# A corrector result for an electrostatic model of interdigitated combs

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**Abstract**

In this paper we consider two opposite and intergitated 2D-combs, each one with $\varepsilon$-periodically distributed fingers, each finger having cross-section of order $\varepsilon$ and fixed height. Via an asymptotic analysis based on the two-scale convergence method, we study, as $\varepsilon$ vanishes, the electrical potential in the vacuum between these two combs when one comb is grounded while the voltage in the other one is assumed of order $\varepsilon$, and we prove a corrector result.

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## 1 Introduction

In this paper we consider two opposite and interdigitated 2D-combs $\Omega^a_\varepsilon$ and $\Omega^b_\varepsilon$, each one with $\varepsilon$-periodically distributed fingers, each finger having cross-section of order $\varepsilon$ and fixed height (see Figure 1). Via an asymptotic analysis based on the two-scale convergence method proposed in [11] and developed in [1] (see also [2] and [10]), we study, as $\varepsilon$ vanishes, the electrical potential in the vacuum $\Omega^c_\varepsilon$ (see Figure 1) between $\Omega^a_\varepsilon$ and $\Omega^b_\varepsilon$, when $\Omega^b_\varepsilon$ is grounded and the voltage in $\Omega^a_\varepsilon$ is assumed of order $\varepsilon$, and we prove a corrector result.

Precisely, let $\zeta_1$, $\zeta_2$, $\zeta_3$, $\zeta_4$ $\in ]0,1[$ be such that $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$, and set $\omega^a = ]\zeta_1, \zeta_2[$ and $\omega^b = ]\zeta_3, \zeta_4[$. Let $L \in ]0,+\infty[$ and $l_1, l_2, l_3 \in ]0,+\infty[$ be such that $l_1 + 2 < l_2 < l_3$.

For every $\varepsilon \in \left\{\frac{L}{n} : n \in \mathbb{N}\right\}$ set (see Figure 1)

$$\Omega^a_\varepsilon = (]0,L[\times]l_2,l_3[) \cup \left( \bigcup_{k=0}^{\frac{L}{\varepsilon}-1} (\varepsilon\omega^a + \varepsilon k) \times ]l_1+1, l_2] \right),$$

$$\Omega^b_\varepsilon = (]0,L[\times]0,l_1[) \cup \left( \bigcup_{k=0}^{\frac{L}{\varepsilon}-1} \left(\varepsilon\omega^b + \varepsilon k\right) \times [l_1, l_2-1[ \right),$$

$$\Omega^c_\varepsilon = (]0,L[\times]0,l_3[) \setminus \left( \overline{\Omega^a_\varepsilon} \cup \overline{\Omega^b_\varepsilon} \right),$$

$$\Gamma^a_\varepsilon = \partial\Omega^a_\varepsilon \cap \partial\Omega^c_\varepsilon, \quad \Gamma^b_\varepsilon = \partial\Omega^b_\varepsilon \cap \partial\Omega^c_\varepsilon, \quad \Gamma_\varepsilon = \Gamma^a_\varepsilon \cup \Gamma^b_\varepsilon, \quad \Gamma = \{0,L\}\times]l_1,l_2[.$$

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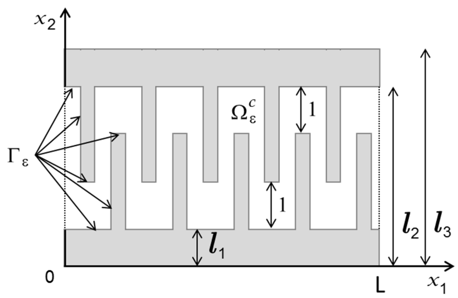


Figure 1: $\Omega_\varepsilon$

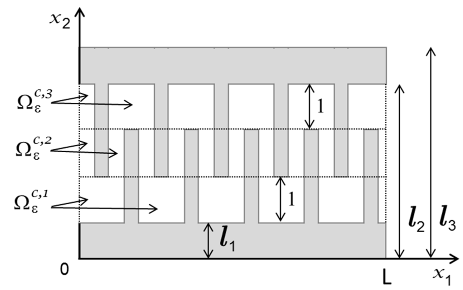


Figure 2: Decomposition of $\Omega_\varepsilon$

In what follows, $x = (x_1, x_2)$ denotes the generic element of $\mathbb{R}^2$, while $y$ belongs to $[0, 1]$.

For every $\varepsilon$, consider the following normalized problem

$$\begin{cases} -\Delta\phi_\varepsilon = 0, \text{ in } \Omega_\varepsilon^c, \\ \phi_\varepsilon = 1, \text{ on } \Gamma_\varepsilon^a, \quad \phi_\varepsilon = 0, \text{ on } \Gamma_\varepsilon^b, \\ \nabla\phi_\varepsilon \cdot \nu = 0, \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where $\nu$ denotes the unit normal to $\Gamma$ exterior to $\Omega_\varepsilon^c$. The solution $\phi_\varepsilon$ represents the electrical potential in the vacuum $\Omega_\varepsilon^c$ between $\Omega_\varepsilon^a$ and $\Omega_\varepsilon^b$, when $\Omega_\varepsilon^b$ is grounded and the voltage in $\Omega_\varepsilon^a$ is assumed equal to 1. The weak formulation of (1.1) is

$$\phi_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^c, \mu_\varepsilon), \qquad \int_{\Omega_\varepsilon^c} \nabla\phi_\varepsilon(x)\nabla\psi(x)dx = 0, \quad \forall\psi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^c, 0), \tag{1.2}$$

where $\mu_\varepsilon = 1$ on $\Gamma_\varepsilon^a$, $\mu_\varepsilon = 0$ on $\Gamma_\varepsilon^b$ (for $g \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$, $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon^c, g) = \{\psi \in H^1(\Omega_\varepsilon^c) : \psi = g, \text{ on } \Gamma_\varepsilon\}$).

The aim of this paper is to study the asymptotic behavior of $\varepsilon\phi_\varepsilon$, as $\varepsilon$ vanishes. Precisely, let

$$\phi_2 : y \in [0,1] \longrightarrow \begin{cases} \dfrac{y+1-\zeta_4}{\zeta_1-\zeta_4+1}, \text{ if } y \in [0,\zeta_1], \\ 1, \text{ if } y \in [\zeta_1,\zeta_2], \\ \dfrac{y-\zeta_3}{\zeta_2-\zeta_3}, \text{ if } y \in [\zeta_2,\zeta_3], \\ 0, \text{ if } y \in [\zeta_3,\zeta_4], \\ \dfrac{y-\zeta_4}{\zeta_1-\zeta_4+1}, \text{ if } y \in [\zeta_4,1]. \end{cases} \tag{1.3}$$

Moreover, split the vacuum $\Omega_\varepsilon^c$ in three parts (see Figure 2) $\Omega_\varepsilon^c = \Omega_\varepsilon^{c,1} \cup \Omega_\varepsilon^{c,2} \cup \Omega_\varepsilon^{c,3}$, with

$\Omega_\varepsilon^{c,1} = \Omega_\varepsilon^c \cap (]0, L[\times]l_1, l_1+1[)$, $\Omega_\varepsilon^{c,2} = \Omega_\varepsilon^c \cap (]0, L[\times[l_1+1, l_2-1])$, $\Omega_\varepsilon^{c,3} = \Omega_\varepsilon^c \cap (]0, L[\times]l_2-1, l_2[)$.

Then, the main result of this paper is the following corrector result.

**Theorem 1.1.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2). *Moreover, let $\phi_2$ be defined by* (1.3). *Then,*

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^c} |\varepsilon\phi_\varepsilon(x)|^2 \, dx = 0, \tag{1.4}$$

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^{c,1}\cup\Omega_\varepsilon^{c,3}} |\varepsilon\nabla\phi_\varepsilon(x)|^2 \, dx = 0, \\ \\ \displaystyle\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon^{c,2}} \left( \left|\varepsilon\partial_{x_1}\phi_\varepsilon(x) - (\partial_y\phi_2)\left(\frac{x_1}{\varepsilon}\right)\right|^2 + |\varepsilon\partial_{x_2}\phi_\varepsilon(x)|^2 \right) dx = 0. \end{cases} \tag{1.5}$$

In section 2, some *a priori* estimates will be proved. These estimate allows us to obtain some two-scale convergence results in section 3 which provide the convergence of the energies in section 4. Eventually, Theorem 1.1 will be proved in section 4.

There is a large bibliography about different problems in a comb-like shaped structure starting from the pioneering PhD thesis [4] (see also [5]). We refer to [8] and references therein for an exhaustive view on the subject. About the homogenization of problems in opposite comb-like shaped structure, we refer to [3], [6], [7], and mainly [9] which strongly inspired the present paper.
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## 2 *A priori* estimates

Let $\varphi^\star : (y, x_2) \in \mathbb{R} \times [l_1, l_2] \to \varphi^\star(y, x_2) \in \mathbb{R}$ be a function such that

$$\varphi^\star \in C^\infty(\mathbb{R} \times [l_1, l_2]) \tag{2.1}$$

$$\begin{cases} \varphi^\star(\cdot, x_2) \text{ is 1-periodic for every } x_2 \in [l_1, l_2], \\ \varphi^\star = 1, \text{ in } \omega^a \times ]l_1 + 1, l_2[, \quad \varphi^\star = 0, \text{ in } \omega^b \times ]l_1, l_2 - 1[, \\ \varphi^\star = 1, \text{ on } \mathbb{R} \times \{l_2\}, \quad \varphi^\star = 0, \text{ on } \mathbb{R} \times \{l_1\}, \end{cases} \tag{2.2}$$

and for every $\varepsilon$ set

$$\varphi^\star_\varepsilon(x_1, x_2) = \varphi^\star\left(\frac{x_1}{\varepsilon}, x_2\right), \text{ in } \mathbb{R} \times [l_1, l_2]. \tag{2.3}$$

**Proposition 2.1.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2)*. Then*

$$\exists c \in ]0, +\infty[ \quad : \quad \|\varepsilon \nabla \phi_\varepsilon\|_{L^2(\Omega^c_\varepsilon)} \le c, \quad \forall \varepsilon. \tag{2.4}$$

*Proof.* For every $\varepsilon$, let $\varphi^\star_\varepsilon$ be defined by (2.1)-(2.3). Moreover, set $Y = ]0, 1[ \times ]l_1, l_2[$. Then, one has

$$\begin{cases} \|\partial_{x_1} \varphi^\star_\varepsilon\|^2_{L^2(\Omega^c_\varepsilon)} = \displaystyle\sum_{k=0}^{\frac{L}{\varepsilon}-1} \frac{1}{\varepsilon} \|\partial_y \varphi^\star\|^2_{L^2(Y)} = \frac{L}{\varepsilon^2} \|\partial_y \varphi^\star\|^2_{L^2(Y)}, \\ \|\partial_{x_2} \varphi^\star_\varepsilon\|^2_{L^2(\Omega^c_\varepsilon)} = \displaystyle\sum_{k=0}^{\frac{L}{\varepsilon}-1} \varepsilon \|\partial_{x_2} \varphi^\star\|^2_{L^2(Y)} = L \|\partial_{x_2} \varphi^\star\|^2_{L^2(Y)}, \end{cases} \quad \forall \varepsilon. \tag{2.5}$$

Now choosing $\psi = \phi_\varepsilon - \varphi^\star_\varepsilon$ in (1.2) and using Young's inequality and (2.5) provide

$$\int_{\Omega^c_\varepsilon} |\nabla \phi_\varepsilon|^2 dx \le L \left( \|\partial_y \varphi^\star\|^2_{L^2(Y)} \varepsilon^{-2} + \|\partial_{x_2} \varphi^\star\|^2_{L^2(Y)} \right), \quad \forall \varepsilon,$$

which implies (2.4). $\square$

**Proposition 2.2.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2)*. Then,*

$$\exists c \in ]0, +\infty[ \quad : \quad \|\phi_\varepsilon\|_{L^2(\Omega^c_\varepsilon)} \le c, \quad \forall \varepsilon. \tag{2.6}$$

*Proof.* First, let us prove that

$$\exists c \in ]0, +\infty[ \quad : \quad \|\phi_\varepsilon\|_{L^2(\Omega^{c,2}_\varepsilon)} \le c, \quad \forall \varepsilon. \tag{2.7}$$

To this aim, set $P = ]0,1[\setminus\left(\overline{\omega^a}\cup\overline{\omega^b}\right) = ]0,\zeta_1[\cup]\zeta_2,\zeta_3[\cup]\zeta_4,1[$. Fix $\varepsilon$. Then, one has

$$\|\phi_\varepsilon\|^2_{L^2(\Omega^{c,2}_\varepsilon)} = \sum_{k=0}^{\frac{L}{\varepsilon}-1}\int_{(\varepsilon P+\varepsilon k)\times]l_1+1,l_2-1[}|\phi_\varepsilon(x)|^2dx. \tag{2.8}$$

Now fix $k\in\left\{0,\cdots,\frac{L}{\varepsilon}-1\right\}$. Then, if $x_1\in\varepsilon P+\varepsilon k$, one of the following three cases holds true:

$$x_1\in]\varepsilon k,\varepsilon\zeta_1+\varepsilon k[,\quad x_1\in]\varepsilon\zeta_2+\varepsilon k,\varepsilon\zeta_3+\varepsilon k[,\quad x_1\in]\varepsilon\zeta_4+\varepsilon k,\varepsilon(1+k)[.$$

In the first case, since $\phi_\varepsilon = 1$ on $\{\varepsilon\zeta_1+\varepsilon k\}\times]l_1+1,l_2-1[$, one has

$$\phi_\varepsilon(x_1,x_2) = 1-\int_{x_1}^{\varepsilon\zeta_1+\varepsilon k}\partial_{x_1}\phi_\varepsilon(t,x_2)dt,\quad \forall x_1\in]\varepsilon k,\varepsilon\zeta_1+\varepsilon k[,\text{ for a.e. } x_2\in]l_1+1,l_2-1[,$$

which implies

$$\int_{l_1+1}^{l_2-1}\int_{\varepsilon k}^{\varepsilon\zeta_1+\varepsilon k}|\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2 \le 2(l_2-l_1-2)\varepsilon+2\varepsilon^2\int_{l_1+1}^{l_2-1}\int_{\varepsilon k}^{\varepsilon\zeta_1+\varepsilon k}|\partial_{x_1}\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2. \tag{2.9}$$

Similarly, since $\phi_\varepsilon = 0$ on $(\{\varepsilon\zeta_3+\varepsilon k\}\cup\{\varepsilon\zeta_4+\varepsilon k\})\times]l_1+1,l_2-1[$, in the last two cases one has

$$\int_{l_1+1}^{l_2-1}\int_{\varepsilon\zeta_2+\varepsilon k}^{\varepsilon\zeta_3+\varepsilon k}|\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2 \le 2\varepsilon^2\int_{l_1+1}^{l_2-1}\int_{\varepsilon\zeta_2+\varepsilon k}^{\varepsilon\zeta_3+\varepsilon k}|\partial_{x_1}\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2 \tag{2.10}$$

and

$$\int_{l_1+1}^{l_2-1}\int_{\varepsilon\zeta_4+\varepsilon k}^{\varepsilon(1+k)}|\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2 \le 2\varepsilon^2\int_{l_1+1}^{l_2-1}\int_{\varepsilon\zeta_4+\varepsilon k}^{\varepsilon(1+k)}|\partial_{x_1}\phi_\varepsilon(x_1,x_2)|^2dx_1dx_2. \tag{2.11}$$

Adding (2.9), (2.10), and (2.11) gives

$$\int_{(\varepsilon P+\varepsilon k)\times]l_1+1,l_2-1[}|\phi_\varepsilon(x)|^2dx \le 2(l_2-l_1-2)\varepsilon+2\varepsilon^2\int_{(\varepsilon P+\varepsilon k)\times]l_1+1,l_2-1[}|\partial_{x_1}\phi_\varepsilon(x)|^2dx,$$

from which, summing up $k\in\left\{0,\cdots,\frac{L}{\varepsilon}-1\right\}$ and using (2.8) and (2.4), one obtains (2.7).

Similarly, one can prove that

$$\exists c\in]0,+\infty[\quad:\quad \|\phi_\varepsilon\|_{L^2(\Omega^{c,1}_\varepsilon)}\le c,\quad \|\phi_\varepsilon\|_{L^2(\Omega^{c,3}_\varepsilon)}\le c\quad\forall\varepsilon. \tag{2.12}$$

Eventually, (2.6) follows from (2.7) and (2.12).

□

# 3 Two-scale convergence results

We refer to [1] for the definition and the main properties of the two-scale convergence (in particular, see Definition 1.1 and Proposition 1.14 in [1]). Set

$$\Omega^{c,1} = ]0,L[\times]l_1,l_1+1[,\quad \Omega^{c,2} = ]0,L[\times]l_1+1,l_2-1[,\quad \Omega^{c,3} = ]0,L[\times]l_2-1,l_2[.$$

**Proposition 3.1.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2). *Set*

$$\overline{\phi_{\varepsilon,2}} = \begin{cases} \phi_{\varepsilon,2}, \ \textit{a.e. in } \Omega_\varepsilon^{c,2}, \\ 1, \ \textit{a.e. in } \bigcup_{k=0}^{\frac{L}{\varepsilon}-1} (\varepsilon\omega^a + \varepsilon k) \times ]l_1+1, l_2-1[, \\ 0, \ \textit{a.e. in } \bigcup_{k=0}^{\frac{L}{\varepsilon}-1} \left(\varepsilon\omega^b + \varepsilon k\right) \times ]l_1+1, l_2-1[, \end{cases} \tag{3.1}$$

*where $\phi_{\varepsilon,2} = \phi_\varepsilon|_{\Omega_\varepsilon^{c,2}}$. Moreover, let $\phi_2$ be defined by* (1.3). *Then, as $\varepsilon$ tends to zero,*

$$\begin{cases} \overline{\phi_{\varepsilon,2}} \ \textit{two-scale converges to } \phi_2, \\ \varepsilon\partial_{x_1}\overline{\phi_{\varepsilon,2}} \ \textit{two-scale converges to } \partial_y\phi_2, \quad \varepsilon\partial_{x_2}\overline{\phi_{\varepsilon,2}} \ \textit{two-scale converges to } 0. \end{cases} \tag{3.2}$$

*Proof.* Proposition 1.14 in [1], Proposition 2.1, and Proposition 2.2 ensure the existence of a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $u_2 \in L^2\left(\Omega^{c,2}, H^1_{\text{per}}(]0,1[)\right)$ (in possible dependence on the subsequence) such that

$$\begin{cases} \overline{\phi_{\varepsilon,2}} \text{ two-scale converges to } u_2, \\ \varepsilon\partial_{x_1}\overline{\phi_{\varepsilon,2}} \text{ two-scale converges to } \partial_y u_2, \quad \varepsilon\partial_{x_2}\overline{\phi_{\varepsilon,2}} \text{ two-scale converges to } 0, \end{cases} \tag{3.3}$$

as $\varepsilon$ tends to zero.

The next step is devoted to proving that

$$u_2 = 1, \ \text{a.e. in } \Omega^{c,2} \times \omega^a. \tag{3.4}$$

Indeed, the definition of $\overline{\phi_{\varepsilon,2}}$ gives

$$\int_{\Omega^{c,2}} \overline{\phi_{\varepsilon,2}}(x_1,x_2)\psi\left(x_1,x_2,\frac{x_1}{\varepsilon}\right)dx_1dx_2 = \int_{\Omega^{c,2}} \psi\left(x_1,x_2,\frac{x_1}{\varepsilon}\right)dx_1dx_2, \forall\psi \in C_0^\infty(\Omega^{c,2}\times\omega^a), \quad \forall\varepsilon. \tag{3.5}$$

Passing to the limit, as $\varepsilon$ tends to zero, in (3.5) and using the first limit in (3.3) provide

$$\int_{\Omega^{c,2}\times\omega^a} u_2(x_1,x_2,y)\psi(x_1,x_2,y)\,dx_1dx_2dy = \int_{\Omega^{c,2}\times\omega^a} \psi(x_1,x_2,y)\,dx_1dx_2dy, \quad \forall\psi \in C_0^\infty(\Omega^{c,2}\times\omega^a),$$

which implies (3.4). Similarly, one proves that

$$u_2 = 0, \ \text{a.e. in } \Omega^{c,2} \times \omega^b. \tag{3.6}$$

Finally, choosing $\psi = \varepsilon^2\chi_1(x_1,x_2)\chi_2\left(\frac{x_1}{\varepsilon}\right)$ with $\chi_1 \in C_0^\infty\left(\Omega^{c,2}\right)$ and $\chi_2 \in H^1_{\text{per}}\left(]0,1[\right)$ such that $\chi_2 = 0$ in $\omega^a \cup \omega^b$ as test function in (1.2) gives

$$\begin{aligned} &\varepsilon^2 \int_{\Omega^{c,2}} \partial_{x_1}\overline{\phi_{\varepsilon,2}}\left(\partial_{x_1}\chi_1(x_1,x_2)\chi_2\left(\frac{x_1}{\varepsilon}\right) + \varepsilon^{-1}\chi_1(x_1,x_2)\partial_y\chi_2\left(\frac{x_1}{\varepsilon}\right)\right)dx_1dx_2 \\ &+\varepsilon^2 \int_{\Omega^{c,2}} \partial_{x_2}\overline{\phi_{\varepsilon,2}}\partial_{x_2}\chi_1(x_1,x_2)\chi_2\left(\frac{x_1}{\varepsilon}\right)dx_1dx_2 = 0, \\ &\forall\chi_1 \in C_0^\infty\left(\Omega^{c,2}\right), \quad \forall\chi_2 \in H^1_{\text{per}}\left(]0,1[\right) \ : \ \chi_2 = 0, \ \text{in } \omega^a \cup \omega^b, \quad \forall\varepsilon. \end{aligned} \tag{3.7}$$

Passing to the limit, as $\varepsilon$ tends to zero, in (3.7) and using the second and third limits in (3.3) provide that, for a.e. $(x_1,x_2)$ in $\Omega^{c,2}$,

$$\int_{]0,1[\setminus(\omega^a\cup\omega^b)} \partial_y u_2(x_1,x_2,y)\partial_y\chi_2(y)dy = 0, \quad \forall \chi_2 \in H^1_{\text{per}}(]0,1[) \ : \ \chi_2 = 0, \text{ in } \omega^a\cup\omega^b. \tag{3.8}$$

Problem (3.4), (3.6), and (3.8) is equivalent to the following problem independent of $(x_1,x_2)$

$$\begin{cases} \partial^2_{y^2} u_2 = 0, \text{ in } ]0,1[\setminus\left(\omega^a\cup\omega^b\right), \\ u_2 = 1, \text{ in } \omega^a, \quad u_2 = 0, \text{ in } \omega^b, \quad u_2(0) = u_2(1), \quad \partial_y u_2(0) = \partial_y u_2(1), \end{cases} \tag{3.9}$$

which admits (1.3) as unique solution. Consequently, limits in (3.3) hold for the whole sequence and (3.2) is satisfied. □

The following result can be proved in a similar way.

**Proposition 3.2.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2). *Set*

$$\widetilde{\phi_{\varepsilon,3}} = \begin{cases} \phi_{\varepsilon,3}, \text{ a.e. in } \Omega^{c,3}_\varepsilon, \\ 1, \text{ a.e. in } \Omega^{c,3}\setminus\Omega^{c,3}_\varepsilon, \end{cases} \quad \text{and} \quad \widehat{\phi_{\varepsilon,1}} = \begin{cases} \phi_{\varepsilon,1}, \text{ a.e. in } \Omega^{c,1}_\varepsilon, \\ 0, \text{ a.e. in } \Omega^{c,1}\setminus\Omega^{c,1}_\varepsilon, \end{cases} \tag{3.10}$$

*where $\phi_{\varepsilon,3} = \phi_\varepsilon|_{\Omega^{c,3}_\varepsilon}$ and $\phi_{\varepsilon,1} = \phi_\varepsilon|_{\Omega^{c,1}_\varepsilon}$. Then, as $\varepsilon$ tends to zero,*

$$\begin{cases} \widetilde{\phi_{\varepsilon,3}} \text{ two-scale converges to } 1, \\ \varepsilon\partial_{x_1}\widetilde{\phi_{\varepsilon,3}} \text{ two-scale converges to } 0, \quad \varepsilon\partial_{x_2}\widetilde{\phi_{\varepsilon,3}} \text{ two-scale converges to } 0, \end{cases} \tag{3.11}$$

$$\begin{cases} \widehat{\phi_{\varepsilon,1}} \text{ two-scale converges to } 0, \\ \varepsilon\partial_{x_1}\widehat{\phi_{\varepsilon,1}} \text{ two-scale converges to } 0, \quad \varepsilon\partial_{x_2}\widehat{\phi_{\varepsilon,1}} \text{ two-scale converges to } 0. \end{cases} \tag{3.12}$$

# 4 Proof of Theorem 1.1

The following proposition is devoted to proving the energies convergence.

**Proposition 4.1.** *For every $\varepsilon$, let $\phi_\varepsilon$ be the unique solution to* (1.2). *Then*

$$\lim_{\varepsilon\to 0}\int_{\Omega^c_\varepsilon} |\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx = L(l_2-l_1-2)\left(\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}\right). \tag{4.1}$$

*Proof.* Let $\varphi^\star_\varepsilon$ be defined by (2.1)-(2.3), and let $\widehat{\phi_{\varepsilon,1}}$, $\widetilde{\phi_{\varepsilon,3}}$, $\overline{\phi_{\varepsilon,2}}$ be defined by (3.10) and (3.1). Choosing $\psi = \varepsilon^2(\phi_\varepsilon - \varphi^\star_\varepsilon)$ as test function in (1.2) and taking into account that $\widehat{\phi_{\varepsilon,1}}$, $\widetilde{\phi_{\varepsilon,3}}$, $\overline{\phi_{\varepsilon,2}}$, and $\varphi^\star_\varepsilon$ are constant on the fingers, give

$$\begin{aligned} &\int_{\Omega^c_\varepsilon} |\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx \\ &= \int_{\Omega^{c,1}} \left(\varepsilon\partial_{x_1}\widehat{\phi_{\varepsilon,1}}(x)\left(\partial_y\varphi^\star\right)\left(\frac{x_1}{\varepsilon},x_2\right) + \varepsilon^2\partial_{x_2}\widehat{\phi_{\varepsilon,1}}(x)\partial_{x_2}\varphi^\star\left(\frac{x_1}{\varepsilon},x_2\right)\right)dx \\ &+ \int_{\Omega^{c,3}} \left(\varepsilon\partial_{x_1}\widetilde{\phi_{\varepsilon,3}}(x)\left(\partial_y\varphi^\star\right)\left(\frac{x_1}{\varepsilon},x_2\right) + \varepsilon^2\partial_{x_2}\widetilde{\phi_{\varepsilon,3}}(x)\partial_{x_2}\varphi^\star\left(\frac{x_1}{\varepsilon},x_2\right)\right)dx \\ &+ \int_{\Omega^{c,2}} \left(\varepsilon\partial_{x_1}\overline{\phi_{\varepsilon,2}}(x)\left(\partial_y\varphi^\star\right)\left(\frac{x_1}{\varepsilon},x_2\right) + \varepsilon^2\partial_{x_2}\overline{\phi_{\varepsilon,2}}(x)\partial_{x_2}\varphi^\star\left(\frac{x_1}{\varepsilon},x_2\right)\right)dx, \quad \forall\varepsilon. \end{aligned} \tag{4.2}$$

Passing to the limit, as $\varepsilon$ tends to zero, in (4.2), and using Proposition 3.1 and Proposition 3.2 imply

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^c}|\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx=\int_{\Omega^{c,2}\times]0,1[}\partial_y\phi_2(y)\partial_y\varphi^\star(y,x_2)dxdy, \tag{4.3}$$

where $\phi_2$ is defined by (1.3).

As the last integral in (4.3) is concerned, the first two lines in (2.2) and (1.3) and an easy computation ensure that

$$\begin{aligned}&\int_{\Omega^{c,2}\times]0,1[}\partial_y\phi_2(y)\partial_y\varphi^\star(y,x_2)dxdy=\left(\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}\right)\int_{\Omega^{c,2}}1dx\\&=\left(\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}\right)L(l_2-l_1-2).\end{aligned} \tag{4.4}$$

Eventually, (4.1) follows from (4.3) and (4.4). □

**Remark 4.2.** *Note that in* (4.1)

$$L(l_2-l_1-2)=|\Omega^{c,2}|, \tag{4.5}$$

*and an easy computation show that*

$$\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}=\int_0^1|\partial_y\phi_2|^2\,dy, \tag{4.6}$$

*where $\phi_2$ is defined by* (1.3).

Now, Proposition 2.2, Proposition 3.1, and Proposition 4.1 allow us to prove Theorem 1.1.

*Proof of Theorem 1.1.* Limit (1.4) follows from Proposition 2.2.

Let us prove (1.5).

Let $\overline{\phi_{\varepsilon,2}}$ be defined by (3.1). Since $\phi_2\left(\frac{x_1}{\varepsilon}\right)$ and $\overline{\phi_{\varepsilon,2}}(x)$ are constant on the fingers, one has

$$\begin{aligned}&\int_{\Omega_\varepsilon^{c,1}\cup\Omega_\varepsilon^{c,3}}|\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx+\int_{\Omega_\varepsilon^{c,2}}\left(\left|\varepsilon\partial_{x_1}\phi_\varepsilon(x)-(\partial_y\phi_2)\left(\frac{x_1}{\varepsilon}\right)\right|^2+|\varepsilon\partial_{x_2}\phi_\varepsilon(x)|^2\right)dx\\&=\int_{\Omega_\varepsilon^c}|\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx+\int_{\Omega^{c,2}}\left|(\partial_y\phi_2)\left(\frac{x_1}{\varepsilon}\right)\right|^2dx-2\int_{\Omega^{c,2}}\varepsilon\partial_{x_1}\overline{\phi_{\varepsilon,2}}(x)\,(\partial_y\phi_2)\left(\frac{x_1}{\varepsilon}\right)dx,\quad\forall\varepsilon.\end{aligned} \tag{4.7}$$

As far as the first integral in the right-hand side of (4.7) is concerned, Proposition 4.1 asserts that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^c}|\varepsilon\nabla\phi_\varepsilon(x)|^2\,dx=L(l_2-l_1-2)\left(\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}\right) \tag{4.8}$$

As far as the second integral in the right-hand side of (4.7) is concerned, the fact that $\partial_y\phi_2$ is a 1-periodic function belonging to $L^\infty(]0,1[)$, (4.5), and (4.6) imply that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega^{c,2}}\left|(\partial_y\phi_2)\left(\frac{x_1}{\varepsilon}\right)\right|^2dx=\int_{\Omega^{c,2}\times]0,1[}|\partial_y\phi_2(y)|^2\,dxdy=|\Omega^{c,2}|\int_0^1|\partial_y\phi_2(y)|^2\,dy\\&=L(l_2-l_1-2)\left(\frac{1}{\zeta_1-\zeta_4+1}+\frac{1}{\zeta_3-\zeta_2}\right).\end{aligned} \tag{4.9}$$

As far as the last integral in the right-hand side of (4.7) is concerned, Proposition 3.1, the fact that $\partial_y \phi_2$ is a 1-periodic function belonging to $L^\infty(]0,1[)$, (4.5), and (4.6) imply that

$$\lim_{\varepsilon\to 0} \int_{\Omega^{c,2}} \varepsilon \partial_{x_1} \overline{\phi_{\varepsilon,2}}(x) \, (\partial_y \phi_2) \left(\frac{x_1}{\varepsilon}\right) dx = \int_{\Omega^{c,2}\times ]0,1[} |\partial_y \phi_2(y)|^2 \, dxdy$$
$$= |\Omega^{c,2}| \int_0^1 |\partial_y \phi_2(y)|^2 \, dy = L(l_2 - l_1 - 2) \left( \frac{1}{\zeta_1 - \zeta_4 + 1} + \frac{1}{\zeta_3 - \zeta_2} \right). \tag{4.10}$$

Passing to the limit, as $\varepsilon$ vanishes, in (4.7) and using (4.9), (4.8), and (4.10) provide (1.5).


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## References


[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math Anal. **23** (1992), 6, pp. 1482-1518.

[2] D. Cioranescu, A. Damlamian, and G. Griso, The Periodic Unfolding Method: Theory and Applications to Partial Differential Problems. Series in Contemporary Mathematics **03**, Springer. 2019.

[3] L. Baffico and C. Conca, *Homogenization of a transmission problem in solid mechanics*, J. Math. Anal. Appl., **233** (1999), pp. 659-680.

[4] R. Brizzi, J.P. Chalot, Homogénéisation de frontière. Ph.D Thesis, University of Nice (1978).

[5] R. Brizzi and J.P. Chalot, *Boundary homogenization and Neumann boundary value problem,* Ricerche Mat. **46** (1997), 2, 341-387.

[6] P. Donato and K. Pettersson *Homogenization of a 2D two-component domain with an oscillating thick interface*, Mathematics and Mechanics of Complex Systems **10** (2022), 2, 103-154

[7] A. Gaudiello, *Homogenization of an elliptic transmission problem*, Adv. Math. Sci. Appl. **5** (1995), 2, 639-657.

[8] A. Gaudiello, O. Guibé, and F. Murat, *Homogenization of the brush problem with a source term in $L^1$*, Arch. Rational Mech. Anal. **225** (2017), pp. 1-64.

[9] A. Gaudiello and M. Lenczner, *A two-dimensional electrostatic model of interdigitated comb drive in longitudinal mode*, SIAM J. Appl. Math. **80** (2020), 2, pp. 792-813.

[10] M. Lenczner, *Homogénéisation d'un circuit électrique.* Comptes Rendus de l'Académie des Sciences-Series IIB-Mechanics-Physics-Chemistry-Astronomy, **324** (1997), pp. 537-542.

[11] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), pp. 608-623.