}$. □

Lemma 3.2 *The operator $\mathcal{B} : H \to H$ defined by (18) is locally Lipschitz.*

Proof. Let $\Omega_r := \{X = (w, u)^\top \in H : \|X\|_H^2 \leq r\}$. For any $X_1 = (w, u)^\top, X_2 = (\hat{w}, \hat{u})^\top \in \Omega_r$, it suffices to show that

$$|\langle \mathcal{B}X_1 - \mathcal{B}X_2, X_1 - X_2\rangle_H| \leq \lambda_r \|X_1 - X_2\|_H^2, \quad (49)$$

where the constant $\lambda_r > 0$ depends on r. Now,

$$\begin{aligned} &|\langle \mathcal{B}X_1 - \mathcal{B}X_2, X_1 - X_2\rangle_H| = \Big|\langle B_2w - B_2\hat{w}, u - \hat{u}\rangle_{-1,1}\Big| \\ &\leq \Big|\int_0^1 M(\|w_x\|^2)w_x(u_x - \hat{u}_x)dx \\ &\quad - \int_0^1 M(\|\hat{w}_x\|^2)\hat{w}_x(u_x - \hat{u}_x)dx\Big| \\ &\quad + v^2\Big|\int_0^1 (w_x - \hat{w}_x)(u_x - \hat{u}_x)dx\Big|. \end{aligned} \quad (50)$$

This yields

$$\begin{aligned} &|\langle \mathcal{B}X_1 - \mathcal{B}X_2, X_1 - X_2\rangle_H| \\ &\leq \Big|\int_0^1 M(\|w_x\|^2)w_x(u_x - \hat{u}_x)dx \\ &\quad - \int_0^1 M(\|w_x\|^2)\hat{w}_x(u_x - \hat{u}_x)dx \\ &\quad + \int_0^1 M(\|w_x\|^2)\hat{w}_x(u_x - \hat{u}_x)dx \\ &\quad - \int_0^1 M(\|\hat{w}_x\|^2)\hat{w}_x(u_x - \hat{u}_x)dx\Big| \\ &\quad + v^2\Big|\int_0^1 (w_x - \hat{w}_x)(u_x - \hat{u}_x)dx\Big| \\ &\leq \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3, \end{aligned} \quad (51)$$

with

$$\begin{aligned} \mathcal{P}_1 &= \Big|\int_0^1 M(\|w_x\|^2)(w_x - \hat{w}_x)(u_x - \hat{u}_x)dx\Big|, \\ \mathcal{P}_2 &= \Big|\int_0^1 [M(\|w_x\|^2) - M(\|\hat{w}_x\|^2)]\hat{w}_x(u_x - \hat{u}_x)dx\Big|, \\ \mathcal{P}_3 &= v^2\Big|\int_0^1 (w_x - \hat{w}_x)(u_x - \hat{u}_x)dx\Big|. \end{aligned}$$

By $X_1 \in \Omega_r$ and the Poincaré inequality: $\|w_x\|^2 \leq L\|w_{xx}\|^2$ with $L = 1$, one has $\|w_x\|^2 \leq r/\beta$. Since

$M \in C^1(0,\infty)$ and $M(s) \geq 1$ for all $s \geq 0$, we obtain $C_r := \max_{0\leq s\leq r/\beta} M(s)$. Thus, for $\mathcal{P}_1$, from the Poincaré inequality, one obtains

$$\begin{aligned}\mathcal{P}_1 &\leq \frac{C_r}{2}\left|\int_0^1 (w_x-\hat{w}_x)^2+(u_x-\hat{u}_x)^2dx\right| \\ &\leq \hat{C}_r(\|w-\hat{w}\|_2^2+\|u-\hat{u}\|_1^2),\end{aligned} \tag{52}$$

for some constant $\hat{C}_r > 0$. In light of the mean value theorem and the Cauchy-Schwartz inequality, one can show that

$$\begin{aligned}&\left|M(\|w_x\|^2)-M(\|\hat{w}_x\|^2)\right| \\ &\leq \max_{0\leq s\leq r/\beta}\left|M_s(s)\right|\left|\int_0^1(w_x^2-\hat{w}_x^2)dx\right| \\ &\leq \tilde{C}_r\|w_x-\hat{w}_x\|,\end{aligned} \tag{53}$$

for some constant $\tilde{C}_r > 0$. For $\mathcal{P}_2$, from the Cauchy-Schwartz inequality, Poincaré inequality and (53), there holds

$$\begin{aligned}\mathcal{P}_2 &\leq \tilde{C}_r\|w_x-\hat{w}_x\|\|w_x\|\|u_x-\hat{u}_x\| \\ &\leq \frac{\tilde{C}_r\sqrt{r}}{2\sqrt{\beta}}(\|w_x-\hat{w}_x\|^2+\|u_x-\hat{u}_x\|^2) \\ &\leq \bar{C}_r(\|w-\hat{w}\|_2^2+\|u-\hat{u}\|_1^2),\end{aligned} \tag{54}$$

for some constant $\bar{C}_r > 0$. Similar to (54), it has

$$\mathcal{P}_3 \leq \mathcal{C}_r(\|w-\hat{w}\|_2^2+\|u-\hat{u}\|_1^2). \tag{55}$$

Substitute (52), (54) and (55) into (51) to get the inequality (49) with $\lambda_r = \hat{C}_r+\bar{C}_r+\mathcal{C}_r$. □

Proof of Lemma 2.1. From (12), taking $y = w_t$ and applying the derivative of energy function $E(t)$ defined in (10), we obtain

$$\begin{aligned}&\int_0^1 w_{tt}w_tdx+[M(\|w_x(t)\|^2)-v^2]\int_0^1 w_{xt}w_xdx \\ &= -\beta\int_0^1 w_{xx}w_{xxt}dx-F_2(w_{xt}(1,t))w_{xt}(1,t) \\ &\quad -2v\int_0^1 w_{xt}wdx-\gamma\int_0^1 w_{xtt}w_xdx \\ &\quad -[vw_t(1,t)-F_1(w_t(1,t))]w_t(1,t),\end{aligned} \tag{56}$$

where we used $\int_0^1 w_{xt}w_tdx = \frac{1}{2}w_t^2(1,t)$ by virtue of $w_t(0,t) = 0$. Hence,

$$\dot{E}(t) = -F_1(w_t(1,t))w_t(1,t)-F_2(w_{xt}(1,t))w_{xt}(1,t). \tag{57}$$

Since $F_i(s)s \geq 0$ $(i = 1,2)$ for any $s \in \mathbb{R}$, $E(t)$ is non-increasing and $E(t) \leq E(0)$ for all $t \geq 0$. Integrating on both sides of (57) from S to T, we obtain the desired results. □

Proof of Theorem 2.1. The proof is split into three steps:

Step 1: Local solution. In view of Lemma 3.1, $\mathcal{A}$ is maximal monotone which implies that $\mathcal{A}$ is an infinitesimal generator of a nonlinear semigroup on H. From Lemma 3.2, it is shown that $\mathcal{B}$ is locally Lipschitz on H. Therefore, invoking theorem 7.2 of Chueshov (2002) or lemma 2.4 of Brezis (1973), for all $X_0 = (f_1,f_2)^\top \in \mathscr{D}_\mathcal{A}$ satisfying the compatibility condition, there exists a unique strong solution $X = (w,w_t)^\top \in W^{1,\infty}((0,t_{\max}),\mathscr{D}_\mathcal{A})$ for the evolution equation (17), which, together with proposition 2.1, leads to $(w,w_t)^\top \in W^{1,\infty}((0,t_{\max}),H_E^3\times H_E^2)$. Moreover, if $X_0 \in H$, it follows that the equation (17) admits a unique weak solution $X \in C((0,t_{\max}),H)$ satisfying $\limsup_{t\to t_{\max}^-}\|X\|_H = \infty$, provided that $t_{\max} < \infty$.

Step 2: Global solutions. Let $X = (w,w_t)^\top \in H$ be a strong solution defined over $[0,t_{\max})$. The Lemma 2.1 leads to $E(t) \leq E(0)$ for all $t \in [0,t_{\max})$. Therefore, the definition of $E(t)$ given in (10) implies that

$$\|X\|_H^2 = \beta\|w_{xx}\|^2+\gamma\|w_{xt}\|^2+\|w_t\|^2 \leq 2E(t) \leq 2E(0),$$

which extends the local solution to the global one. The above inequality also holds for weak solutions, due to a density argument (weak solutions are limits of strong solutions). Therefore, $t_{\max} = \infty$.

Step 3: Continuous dependence. Let $X_1 = (w,u)^\top, X_2 = (\hat{w},\hat{u})^\top \in H$ be two weak solutions of (17). Then

$$\frac{d}{dt}(X_1-X_2)+\mathcal{A}X_1-\mathcal{A}X_2 = \mathcal{B}(X_1)-\mathcal{B}(X_2). \tag{58}$$

Taking the inner product with X_1-X_2 in H on both sides of (58), and integrating from 0 to t, we obtain

$$\begin{aligned}\|X_1-X_2\|_H^2 =& -\int_0^t\langle\mathcal{A}X_1-\mathcal{A}X_2,X_1-X_2\rangle_Hds \\ &+\int_0^t\langle\mathcal{B}X_1-\mathcal{B}X_2,X_1-X_2\rangle_Hds \\ &+\|X_1(0)-X_2(0)\|_H^2,\end{aligned} \tag{59}$$

from which and Lemma 3.1, we obtain further that

$$\begin{aligned}\|X_1-X_2\|_H^2 \leq& \int_0^t\langle\mathcal{B}X_1-\mathcal{B}X_2,X_1-X_2\rangle_Hds \\ &+\|X_1(0)-X_2(0)\|_H^2.\end{aligned} \tag{60}$$

By Lemma 3.2 and (60), it has

$$\begin{aligned}\|X_1-X_2\|_H^2 \leq& C_E\int_0^t\|X_1-X_2\|_H^2ds \\ &+\|X_1(0)-X_2(0)\|_H^2,\end{aligned} \tag{61}$$

where $C_E > 0$ depends on $E(0)$. From Gronwall's inequality, we obtain

$$\|X_1 - X_2\|_H^2 \leq e^{C_E t}\|X_1(0) - X_2(0)\|_H^2,\ \forall\, t \in [0,T], \quad (62)$$

which leads to the continuous dependence of solutions with respect to the initial values. □

3.2 Proof of stability

In this subsection, we prove Theorem 2.2. To this purpose, we need two auxiliary Lemmas 3.3 and 3.4 following.

Lemma 3.3 *Let $w(\cdot,\cdot)$ be the strong solution of the closed-loop system (9), and $\eta > 0$ be a constant. Then the following identities hold true:*

$$\begin{aligned}\int_0^1 w_{tt}(xw_x - \eta w)dx &= \int_0^1 [w_t(xw_x - \eta w)]_t\, dx \\ &- \frac{1}{2}w_t^2(1,t) + \left(\frac{1}{2} + \eta\right)\int_0^1 w_t^2 dx,\end{aligned} \quad (63)$$

and

$$\int_0^1 (xw_x - \eta w)_{xx} w_{xx} dx = \left(\frac{3}{2} - \eta\right)\int_0^1 w_{xx}^2 dx - \frac{1}{2}w_{xx}^2(1,t). \quad (64)$$

Proof. For all $t > 0$, a direct computation shows that

$$\begin{aligned}\int_0^1 w_{tt}(xw_x - \eta w)dx &= \int_0^1 [w_t(xw_x - \eta w)]_t\, dx \\ &- \int_0^1 w_t(xw_{xt} - \eta w_t)dx,\end{aligned} \quad (65)$$

and

$$\begin{aligned}\int_0^1 xw_t w_{xt} dx &= \frac{1}{2}\int_0^1 (xw_t^2)_x dx - \frac{1}{2}\int_0^1 w_t^2 dx \\ &= \frac{w_t^2(1,t)}{2} - \frac{1}{2}\int_0^1 w_t^2 dx.\end{aligned} \quad (66)$$

Substitution of (66) into (65), we get immediately equation (63). Likewise (65), we can find

$$\begin{aligned}&\int_0^1 (xw_x - \eta w)_{xx} w_{xx} dx \\ &= -\int_0^1 [xw_{xxx} + (2-\eta)w_{xx}]w_{xx} dx,\end{aligned} \quad (67)$$

and

$$\begin{aligned}\int_0^1 xw_{xx}w_{xxx}dx &= \frac{1}{2}\int_0^1 (xw_{xx}^2)_x dx - \frac{1}{2}\int_0^1 w_{xx}^2 dx \\ &= \frac{1}{2}w_{xx}^2(1,t) - \frac{1}{2}\int_0^1 w_{xx}^2 dx.\end{aligned} \quad (68)$$

Plugging (68) into (67) and performing the integration by parts, we obtain equation (64). □

Lemma 3.4 *Under the assumption of Lemma 3.3, the following identities hold true:*

$$\begin{aligned}\int_0^1 (xw_x - \eta w)_x w_{xtt} dx &= -C_\eta \int_0^1 w_{xt}^2 dx - \frac{1}{2}w_{xt}^2(1,t) \\ &+ \int_0^1 [(1-\eta)w_{xt}w_x + xw_{xt}w_{xx}]_t\, dx,\end{aligned} \quad (69)$$

$$\begin{aligned}\int_0^1 (xw_x - \eta w)w_{xt} dx &= \eta\int_0^1 w_x w_t dx \\ &+ \int_0^1 \left[\frac{xw_x^2}{2} - \eta w_x w\right]_t dx,\end{aligned} \quad (70)$$

and

$$\begin{aligned}&\Phi(t)\int_0^1 (xw_x - \eta w)w_{xx} dx \\ &= C_\eta \Phi(t)\int_0^1 w_x^2 dx + \frac{\Phi(t)}{2}w_x^2(1,t),\end{aligned} \quad (71)$$

where $\Phi(t) := M(\|w_x\|^2) - v^2$ and $C_\eta = 1/2 - \eta$.

Proof. Similar to (67), we can deduce that

$$\int_0^1 (xw_x - \eta w)_x w_{xtt} dx = \int_0^1 [xw_{xx} + (1-\eta)w_x]w_{xtt} dx. \quad (72)$$

Hence,

$$\begin{aligned}&\int_0^1 xw_{xx}w_{xtt} dx \\ &= \int_0^1 [xw_{xx}w_{xt}]_t dx - \int_0^1 xw_{xt}w_{xxt} dx \\ &= \int_0^1 [xw_{xx}w_{xt}]_t dx - \frac{1}{2}w_{xt}^2(1,t) + \frac{1}{2}\int_0^1 w_{xt}^2 dx,\end{aligned} \quad (73)$$

and

$$\int_0^1 w_x w_{xtt} dx = \int_0^1 [w_x w_{xt}]_t dx - \int_0^1 w_{xt}^2 dx. \quad (74)$$

Substituting (73) and (74) into (72), we get immediately equation (69). Likewise, we can also show that

$$\int_0^1 xw_x w_{xt} dx = \frac{1}{2}\int_0^1 [xw_x^2]_t dx, \quad (75)$$

and

$$\int_0^1 \eta w w_{xt} dx = \int_0^1 [\eta w w_x]_t dx - \int_0^1 \eta w_x w_t dx. \quad (76)$$

It is easy to see that (70) holds by virtue of (75) and (76). Since

$$\int_0^1 xw_x w_{xx} dx = \frac{1}{2}w_x^2(1,t) - \frac{1}{2}\int_0^1 w_x^2 dx, \quad (77)$$

the (71) follows easily. □

Proof of Theorem 2.2. Since $w(\cdot,\cdot)$ is the strong solution of the closed-loop system (9), by the definition of $E(\cdot)$ as in (10), the Poincaré inequality and Lemma 2.1, we obtain

$$\|w_x\|^2 \le \|w_{xx}\|^2 \le 2E(0)/\beta. \tag{78}$$

Set

$$C_\beta := \frac{3\beta + (1-\alpha)(M_* - v^2)}{2M_* + 2\beta} \tag{79}$$

where $M_* := \max_{0\le s\le 2E(0)/\beta} M(s)$, β is the constant given in (9), and α is a regulating constant satisfying $0 < \alpha < \min\{\frac{2\beta - v^2}{M_* - v^2}, 1\}$. From $\beta > v^2/2$, $M_* > v^2$ and $\alpha < \frac{2\beta-v^2}{M_*-v^2}$, it is easy to get $\frac{1}{2} < C_\beta < \frac{3}{2}$. Replacing y in (12) with $\Upsilon := xw_x - \eta w$ $(1/2 < \eta < C_\beta)$ gives

$$\begin{aligned} &\langle\Upsilon, w_{tt}\rangle + \langle\Upsilon, 2vw_{xt}\rangle + \beta\langle\Upsilon_{xx}, w_{xx}\rangle + \gamma\langle\Upsilon_x, w_{xtt}\rangle \\ &+\Phi(t)\langle\Upsilon_x, w_x\rangle + F_2(w_{xt}(1,t))\Upsilon_x(1) \\ &= [vw_t(1,t) - F_1(w_t(1,t))]\Upsilon(1), \end{aligned} \tag{80}$$

where $\Phi(t) = M(\|w_x\|^2) - v^2$. Set

$$\begin{aligned} \Xi : =& \left(\frac{1}{2}+\eta\right)\int_0^1 w_t^2 dx + \left(\frac{3}{2}-\eta\right)\beta\int_0^1 w_{xx}^2 dx \\ &+ \gamma\left(\eta - \frac{1}{2}\right)\int_0^1 w_{xt}^2 dx + 2v\eta\int_0^1 w_x w_t dx \\ &+ \left(\frac{1}{2}-\eta\right)\Phi(t)\int_0^1 w_x^2 dx, \end{aligned} \tag{81}$$

and

$$\begin{aligned} \Delta(x,t) : =& w_t(xw_x - \eta w) + \gamma(1-\eta)w_{xt}w_x \\ &+ vxw_x^2 - 2\eta v w_x w + \gamma x w_{xt} w_{xx}. \end{aligned} \tag{82}$$

Substitution of (63), (64) and (69)-(71) into (80) gives

$$\begin{aligned} \Xi =& \frac{\beta}{2}w_{xx}^2(1,t) + \frac{1}{2}w_t^2(1,t) - \frac{\Phi(t)}{2}w_x^2(1,t) \\ &+ \frac{\gamma}{2}w_{xt}^2(1,t) + (1-\eta)\beta w_x(1,t)w_{xx}(1,t) \\ &+ [\Phi(t)w_x(1,t) - \beta w_{xxx}(1,t) + \gamma w_{xtt}(1,t)] \\ &\times \Upsilon(1,t) - \int_0^1 [\Delta(x,t)]_t dx, \end{aligned} \tag{83}$$

where the boundary value conditions of (9) were applied and

$$\Upsilon(1,t) = w_x(1,t) - \eta w(1,t). \tag{84}$$

Next, we first treat equation (81). By setting

$$\begin{aligned} \mathscr{R} : =& -\frac{\alpha}{2}\widetilde{M}(\|w_x\|^2) + \left(\frac{1}{2}-\eta\right)\Phi(t)\int_0^1 w_x^2 dx \\ &+ \eta\int_0^1 w_t^2 dx + 2v\eta\int_0^1 w_x w_t dx, \end{aligned} \tag{85}$$

and the energy $E(\cdot)$ given as in (10), we can show that

$$\begin{aligned} \Xi =& \frac{1}{2}\int_0^1 w_t^2 dx + \left(\frac{3}{2}-\eta\right)\beta\int_0^1 w_{xx}^2 dx \\ &+\gamma\left(\eta - \frac{1}{2}\right)\int_0^1 w_{xt}^2 dx + \frac{\alpha}{2}\widetilde{M}(\|w_x\|^2) + \mathcal{R}, \end{aligned} \tag{86}$$

where α is the constant given in (79), an adjustment parameter adapted to the existence interval of η in (80). The introduction of the adjustment term $\frac{\alpha}{2}\widetilde{M}(\|w_x\|^2)$ in (86) is to get a proportional form of the energy function, which leads to estimate (90) later.

From (85) and $\Phi(t) = M(\|w_x\|^2) - v^2$, it is easy to see that

$$\begin{aligned} \mathscr{R} =& \left(\frac{1}{2}-\eta\right)M(\|w_x\|^2)\int_0^1 w_x^2 dx - \frac{\alpha}{2}\widetilde{M}(\|w_x\|^2) \\ &+ \eta\int_0^1 (w_t + vw_x)^2 dx - \frac{v^2}{2}\|w_x\|^2. \end{aligned} \tag{87}$$

Since $\|w_x\| \le \|w_{xx}\|$, $\eta > 1/2$ and $\widetilde{M}(\|w_x\|^2) \le (M^* - v^2)\|w_x\|^2$, one has

$$\begin{aligned} &\left(\eta - \frac{1}{2}\right)M(\|w_x\|^2)\|w_x\|^2 + \frac{v^2}{2}\|w_x\|^2 + \frac{\alpha}{2}\widetilde{M}(\|w_x\|^2) \\ &\le \hat{\mathcal{C}}\|w_x\|^2 \le \hat{\mathcal{C}}\|w_{xx}\|^2 \end{aligned} \tag{88}$$

with $\hat{\mathcal{C}} = (\eta - \frac{1-\alpha}{2})M^* + \frac{1-\alpha}{2}v^2$. Plug (88) into (87) to obtain

$$\mathscr{R} \ge -\hat{\mathcal{C}}\|w_{xx}\|^2. \tag{89}$$

Substitution of (89) into (86) with $E(\cdot)$ given in (10) yields

$$\Xi \ge \mathscr{C}E(t) \tag{90}$$

where $\mathscr{C} := \min\{1, 3 - 2\eta - 2\hat{\mathcal{C}}/\beta, 2\eta - 1, \alpha\}$. Thanks to $1/2 < \eta < C_\beta$, we see that $\mathscr{C} > 0$.

In what follows, we deal with the equation (83). In light of the boundary condition of (9), we can deduce that

$$\begin{aligned} \Xi =& \frac{1}{2\beta}F_2^2(w_{xt}(1,t)) + \frac{1}{2}w_t^2(1,t) - \frac{\Phi(t)}{2}w_x^2(1,t) \\ &- \int_0^1 [\Delta(x,t)]_t dx - (1-\eta)w_x(1,t)F_2(w_{xt}(1,t)) \\ &+ \frac{\gamma}{2}w_{xt}^2(1,t) + [vw_t(1,t) - F_1(w_t(1,t))]\Upsilon(1,t). \end{aligned} \tag{91}$$

By $\Phi = M(s) - v^2 \geq 1 - v^2$ and Young's inequality, it follows from (91) that

$$\begin{aligned} \Xi \leq & \frac{2\varepsilon+1}{4\varepsilon} w_t^2(1,t) + \frac{1}{4\varepsilon} F_1^2(w_t(1,t)) + \frac{\gamma}{2} w_{xt}^2(1,t) \\ & + \left(\frac{1}{4\varepsilon} + \frac{1}{2\beta}\right) F_2^2(w_{xt}(1,t)) - \int_0^1 [\Delta(x,t)]_t dx \\ & - \frac{1-v^2}{2} w_x^2(1,t) + 4\varepsilon\eta^2 w^2(1,t) + 8\varepsilon w_x^2(1,t), \end{aligned} \tag{92}$$

where $\varepsilon > 0$ is the Young's parameter. According to Poincaré's inequality: $|w(1,t)| \leq \|w_{xx}\|$, one has from (10) that $|w(1,t)|^2 \leq \frac{2}{\beta} E(t)$. Set

$$\begin{aligned} \mathcal{R}_1 &:= w_t^2(1,t) + F_1^2(w_t(1,t)), \\ \mathcal{R}_2 &:= w_{xt}^2(1,t) + F_2^2(w_{xt}(1,t)). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can choose $\varepsilon < \min\{\frac{1-v^2}{16}, \frac{\beta\mathscr{C}}{16\eta^2}\}$ in (92) to get

$$\Xi \leq \mathcal{C}_1 \mathcal{R}_1 - \int_0^1 [\Delta(x,t)]_t dx + \mathcal{C}_2 \mathcal{R}_2 + \frac{\mathscr{C}}{2} E(t), \tag{93}$$

where $\mathcal{C}_1 = \frac{2\varepsilon+1}{4\varepsilon}$ and $\mathcal{C}_2 := \max\{\frac{\gamma}{2}, \frac{1}{4\varepsilon} + \frac{1}{2\beta}\}$. By (90), it follows from (93) that

$$\frac{\mathscr{C}}{2} E(t) \leq \mathcal{C}_1 \mathcal{R}_1 - \int_0^1 [\Delta(x,t)]_t dx + \mathcal{C}_2 \mathcal{R}_2. \tag{94}$$

Recalling $E(t)$ and applying the mean value inequality and Poincaré's inequality yields

$$\begin{aligned} & \int_0^1 \Delta(x,t) dx \\ & = \int_0^1 [w_t(x w_x - \eta w) + \gamma(1-\eta) w_{xt} w_x] dx \\ & \quad + \int_0^1 [\gamma x w_{xt} w_{xx} + v x w_x^2 - 2\eta v w_x w] dx \\ & \leq \frac{1+\eta}{2} \|w_t\|^2 + \frac{3+5\eta}{2} \|w_x\|^2 \\ & \quad + \frac{\gamma}{2} \|w_{xx}\|^2 + \frac{(2+\eta)\gamma}{2} \|w_{xt}\|^2 \leq C_a E(t), \end{aligned} \tag{95}$$

with

$$C_a := \max\{\frac{\gamma}{\beta}, \frac{3+5\eta}{1-v^2}\}.$$

It is easy to check that for any $T > S$,

$$\begin{aligned} & \Big| \int_S^T \int_0^1 [\Delta(x,t)]_t dx dt \Big| \\ & \leq \Big| \int_0^1 \Delta(x,T) dx \Big| + \Big| \int_0^1 \Delta(x,S) dx \Big|. \end{aligned} \tag{96}$$

Thanks to (95) and $E(T) \leq E(S)$ for any $T > S$, we can derive that

$$\left| \int_S^T \int_0^1 [\Delta(x,t)]_t dx dt \right| \leq 2 C_a E(S). \tag{97}$$

Now, integrate simultaneously both sides of (94) from 0 to T, and invoke (97) to obtain

$$\int_0^T E(t) dt \leq \mathcal{C}_3 \int_0^T \mathcal{R}_1 dt + \frac{4C_a}{\mathscr{C}} E(0) + \mathcal{C}_4 \int_0^T \mathcal{R}_2 dt, \tag{98}$$

where $\mathcal{C}_3 = 2\mathcal{C}_1/\mathscr{C}$, $\mathcal{C}_4 = 2\mathcal{C}_2/\mathscr{C}$, and $\mathscr{C}, C_a$ are positive constants given by (90) and (95), respectively. Furthermore, thanks to (26), one gets

$$\begin{aligned} E(0) &= E(T) + \int_0^T F_1(w_t(1,t)) w_t(1,t) dt \\ & \quad + \int_0^T F_2(w_{xt}(1,t)) w_{xt}(1,t) dt, \end{aligned} \tag{99}$$

and

$$T E(T) \leq \int_0^T E(t) dt. \tag{100}$$

Plugging (99) and (100) into (98) yields

$$E(T) \leq C_T \int_0^T \mathcal{R}_1 dt + \hat{C}_T \int_0^T \mathcal{R}_2 dt, \tag{101}$$

for $T > \frac{4C_a}{\mathscr{C}}$, where C_T and $\hat{C}_T$ are positive constants depending on T. Denote

$$\Pi_N := \{t \in [0,T]; |w_t(1,t)| \leq N\}$$

and

$$\Lambda_N := \{t \in [0,T]; |w_{xt}(1,t)| \leq N\}$$

with the constant $N > 0$ given by (22). From (8), it is easy to verify that

$$\int_{[0,T]\setminus\Pi_N} \mathcal{R}_1 dt \leq \mathcal{C}_5 \int_{[0,T]\setminus\Pi_N} w_t(1,t) F_1(w_t(1,t)) dt,$$

where $\mathcal{C}_5 = \frac{1+k_2^2}{k_1^2}$. Similarly,

$$\int_{[0,T]\setminus\Lambda_N} \mathcal{R}_2 dt \leq \mathcal{C}_5 \int_{[0,T]\setminus\Lambda_N} w_{xt}(1,t) F_2(w_{xt}(1,t)) dt.$$

On the other hand, from (22) we have

$$\int_{\Pi_N} \mathcal{R}_1 dt \leq \int_{\Pi_N} W_1(w_t(1,t) F_1(w_t(1,t))) dt. \tag{102}$$

and

$$\int_{\Lambda_N} \mathcal{R}_2 dt \le \int_{\Lambda_N} W_2(w_{xt}(1,t)F_2(w_{xt}(1,t)))dt. \quad (103)$$

In view of Jensen's inequality, there holds

$$\begin{aligned} &\int_{\Pi_N} W_1(w_t(1,t)F_1(w_t(1,t)))dt \\ &\le T\cdot W_1\left(\int_0^T \frac{w_t(1,t)F_1(w_t(1,t))}{T}dt\right) \\ &\le T\cdot \hat{W}_1\left(\int_0^T w_t(1,t)F_1(w_t(1,t))dt\right). \end{aligned} \quad (104)$$

As a result,

$$\begin{aligned} &\int_{\Lambda_N} W_2(w_{xt}(1,t)F_2(w_{xt}(1,t)))dt \\ &\le T\cdot \hat{W}_2\left(\int_0^T w_{xt}(1,t)F_2(w_{xt}(1,t))dt\right). \end{aligned} \quad (105)$$

Set

$$\begin{aligned} Y_1 &:= \int_0^T w_t(1,t)F_1(w_t(1,t))dt, \\ Y_2 &:= \int_0^T w_{xt}(1,t)F_2(w_{xt}(1,t))dt. \end{aligned}$$

From (101), one can show that

$$E(T)\le C_T T\hat{W}_1(Y_1)+\hat{C}_T T\hat{W}_2(Y_2)+C_T\mathcal{C}_5(Y_1+Y_2). \quad (106)$$

Since the maps $W_i(s), i=1,2$, are concave and strictly increasing for $s\ge 0$, it follows from (106) that

$$\begin{aligned} E(T) &\le 2C_T T\cdot \hat{W}_1\left(\frac{1}{2}(Y_1+Y_2)\right)+C_T\mathcal{C}_5 Y_1 \\ &+2\hat{C}_T T\cdot \hat{W}_2\left(\frac{1}{2}(Y_1+Y_2)\right)+C_T\mathcal{C}_5 Y_2 \\ &\le h_1[\frac{1}{2}(\hat{W}_1+\hat{W}_2)]\left(\frac{1}{2}(Y_1+Y_2)\right) \\ &+h_2\frac{1}{2}(Y_1+Y_2), \end{aligned} \quad (107)$$

where the map $\frac{1}{2}(\hat{W}_1+\hat{W}_2)$ is defined as in (23), $h_1 = 4T\max\{C_T,\hat{C}_T\}$ and $h_2 = 2C_T\mathcal{C}_5$. According to Lemma 2.1, we have $Y_1+Y_2 = E(0)-E(T)$. Set $\hat{\delta} = \frac{1}{h_1}$ and $\delta = \frac{h_2}{h_1}$ in (24). Then, from (107) for $P(s)$ defined by (24), we can show that

$$P(E(T))+E(T)\le E(0). \quad (108)$$

Repeating this process, we arrive at

$$P(E((n+1)T))+E((n+1)T)\le E(nT), n=0,1,\cdots.$$

Following lemma 3.3 of Lasiecka (1993) with $s_n = E(nT)$, $S_0 = E(0)$, $n=1,2,\cdots$, we obtain $E(nT)\le S(n)$, where S is the solution of the ODE (25) satisfying $\lim_{t\to\infty} S(t)=0$. For $t\ge T$, let $t=nT+\xi$, with $0\le\xi<T$ and $n=0,1,2,\cdots$. We can thus arrive at

$$E(t)\le E(nT)\le S(n)=S\left(\frac{t-\xi}{T}\right)\le S\left(\frac{t}{T}-1\right). \quad (109)$$

Owing to the boundary conditions $w(0,t)=w_x(0,t)=0$ and applying the Cauchy-Schwarz inequality, there holds

$$\begin{aligned} \left|w(x,t)\right|^2 &= \left(\int_0^x w_x(s,t)ds\right)^2 \le \left(\int_0^1 |w_x(x,t)|dx\right)^2 \\ &\le \int_0^1 |w_x(x,t)|^2dx \le \int_0^1 |w_{xx}(x,t)|^2dx \\ &\le \frac{2}{\beta}E(t), \end{aligned} \quad (110)$$

for all $t\ge 0$ and $x\in[0,1]$. The result then follows from (109) and (110). □

4 Numerical simulations

In this section, we present some numerical simulations for the closed-loop system (9) to validate the effectiveness of the proposed control (6). The finite element method is used to simulate the system performance with the proposed boundary schemes by using the quadratic Lagrange basis. The specific discrete equation of the closed-loop system is similar to the discrete equation in (Cheng-Wu-Guo, 2021), which is omitted here. From the simulation results, it is seen that the stability of the closed-loop system is consistent with respect to the spatial step size. The uniform stability of the discrete scheme is another interesting problem to be investigated in the future.

Several different feedback functions satisfying the condition (6) are applied in simulations as follows:

$$\mathbb{L}(s) = 3s, \quad (111)$$

$$F(s) = \begin{cases} 4s-1, & s\le -1, \\ 5s^3, & -1<s<1, \\ 2s+2+\cos(s-1), & 1\le s, \end{cases} \quad (112)$$

$$G(s) = \begin{cases} 3s-1+\sin(s+1), & s\le -1, \\ 4s^5, & -1<s<1, \\ 2s+2, & 1\le s, \end{cases} \quad (113)$$

$$K(s) = \begin{cases} 3s+3-\frac{1}{e}+m(s), & s\le -1, \\ s^5 e^{-\frac{1}{s^4}}, & -1<s<1, \\ 2s-2+\frac{1}{e}, & 1\le s, \end{cases} \quad (114)$$

where $m(s) = \ln([s+1]^2+1)$.

From Fig. 2, it is seen that three nonlinear feedback functions (112), (113) and (114) satisfy the restrictive

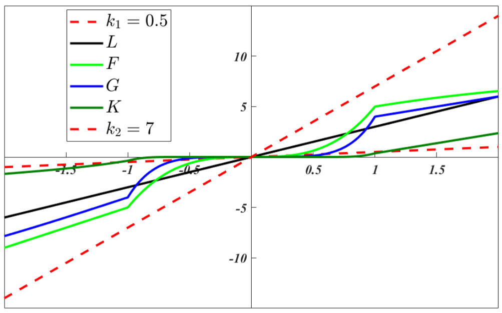


Fig. 2. The boundary feedback laws in simulation.

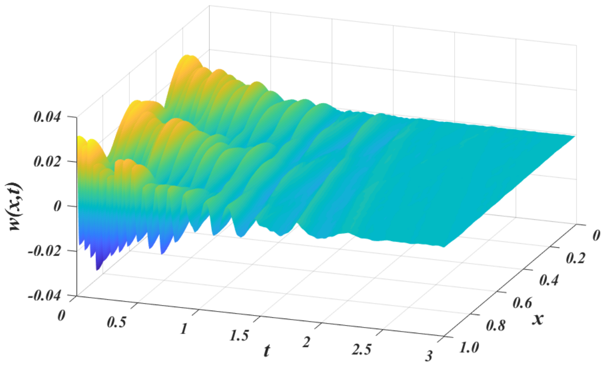


Fig. 3. Transverse displacements of the closed-loop system with the feedback case I.

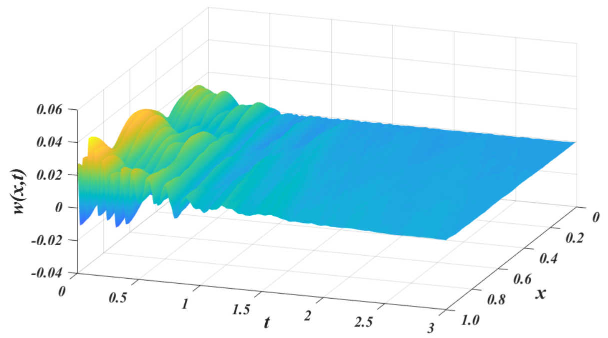


Fig. 4. Transverse displacements of the closed-loop system with the feedback case II.

condition (8) near infinity with $k_1 = 0.5, k_2 = 7$, the condition (28) of Corollary 2.1 with $p = 3$ and $p = 5$, and the condition (35) of Corollary 2.3 near zero with $p = 2$, respectively.

The dynamic responses of the closed-loop system (9) are simulated by three control strategies.

- Case I: Linear control laws: $F_1 = F_2 = \mathbb{L}$.
- Case II: Nonlinear control laws: $F_1 = F,\ F_2 = G$.
- Case III: Nonlinear control laws: $F_1 = K,\ F_2 = K$.

For any given $\mathcal{N} \in \mathbb{N}$, the uniform grid $T_{\mathcal{N}}$ on $[0, 1]$ is defined as simply as a set of points $(x_j = hj), 0 \le j \le \mathcal{N}$, where $h = 1/\mathcal{N}$ and $0 = x_0 < x_1 < \cdots < x_{\mathcal{N}} =$

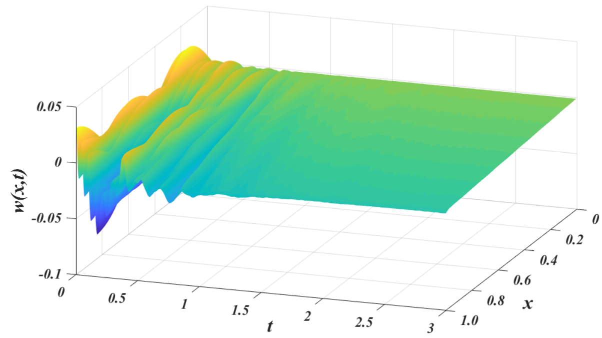


Fig. 5. Transverse displacements of the closed-loop system with the feedback case III.

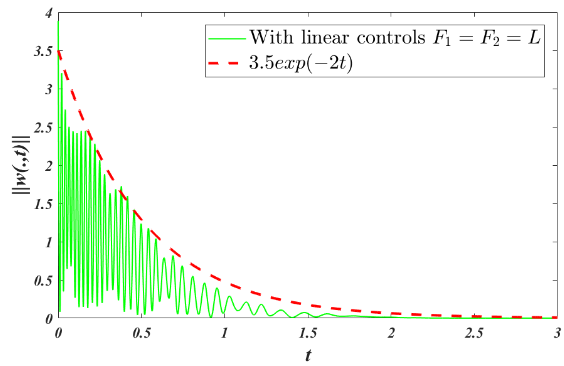


Fig. 6. Norm $\|w(\cdot, t)\|$ of the closed-loop system with the feedback case I.

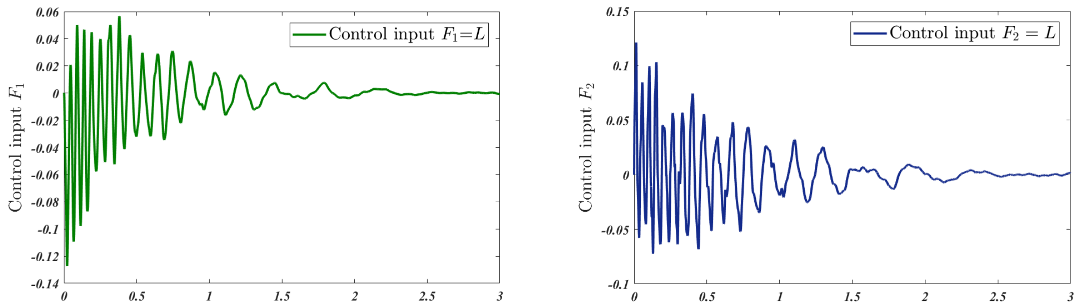
1. To show the numerical results, the non-dimensional parameters of the closed-loop system (9) are assigned as follows: $M(\|w_x(\cdot, t)\|) = 1 + 2\|w_x(\cdot, t)\|^2$, $\mathcal{N} = 50$, $v = 0.06, \beta = 0.4$, $\gamma = 0.03$. The initial displacement and velocity in simulation are chosen as

$$f_1(x) = 0.04 \sin(8x),\ f_2(x) = 0.02 \cos(3x).$$

As depicted in Fig. 3, Fig. 4 and Fig. 5, the transverse vibration $w(\cdot, \cdot)$ of the beam (4) for three cases of boundary control laws is quickly suppressed. In addition, the corresponding control inputs $F_1(\cdot)$ and $F_2(\cdot)$ and the norm $\|w(\cdot, t)\|$ of the closed-loop system (9) for three boundary control laws are shown in Fig. 6-Fig. 11, which is consistent with the theoretical results obtained in Theorem 2.2 and Corollaries 2.1 and 2.3.

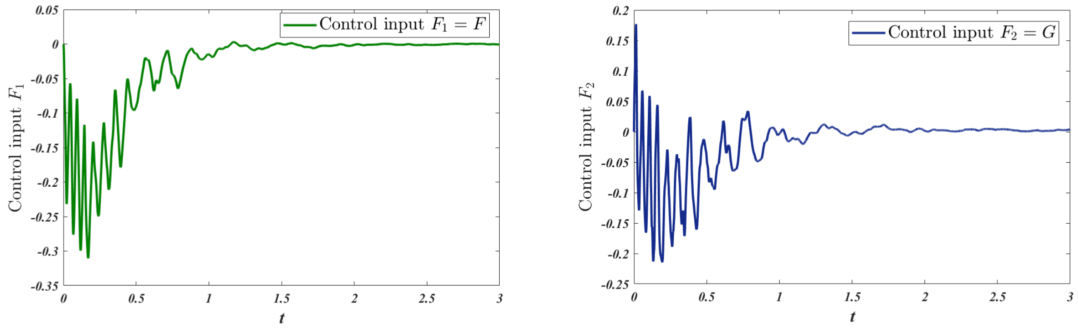
5 Conclusions

In this paper, the stabilization of an axially moving Kirchhoff-type beam with rotational inertia under double nonlinear controls is considered. Both the decay rates of the solution and the energy are estimated. Instead of the Faedo-Galerkin approximation method, the nonlinear semigroup method is used to establish the well posedness of the resulting closed-loop system for which the solutions depend continuously on the initial values.



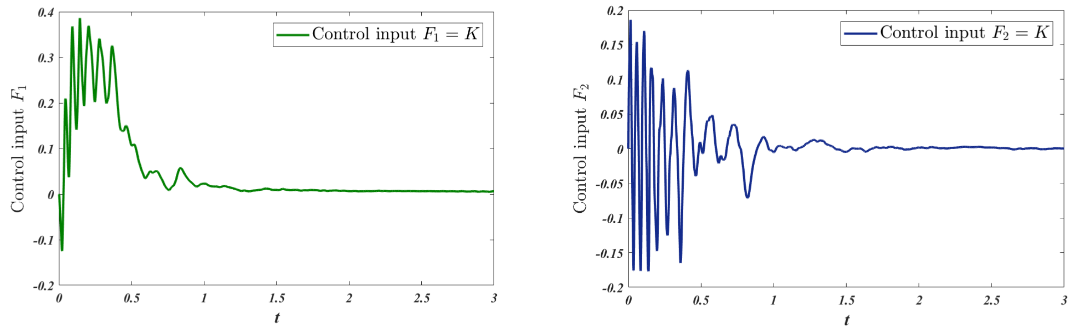
(I) Control input $F_1 = \mathbb{L}$. (II) Control input $F_2 = \mathbb{L}$.

Fig. 9. Control inputs of the closed-loop system with the feedback case I.



(I) Control input $F_1 = F$. (II) Control input $F_2 = G$.

Fig. 10. Control inputs of the closed-loop system with the feedback case II.



(I) Control input $F_1 = K$. (II) Control input $F_2 = K$.

Fig. 11. Control inputs of the closed-loop system with the feedback case III.

Then the asymptotic stability of the resulting closed-loop system is guaranteed by solving a dissipative ODE system and decay rates for the system can be calculated when the changing behavior of the nonlinear feedbacks near zero point is available. The method adopted in this paper can be potentially extended to deal with three or more nonlinear controls in PDEs such as coupled systems. The explicit or optimal decay rate of the system in this paper is still an open problem, which will be investigated in the future.

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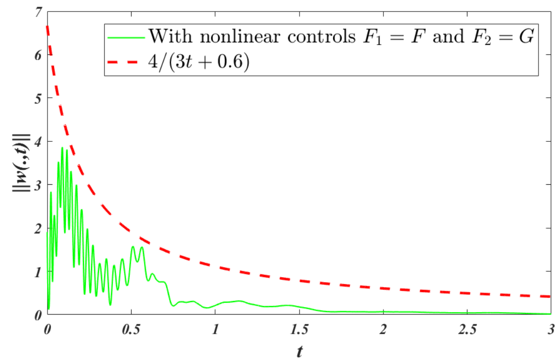

Fig. 7. Norm $\|w(\cdot,t)\|$ of the closed-loop system with the feedback case II.

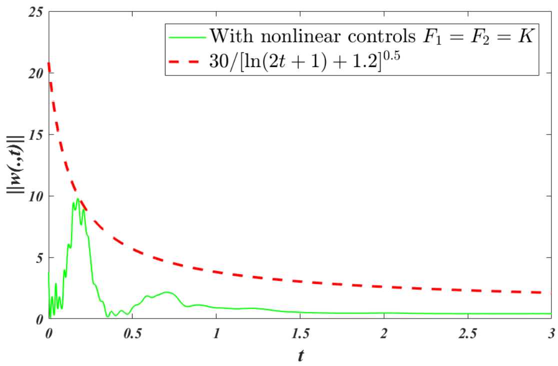

Fig. 8. Norm $\|w(\cdot,t)\|$ of the closed-loop system with the feedback case III.

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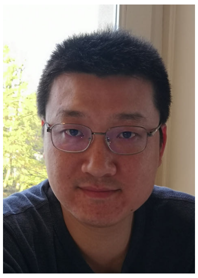

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