An Asymptotic Electrostatic Model of an Array of Micro Mirrors.

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Abstract

This paper reports a multiscale electrostatic model of a two-dimensional Micro-Mirror Array. It is applicable to very large arrays with several zone of electrical actuation. The model is made with periodic solutions and four kinds of boundary layer effects at outer boundaries, interfaces between different actuation zones and also to outer and inner edges. This work is done in the context of the development of a symbolic calculation software based on an extension-combination principle, so that the model derivations are constructed in such a way as to follow a same algorithm.

1 Introduction

Micro-Mirror Arrays, abbreviated as MMAs, are devices related to Micro-Optical-Electromechanical Systems (MOEMS) family with mirrors in their components. The size of the mirror is very small, millimeter-sized, micro-sized, or smaller, with the principal goal being steering or monitoring light phase or amplitude. According to the statistics in 2018 of authors in [33], there are about 277 MMA designs from 49 companies and 23 academic research groups. They are widely used in various fields such as optics, telecommunications, astronomy, biology, etc.

MMAs can be categorized according to the type of their actuators into four groups: electrostatic, electrothermal, piezoeletric, and magnetic. Another aspect of the classification is based on the kind of mirror surface. Two groups are distinguished, the discrete and the continuous one. In the former, the mirrors are disconnected from that of the adjacent cells, so their movements are independent. In the latter, the mirrors in each cell are continuously linked to each other. In other words, there is only one mirror in the structure of the devices in this group. The number of mirrored elements in the array depends on the function of the device, can vary from one cell to thousands and can be placed in a one or two dimensional array. These arrays can be operated by one of the command algorithms: direct addressing, line addressing, or the line-column addressing, see more in [11, 12, 13].

The MMA for which the model of this paper has been developped is with electrostatically actuated tilting mono-crystalline silicon micro-mirrors called MIRA, see its top view in Figure 1. It is actuated according the line-column addressing scheme. It has been designed with stringent requirements such as a mirror size of $200 \times 100 \mu m^2$, a title angle of more than 20^o, a filling factor of more than 80%, a contrast ratio of more than 1000, a wavelength bandwidth from visible to IR, an actuation voltage lower than $100V$ and an operating temperature ranging from room temperature to less than $100K$. For more details see [13, 37].

The direct simulation of physical phenomena in such a micromirror array is very computationally expensive due to the large number of degrees of freedom, its enormous size and the existence of several scales. The approach adopted in this paper to overcome this difficulty is to use an approximate model obtained by deploying asymptotic methods for periodic problems, see introductions to the field through historical references as [9, 34, 18] among others. Precisely, we use the unfolding method [24, 17, 15, 16, 4, 14] also called two-scale convergence since it generalizes the two scale convergence introduced in [27] and developed in [1]. A preliminary work was done for a one-dimensional array in [29]. Here, we report results for two-dimensional arrays governed by the equations of electrostatics. Similar results for the coupling with the system of linear elasticity are available in the PhD thesis [35]. They are not reported here due to the paper length limitation, however their statement and derivation follow similar principles.

We assume that the array is divided into two zones where the actuation voltage is uniform. The electrostatic potential of the asymptotic model is periodic, with different periods, in each of these zones. Compared to the solution of a standard periodic homogenisation problem, here the periodic model solution corresponds to the periodic correctors only. This is due to the fact that each cell is grounded and a potential difference governs its behaviour. As a result, the electrostatic potential and its normal derivative are discontinuous at the interfaces between the uniform actuation zones. In addition, they do not satisfy the boundary condition at the lateral boundaries of the array. To get rid of these defects, boundary layer correctors are introduced at the interfaces and at the lateral boundaries. Besides, the corrections are formulated separately on each face of the interfaces and of the lateral boundaries, which led to the discontinuity of the sum of their contribution at the face junctions, namely at the edges. This is why, additional boundary layer correctors are also introduced at the edges.

Boundary layer problems in periodic homogenization problems have been much investigated, see [10, 2, 31, 21, 23, 32, 20, 19, 3, 26] to cite only few. In this work, our contribution is to outer edge and internal edge corrector models which have not been studied yet. In total, we derive five kinds of models with the following features: periodic solution, lateral (i.e. outer) boundary layer, interface boundary layer, internal edge boundary layer, and exterior edge boundary layer, see in Figure 2. For each kind, we provide only one model instance for one boundary, interface or edge, the other ones being obtained without difficulty. Due to the length of the paper, the results of our numerical implementation of the models are not presented here. The interested reader can find them in the PhD thesis [35] while older ones for a one-dimensional array were reported in [30] in an optimization context.

An other point is that this work is carried out with the perspective of developing symbolic computation algorithms for model building in continuation of the works [36, 5, 28, 8, 7]. Thus, a particular attention is paid to the algorithmic structure of the model proofs and here we have endeavored to write them all following the framework of a single algorithm. Variations from this reference algorithm can be expressed by the extension-combination method. Here, we do not expose this aspect but it has been the subject of our work [7] achieved for simpler models with for the same algorithm. Notice that a complete theory of extension-combination is available in [6] while an extended version is submitted for publication.

It can be observed that in the above mentioned algorithm, most of the operations are done on a very weak formulation instead on a weak formulation as it is usual. This leads to shortened proof lengths due to the absence of need of weak convergences of derivatives.

Another characteristic of our choice in designing symbolic computation algorithm is to adopt a compromise between imposing assumptions and doing more mathematical analysis. Thus our attention is more on developping calculations that can be algebraized than on fine mathematical analysis deployment. Precisely, in our algorithm, we assume a priori estimates, or equivalently weak convergences of subsequences, on the physical solutions. Thus in the following model derivations, we adopt the same assumptions which apply to the solution as well as to the boundary layer correctors. In addition, the boundary layer correctors and their gradients are assumed to converge exponentially to zero at infinity. This might be proven as e.g. [2, 34]. Another characteristic of this work, which shows the interest of having models automatically derived, is the choice to deal with a real problem whose complexity exceeds by far the one usually treated in academic works. While the complexity of the MIRA cells is not so high, nevertheless its handling in the framework of asymptotic methods quickly leads to having to manage extremely heavy notations, which is quickly prohibitive for a manual treatment. In this sense, this work provides a very interesting (indeed precious) family of models to guide the development of a rather general symbolic computation tool.

Still in the perspective of developing systematic proofs, despite the fact that the imposed electric voltage is assumed to be piecewise constant in the array MIRA, it is treated with the minimal conditions necessary for the validity of the models. In particular, it can have smooth variations inside some zones and abrupt changes at their interface. In the paper we do not discuss further the other possible cases. The electrostatic potential of the two-scale model in a cell is then solution to a periodic problem depending on the local actuation voltage. The latter varies continuously in each zone and is discontinuous at their interface. This yields additional boundary layer effects that could find applications for other devices.

As the model proofs all follow the same pattern, it would be unnecessarily long to write them all in detail. It has been chosen to provide all details for the first models, then to focus on the special features for the next ones.

2 Problem Statement

We start by providing more details on the operation of a MMA cell. Then, the electrostatic equations are recalled in their strong, weak and very weak forms. Since the principle of asymptotic methods deals with small parameters, it is necessary to distinguish the small physical dimensions of the small parameters to be taken into account for the asymptotic analysis. This is why the whole system is scaled to a length of the order of unity. Finally, the algorithm followed by the model constructions is detailed. It uses operators related to scale transformations which can be specific to certain problems. Here those used for the construction of the periodic model are recalled to illustrate the algorithm.

2.1 Structure of a Cell of the MMA

The structure of one cell of MIRA is illustrated in Figure 3. It is composed of two components: the mirror part and the electrode part. The mirror part is made with a micromirror supported by two flexible beams. The latter are attached to a frame enabling a displacement of the mirror when a voltage is applied. A stopper beam is situated under the frame to guarantee that a tilt angle satisfies a given value after actuation. Two landing beams are under the tilting edge of the micromirror to avoid the generation of a short-circuit between the mirror and the electrode throughout the actuation. The electrode part includes the electrode base which is electrically grounded; landing pads are where the landing beams contact; two pillars separate the mirror and electrode parts defining an electrostatic gap. The electrostatic force applied to the mirror results from its difference of potential with the electrode base.

2.2 Geometry and Mathematical Equations

We begin by describing the geometry of the MIRA array. It occupies the region Ω decomposed into Ω^{mec} and Ω^{vac} where the mechanical part and the vacuum surrounding it are located. Its width, length and thickness are respectively L_1, L_2 and L_3, see Figure 4. It includes $n_1 \times n_2$ cells Ω_c of sizes l_1, l_2, and l_3.

Thus $\Omega = \cup_c \Omega_c$, where c is a multi-index belonging to $\mathcal{I}_{mul} = \{c = (c_1, c_2),\ c_1 \in 1, ..., n_1$ and $c_2 \in 1, ..., n_2\}$. Each cell Ω_c, includes the mechanical part Ω_c^{mec} and the vacuum Ω_c^{vac}, see Figure 5. The mechanical structure consists of two parts, the mirror $\Omega_{mir,c}^{mec}$ and the electrode $\Omega_{ele,c}^{mec}$ so that $\Omega_c^{mec} = \Omega_{mir,c}^{mec} \cup \Omega_{ele,c}^{mec}$. We also use the decomposition of the domains Ω^{mec} and Ω^{vac} of the array as the unions $\Omega^{mec} = \cup_c \Omega_c^{mec}$ and $\Omega^{vac} = \cup_c \Omega_c^{vac}$, and the same for the domains consisting of all mirrors and electrodes $\Omega_{mir}^{mec} = \cup_c \Omega_{mir,c}^{mec}$ and $\Omega_{ele}^{mec} = \cup_c \Omega_{ele,c}^{mec}$.

The boundary of Ω^{mec} is the union $\Gamma_0^{mec} \cup \Gamma_1^{mec} \cup \Gamma_{lat}^{mec}$, where Γ_{lat}^{mec} is the boundary of Ω^{mec} intersecting with this of Ω, while Γ_0^{mec} and Γ_1^{mec} are the complementary parts of the boundaries of Ω_{ele}^{mec} and Ω_{mir}^{mec}. The lateral part Γ_{lat}^{mec} does not play any role for the electrostatic models, thus it is not discussed further. Moreover, $\Gamma_0^{mec} = \cup_c \Gamma_{0,c}^{mec}$ and $\Gamma_1^{mec} = \cup_c \Gamma_{1,c}^{mec}$ where $\Gamma_{0,c}^{mec}$ and $\Gamma_{1,c}^{mec}$ denote respectively the boundary of the electrode $\Omega_{ele,c}^{mec}$ and of the mirror $\Omega_{mir,c}^{mec}$ of the mechanical body in a cell Ω_c^{mec}. The boundary $\partial\Omega^{vac}$ of Ω^{vac} is the union of the internal boundary Γ_{int}^{vac} and the external boundary Γ_{ext}^{vac}, where Γ_{int}^{vac} is defined by $\Gamma_0^{mec} \cup \Gamma_1^{mec}$ and Γ_{ext}^{vac} is the union of the lateral boundary Γ_{lat}^{vac} and of the top boundary Γ_{top}^{vac} of the vacuum part, $\Gamma_{ext}^{vac} = \Gamma_{lat}^{vac} \cup \Gamma_{top}^{vac}$. For the sake

of simplicity but without losing generality, we consider that Ω is split into two zones Ω_1 and Ω_2 in which the imposed voltages noted as V_1 and V_2 are different. Hereafter, we add the subscripts $1, 2$ in geometrical notations to represent to which zones they belong, for example, Ω_1^{vac} and Ω_2^{vac} is a vacuum part of Ω_1 and Ω_2, $\Gamma_{1,int}^{vac}$ and $\Gamma_{2,int}^{vac}$ is the internal boundary of Ω_1^{vac} and Ω_2^{vac}, and note that all previous geometrical notations without the subscripts $1, 2$ now are understood as a union of two elements related to zones Ω_1 and Ω_2, e.g. $\Gamma_{int}^{vac} = \Gamma_{1,int}^{vac} \cup \Gamma_{2,int}^{vac}$.
The field of electric potential ϕ in the vacuum is governed by the equation of electrostatics, see [22],

$$\begin{cases} -\Delta\phi & = 0 \quad \text{in } \Omega^{vac} \\ \phi & = V \quad \text{on } \Gamma_{int}^{vac} \\ \nabla\phi \cdot \mathbf{n} & = 0 \quad \text{on } \Gamma_{ext}^{vac} \end{cases}, \tag{2.1}$$

where V is the imposed voltage taking two distinct constant values V_1 in Ω_1 and V_2 and Ω_2, and $\mathbf{n}$ is the outward unit normal vector. The continuity of the potential and the electrostatic field at the interface Γ_{interf}^{vac} of Ω_1^{vac} and Ω_2^{vac} are given as

$$\phi_{|\Omega_1^{vac}} = \phi_{|\Omega_2^{vac}} \text{ and } \nabla\phi_{|\Omega_1^{vac}} \cdot \mathbf{n}^1 = -\nabla\phi_{|\Omega_2^{vac}} \cdot \mathbf{n}^2,$$

where $\mathbf{n}^1$ and $\mathbf{n}^2$ are the outward unit normal vectors of Ω_1^{vac} and Ω_2^{vac} on Γ_{interf}^{vac}, $\mathbf{n}^1 = -\mathbf{n}^2$.

Let us introduce a Hilbert space $H^1_{\Gamma_{int}^{vac},0}(\Omega^{vac}) \doteq \{v \in H^1(\Omega^{vac}),\, v = 0 \text{ in } \Gamma_{int}^{vac}\}$ endowed with the norm

$$\|v\|_{H^1_{\Gamma_{int}^{vac},0}(\Omega^{vac})} = \|\nabla v\|_{L^2(\Omega^{vac})},$$

for all $v \in H^1_{\Gamma_{int}^{vac},0}(\Omega^{vac})$.

Then a variational problem of (2.1) is to find $\phi \in H^1_{\Gamma_{int}^{vac},V}(\Omega^{vac}) \doteq \{\phi \in H^1(\Omega^{vac}),\, \phi = V \text{ in } \Gamma_{int}^{vac}\}$ such that

$$\int_{\Omega^{vac}} \nabla\phi\nabla v \,\mathrm{d}x = 0,$$

for all $v \in H^1_{\Gamma_{int}^{vac},0}(\Omega^{vac})$. Assuming more regularity of the test function and applying Green's formula, we have a very weak formulation of the problem,

$$\int_{\Omega^{vac}} \phi\Delta_x v \,\mathrm{d}x = \int_{\Gamma_{int}^{vac}} V\nabla_x v \cdot \mathbf{n} \,\mathrm{d}s\,(x) + \int_{\Gamma_{ext}^{vac}} \phi\nabla_x v \cdot \mathbf{n} \,\mathrm{d}s\,(x)\,, \tag{2.2}$$

for all v in $H^2_{\Gamma_{int}^{vac},0}(\Omega^{vac}) = \{v \in H^2(\Omega^{vac}), v = 0 \text{ on } \Gamma_{int}^{vac}\}$.

2.3 Global Scalings

The asymptotic analysis is conducted for the small parameter ε specified below but which is of the order of the l_i/L_i assumed to remain in the same order of magnitude. All the geometrical notations, normal vectors, variables, functions, etc of the physical problem are written with the superscript ε, for example one writes Ω^ε, $\Gamma_{int}^{\varepsilon,vac}$, n^ε, x^ε, and ϕ^ε instead of Ω, Γ_{int}^{vac}, n, x, and ϕ. Then, all the geometrical data are scaled by the largest length L of the array, e.g. $\widehat{x^\varepsilon} = x^\varepsilon/L$ yielding the scaling of Ω^ε into $\widehat{\Omega^\varepsilon}$ and Ω_c^ε into $\widehat{\Omega_c^\varepsilon}$ with respective sizes $\widehat{L_i} = L_i/L$ and $\widehat{l_i} = l_i/L$ for $i = 1, 2, 3$. All the other geometrical notations are then decorated by a hat $\widehat{\cdot}$ to represent scaled domains and boundaries, e.g. $\widehat{\Omega^{\varepsilon,vac}}, \widehat{\Gamma_{int}^{\varepsilon,vac}}$ are scaled regions from $\Omega^{\varepsilon,vac}, \Gamma_{int}^{\varepsilon,vac}$. Moreover, the derivation variables are added as subscripts to operators such as Laplace Δ, divergence div. For instance, $\Delta_{\widehat{x^\varepsilon}}$, $\mathrm{div}_{\widehat{x^\varepsilon}}$ are the Laplace and divergence operators with respect to the variable $\widehat{x^\varepsilon}$.

Now, we define the small asymptotic parameter as $\varepsilon = \max\{\widehat{l_i}/\widehat{L_i} = 1/n_i\}$ over $i \in \{1,2,3\}$. We say that it tends to 0 with the meaning that the numbers n_1 and n_2 of cells tend to infinity. Another constraint on n_1 and n_2 is that the positions and sizes of Ω_1^ε and Ω_2^ε in the x_1 and x_2 directions remain fixed when $\varepsilon \to 0$. Finally, to simplify the formulations, we assume that $\widehat{l_i} = \widehat{L_3} = \varepsilon$ for all $i = 1,2,3$ so the volume of a scaled cell is $|\Omega_c^\varepsilon| = \prod_i \widehat{l_i} = \varepsilon^3$, and that $\widehat{L_1} = \widehat{L_2} = 1$ so the volume of the scaled array is $|\Omega^\varepsilon| = \prod_i \widehat{L_i} = \varepsilon$. This avoids unnecessary complications in the calculation writing without affecting the principle of the final models.

We now deal with the scaling for the electrostatic potential and the mechanical displacement. In the electrostatic model part, the space scale L is reused, we set $\widehat{V^\varepsilon} = V^\varepsilon/L$ and $\widehat{\phi^\varepsilon} = \phi^\varepsilon/L$. Plugging these new scaled fields into the equation (2.1), we obtain the following equations for the scaled potential $\widehat{\phi^\varepsilon}$,

$$\left\{\begin{array}{rcll} -\Delta_{\widehat{x^\varepsilon}}\widehat{\phi^\varepsilon} & = & 0 & \text{in } \widehat{\Omega^{\varepsilon,vac}} \\ \widehat{\phi^\varepsilon} & = & \widehat{V^\varepsilon} & \text{on } \widehat{\Gamma_{int}^{\varepsilon,vac}} \\ \nabla_{\widehat{x^\varepsilon}}\widehat{\phi^\varepsilon}\cdot\widehat{\mathbf{n}^\varepsilon} & = & 0 & \text{on } \widehat{\Gamma_{ext}^{\varepsilon,vac}}. \end{array}\right. \tag{2.3}$$

Remark 2.1 *For simplicity of notation, we hereafter remove the hat* $\widehat{\cdot}$ *from all the notations, for instance,* $\Omega^{\varepsilon,mec}$*,* ϕ^ε *replaces* $\widehat{\Omega^{\varepsilon,mec}}$*, resp.* $\widehat{\phi^\varepsilon}$*, and we employ the notation* Γ *referring to the boundary of a domain with name the domain name, for example,* $\Gamma^{\varepsilon,vac}$ *is the boundary of* $\Omega^{\varepsilon,vac}$*.*

2.4 Two-Scale Transform Operators for the Periodic Model

We recall the two-scale transform operator or unfolding operator in a domain as introduced in [24, 17, 15, 16, 4, 14]. This operator is used to build the periodic solution model. The definitions and properties of this section are adapted from [25]

Let us begin by introducing $\Omega^\sharp \subset \mathbb{R}^2$ such that $\Omega^\varepsilon = \Omega^\sharp \times]0,\varepsilon[$ with a partition $\{\Omega_c^\sharp\}_c$ where $\Omega_c^\sharp = [(c_1-1)\varepsilon, c_1\varepsilon[\, \times [(c_2-1)\varepsilon, c_2\varepsilon[$, $c = (c_1,c_2) \in \mathcal{I}_{mul}$, and $x^{\sharp,c}$ is the center of the cell $\Omega_c^\sharp$ defined as $x^{\sharp,c} = (c_1\varepsilon - \varepsilon/2, c_2\varepsilon - \varepsilon/2)$. It follows that $\Omega_c^\varepsilon = \Omega_c^\sharp \times]0,\varepsilon[$ and that $x^{\varepsilon,c} = (x^{\sharp,c}, \varepsilon/2)$ where $x^{\varepsilon,c}$ is the center of the cell Ω_c^ε.

We now represent the reference cell also called the unit periodicity cell Ω^1 residing at the position $]-1/2,1/2[^3$, see Figure 6. Its boundaries of the vacuum and mechanical parts are denoted by $\partial\Omega^{1,vac} = \Gamma_{int}^{1,vac} \cup \Gamma_{per}^{1,vac} \cup \Gamma_{top}^{1,vac}$ and $\partial\Omega^{1,mec} = \Gamma_0^{1,mec} \cup \Gamma_1^{1,mec} \cup \Gamma_{per}^{1,mec}$. Obviously, if $x^\varepsilon \in \Omega_c^\varepsilon$, $c \in \mathcal{I}_{mul}$ then $(x^\varepsilon - x^{\varepsilon,c})/\varepsilon \in \Omega^1$, and $\Omega^\varepsilon = \cup_c \varepsilon((c_1-1/2, c_2-1/2, 1/2)+\Omega^1)$. Similarly, we also use Γ^1 representing any surface in $\overline{\Omega^1}$ and the associated periodic surface $\Gamma^\varepsilon = \cup_{c\in\mathcal{I}_{mul}}\varepsilon((c_1-1/2, c_2-1/2, 1/2)+\Gamma^1)$ in $\overline{\Omega^\varepsilon}$.

In the following definitions and properties the pair (X^ε, X^1) stands both for $(\Omega^\varepsilon,\Omega^1)$ and for $(\Gamma^\varepsilon,\Gamma^1)$. The same notation for operators defined on functions with variables in domains or their boundary because they are defined by the same formulae.

Definition 2.2 *The two-scale transform operator* T^ε *operating on functions with variable in* X^ε *is defined by*

$$T^\varepsilon(\varphi)(x^\sharp, x^1) = \sum_c \chi_{\Omega_c^\sharp}(x^\sharp)\varphi(x^{\varepsilon,c} + \varepsilon x^1),$$

for a.e. $x^\sharp \in \Omega^\sharp$ *and* $x^1 \in X^1$*, where* χ_A *is the characteristic function over a set* A*.*

Proposition 2.3 *The two-scale transform operator has the following properties.*

1. *T^ε is a linear and continuous operator from $L^2(X^\varepsilon)$ to $L^2(\Omega^\sharp \times X^1)$.*

2. *For $\varphi, \psi \in L^2(X^\varepsilon)$, $T^\varepsilon(\varphi\psi) = T^\varepsilon(\varphi)T^\varepsilon(\psi)$.*

3. *For $\varphi \in L^1(\Omega^\varepsilon)$*
$$\int_{\Omega^\varepsilon} \varphi \, dx^\varepsilon = \varepsilon \int_{\Omega^\sharp \times \Omega^1} T^\varepsilon(\varphi) \, \mathrm{d}x^\sharp \mathrm{d}x^1.$$

4. *For $\varphi \in L^1(\Gamma^\varepsilon)$*
$$\int_{\Gamma^\varepsilon} \varphi \, dx^\varepsilon = \int_{\Omega^\sharp \times \Gamma^1} T^\varepsilon(\varphi) \, \mathrm{d}x^\sharp \mathrm{d}s(x^1).$$

5. *For $\varphi \in L^2(\Omega^\varepsilon)$, $\|\varphi\|_{L^2(\Omega^\varepsilon)} = \sqrt{\varepsilon}\|T^\varepsilon(\varphi)\|_{L^2(\Omega^\sharp \times \Omega^1)}$.*

6. *For $\varphi \in L^1(\Gamma^\varepsilon)$, $\|\varphi\|_{L^2(\Gamma^\varepsilon)} = \|T^\varepsilon(\varphi)\|_{L^2(\Omega^\sharp \times \Gamma^1)}$.*

Remark 2.4 *We introduce the norm $||| \cdot |||= \varepsilon^{-1/2}\| \cdot \|$ to include the factor $\varepsilon^{1/2}$ of the height of a thin domain.*

Let us introduce the operator

$$T^{\varepsilon*}(\psi)\,(x^\varepsilon) = \frac{1}{\varepsilon^2} \sum_c \int_{\Omega_c^\sharp} \psi\left(x^\sharp, \frac{x^\varepsilon - x^{\varepsilon,c}}{\varepsilon}\right) \mathrm{d}x^\sharp \chi_{\Omega_c^\varepsilon}(x^\varepsilon) \text{ for any } x^\varepsilon \in \Omega^\varepsilon \tag{2.4}$$

operating on functions ψ with variables in $\Omega^\sharp \times X^1$ and returning a function with variables in X^ε.

Property 2.5 *The operator $T^{\varepsilon*}$ is the adjoint of T^ε in the sense*

$$\frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \varphi T^{\varepsilon*}(\psi) \, \mathrm{d}x^\varepsilon = \int_{\Omega^\sharp \times \Omega^1} T^\varepsilon(\varphi)\psi \, \mathrm{d}x^\sharp \mathrm{d}x^1,$$

for all $\psi \in L^2(\Omega^\sharp \times \Omega^1)$ and $\varphi \in L^2(\Omega^\varepsilon)$, and in the sense

$$\int_{\Gamma^\varepsilon} \varphi T^{\varepsilon*}(\psi) \, \mathrm{ds}(x^\varepsilon) = \int_{\Omega^\sharp \times \Gamma^1} T^\varepsilon(\varphi)\psi \, \mathrm{d}x^\sharp \mathrm{ds}(x^1),$$

for all $\psi \in L^2(\Omega^\sharp \times \Gamma^1)$ and $\varphi \in L^2(\Gamma^\varepsilon)$.

We observe that $T^{\varepsilon*}(\psi)$ is not regular, thus we introduce a smooth approximation B^ε.

Definition 2.6 *The operator B^ε is defined on functions ψ with variables in $\Omega^\sharp \times X^1$ as*

$$B^\varepsilon(\psi)(x^\varepsilon) = \psi\left(P(x^\varepsilon), \frac{x^\varepsilon}{\varepsilon} - \frac{1}{2}\right),$$

where $P(x^\varepsilon) = (x_1^\varepsilon, x_2^\varepsilon)$ and returns a function with variables in X^ε.

For derivable functions ψ, the derivation property of $B^\varepsilon\psi$ reads as

$$\frac{\partial B^\varepsilon \psi}{\partial x_i^\varepsilon} = B^\varepsilon \left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial \psi}{\partial x_i^\sharp} + \frac{1}{\varepsilon}\frac{\partial \psi}{\partial x_i^1}\right) \tag{2.5}$$

for all $i \in \mathcal{I} = \{1, 2, 3\}$, $\mathcal{I}^\sharp = \{1, 2\}$.

In the following, a function $x^1 \to \psi(x^1)$ is said to be Ω^1-periodic in the directions x_1^1 and x_2^2 if it is defined in $\mathbb{R}^2\times] - \frac{1}{2}, \frac{1}{2}[$ and such that $\psi(x_1^1 + k_1, x_2^1 + k_2, x_3^1) = \psi(x_1^1, x_2^1, x_3^1)$ for all $k_1, k_2 \in \mathbb{Z}$.

Proposition 2.7 *For all ψ in $C^1(\Omega^\sharp \times X^1)$ and Ω^1-periodic in the directions x_1^1 and x_2^1,*

$$T^{\varepsilon *}(\psi)(x^\varepsilon) = B^\varepsilon(\psi)(x^\varepsilon) + O(\varepsilon) \text{ for all } x^\varepsilon \in X^\varepsilon,$$

where $O(\varepsilon)$ is the Landau notation for a sequence bounded by ε up to a multiplicative constant.

Remark 2.8 *In the following, C represent a constant that may be different from place to place.*

Proposition 2.9 *Let φ^ε be a sequence in $L^2(\Omega^\varepsilon)$ that satisfies*

$$||| \varphi^\varepsilon |||_{L^2(\Omega^\varepsilon)} \leq C \quad \text{and} \quad \varepsilon ||| \nabla_{x^\varepsilon} \varphi^\varepsilon |||_{L^2(\Omega^\varepsilon)} \leq C,$$

then, there exists a function φ^0 in $L^2(\Omega^\sharp; H^1(\Omega^1))$, Ω^1- periodic in the directions x_1^1, x_2^1 such that, up to the extraction of a subsequence, when $\varepsilon \to 0$

i. $T^\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi^0$ weakly in $L^2(\Omega^\sharp \times \Omega^1)$,

ii. $\varepsilon T^\varepsilon(\nabla_{x^\varepsilon} \varphi^\varepsilon) \rightharpoonup \nabla_{x^1} \varphi^0$ weakly in $L^2(\Omega^\sharp \times \Omega^1)$.

Remark 2.10 *One can show that $T^{\varepsilon *}$ is a left inverse of T^ε namely that $T^{\varepsilon *} T^\varepsilon = Id$. Using this remark and the fact that B^ε is an approximation of $T^{\varepsilon *}$, the principle of building a two-scale model is done by the following steps. We start from a physical field ϕ^ε solution of a problem $\mathcal{P}^\varepsilon(\phi^\varepsilon)$, and look for the problem $\mathcal{P}^0(\phi^0)$ verified by the limit ϕ^0 of $T^\varepsilon \phi^\varepsilon$ when $\varepsilon \to 0$. Then, the approximation to ϕ^ε is $B^\varepsilon \phi^0$. The same principle applies to all the subsequent models and will not be repeated.*

2.5 The Reference Algorithm for Model Proofs

Here we recall the symbolic computation algorithm that served as a reference proof for the construction of the models reported in [7] and based on the extension-combination method. It is this same algorithm that drives the construction of the five models of this paper. The operations described therein are high level, the implementation details not being explained because they strongly depend on the special case considered as well as how the way partial differential equations are represented in a symbolic computing environment, see the two approaches in [36] and in the PhD Thesis [35].

The starting point of the algorithm is a boundary value problem either in strong form or in weak form. It uses the definition of a two-scale transformation T^ε and its associated operators $T^{\varepsilon *}$ and B^ε. These operators and their properties depend on each model.

i) `Define`
  - `a two-scale transform (or unfolding) operator` T^ε,
  - `its adjoint` $T^{\varepsilon *}$,
  - `and a smooth approximation` B^ε `of` $T^{\varepsilon *}$.

ii) `Derive the very weak form of the boundary value problem with`
  - `solution` Ψ^ε,
  - `and test function` v.

iii) `Replace` v `by` $\varepsilon^k B^\varepsilon(w)$ `for some` $k \in \mathbb{Z} \backslash \{0\}$`, and apply the rule of the derivative of` $B^\varepsilon(w)$.

iv) Replace B^ε by an approximation in terms of $T^{\varepsilon *}$.

v) Apply the adjoint rule to replace the instances of $T^{\varepsilon *}$ by instances of T^ε on expressions of Ψ^ε.

vi) Assuming that $T^\varepsilon(\Psi^\varepsilon)$ is bounded for an appropriate L^2-norm when ε vanishes, an extracted subsequence weakly converges to a limit Ψ^0.

vii) Convert the very weak form satisfied by Ψ^0 into a strong form.

viii) Finally, the approximation of Ψ^ε is $B^\varepsilon \Psi^0$.

The rest of the paper is devoted to the construction of the main model whose solution is periodic in each subdomain where the applied voltage is constant and of its boundary layer correctors on the outer boundary, on the interfaces and on their edges. For each of these cases, the construction follows the above algorithm.

3 Periodic Model

We start with an assumption on the voltage source which expressed in terms of the weak limit of its two-scale transform.

Assumption 3.1 *$T^\varepsilon(V^\varepsilon)$ converges weakly to V^0 in $L^2(\Omega^\sharp \times \Gamma^{1,vac}_{int})$ which is continuous in $\Omega^\sharp$ except at the interfaces between some subdomains that are specified in the section of boundary layer models.*

Then, we make an assumption on ϕ^ε the solution of (2.3) that could be easily proved using a priori estimates techniques. However, we skip this step since we do not take it into account in the algorithm. The same principle is adopted for each of the following models.

Assumption 3.2 *$|||\phi^\varepsilon|||_{L^2(\Omega^{\varepsilon,vac})}$ and $\varepsilon|||\nabla_{x^\varepsilon}\phi^\varepsilon|||_{L^2(\Omega^{\varepsilon,vac})}$ are bounded uniformly with respect to ε.*

Proposition 3.3 *If ϕ^ε satisfies Assumptions 3.2 and 3.1, there exists $\phi^0 \in L^2(\Omega^\sharp, H^1(\Omega^{1,vac}))$ $\Omega^{1,vac}$-periodic in the directions x_1^1, x_2^1 such that $T^\varepsilon\phi^\varepsilon \rightharpoonup \phi^0$ weakly in $L^2(\Omega^\sharp \times \Omega^{1,vac})$. Moreover for a.e $x^\sharp \in \Omega^\sharp$, ϕ^0 is solution to*

$$\begin{cases} -\Delta_{x^1}\phi^0 = 0 & \text{in } \Omega^{1,vac} \\ \phi^0 = V^0 & \text{on } \Gamma^{1,vac}_{int} \\ \nabla_{x^1}\phi^0 \cdot \mathbf{n}^1 = 0 & \text{on } \Gamma^{1,vac}_{top} \\ \nabla_{x^1}\phi^0 \cdot \mathbf{n}^1 & \text{is } \Gamma^{1,vac}_{per}\text{-antiperiodic} \\ \phi^0 & \text{is } \Gamma^{1,vac}_{per}\text{-periodic.} \end{cases}$$

Proof. Thanks to Proposition 2.9 and Assumption 3.2, we obtain the existence and the periodicity of ϕ^0. The proof is completed by showing that ϕ^0 satisfies the above equations.

Let us take w sufficiently regular in $\Omega^\sharp \times \Omega^{1,vac}$ such that $w = 0$ on $\Gamma^{1,vac}_{int}$ and $\nabla_{x^1} w \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{top} \cup \Gamma^{1,vac}_{per}$. Obviously, $B^\varepsilon w = 0$ on $\Gamma^{\varepsilon,vac}_{int}$ then we can replace v^ε in (2.2) by $\varepsilon B^\varepsilon w$,

$$\varepsilon \int_{\Omega^{\varepsilon,vac}} \phi^\varepsilon \Delta_{x^\varepsilon} B^\varepsilon w \, \mathrm{d}x^\varepsilon = \varepsilon \int_{\Gamma^{\varepsilon,vac}_{int}} V^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon w \cdot \mathbf{n}^\varepsilon \, \mathrm{d}s\,(x^\varepsilon) + \varepsilon \int_{\Gamma^{\varepsilon,vac}_{ext}} \phi^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon w \cdot \mathbf{n}^\varepsilon \, \mathrm{d}s\,(x^\varepsilon). \quad (3.1)$$

From the property (2.5) of the derivative of B^ε,

$$\frac{\partial}{\partial x_i^\varepsilon}\frac{\partial B^\varepsilon w}{\partial x_i^\varepsilon} = B^\varepsilon\left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^\sharp} + \chi_{\mathcal{I}^\sharp}(i)\frac{2}{\varepsilon}\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^1} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial x_i^1}\frac{\partial w}{\partial x_i^1}\right)$$

for all $i \in \mathcal{I} = \{1,2,3\}$, $\mathcal{I}^\sharp = \{1,2\}$.
By a calculation, the left-hand side ($l.h.s$) of (3.1) becomes

$$\begin{aligned} l.h.s &= \varepsilon\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^\sharp} + \frac{2}{\varepsilon}\sum_{i=1}^{2}\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^1} + \frac{1}{\varepsilon^2}\Delta_{x^1} w\right)\mathrm{d}x^\varepsilon \\ &= \frac{1}{\varepsilon}\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\Delta_{x^1} w\right)\mathrm{d}x^\varepsilon + O(\varepsilon), \end{aligned} \tag{3.2}$$

where

$$O(\varepsilon) = \varepsilon\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^\sharp}\right)\mathrm{d}x^\varepsilon + 2\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i^\sharp}\frac{\partial w}{\partial x_i^1}\right)\mathrm{d}x^\varepsilon.$$

Similarly, the right-hand side ($r.h.s$) of (3.1) becomes

$$\begin{aligned} r.h.s &= \varepsilon\int_{\Gamma_{int}^{\varepsilon,vac}} V^\varepsilon\left[\sum_{i=1}^{2} B^\varepsilon\left(\frac{\partial w}{\partial x_i^\sharp}\right) n_i^\varepsilon + \frac{1}{\varepsilon}B^\varepsilon\left(\nabla_{x^1} w\right)\cdot\mathbf{n}^\varepsilon\right]\mathrm{d}s\left(x^\varepsilon\right) \\ &\quad + \varepsilon\int_{\Gamma_{ext}^{\varepsilon,vac}}\phi^\varepsilon\left[\sum_{i=1}^{2} B^\varepsilon\left(\frac{\partial w}{\partial x_i^\sharp}\right) n_i^\varepsilon + \frac{1}{\varepsilon}B^\varepsilon\left(\nabla_{x^1} w\right)\cdot\mathbf{n}^\varepsilon\right]\mathrm{d}s\left(x^\varepsilon\right). \end{aligned}$$

It is clear from $\nabla_{x^1} w\cdot\mathbf{n}^1 = 0$ on $\Gamma_{top}^{1,vac}\cup\Gamma_{per}^{1,vac}$ that $B^\varepsilon\left(\nabla_{x^1} w\right)\cdot\mathbf{n}^\varepsilon = 0$ on $\Gamma_{ext}^{\varepsilon,vac} = \Gamma_{top}^{\varepsilon,vac}\cup\Gamma_{lat}^{\varepsilon,vac}$, then

$$r.h.s = \int_{\Gamma_{int}^{\varepsilon,vac}} V^\varepsilon B^\varepsilon\left(\nabla_{x^1} w\cdot\mathbf{n}^1\right)\mathrm{d}s\left(x^\varepsilon\right) + O(\varepsilon), \tag{3.3}$$

where

$$O(\varepsilon) = \varepsilon\sum_{i=1}^{2}\int_{\partial\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\frac{\partial w}{\partial x_i^\sharp}\right) n_i^\varepsilon\,\mathrm{d}s(x^\varepsilon).$$

Combining with (3.2) and (3.3), we can assert that

$$\frac{1}{\varepsilon}\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon B^\varepsilon\left(\Delta_{x^1} w\right)\mathrm{d}x^\varepsilon = \int_{\Gamma_{int}^{\varepsilon,vac}} V^\varepsilon B^\varepsilon\left(\nabla_{x^1} w\cdot\mathbf{n}^1\right)\mathrm{d}s\left(x^\varepsilon\right) + O(\varepsilon).$$

Approximating B^ε by $T^{\varepsilon*}$ from Proposition 2.7 it follows that

$$\frac{1}{\varepsilon}\int_{\Omega^{\varepsilon,vac}}\phi^\varepsilon T^{\varepsilon*}\left(\Delta_{x^1} w\right)\mathrm{d}x^\varepsilon = \int_{\Gamma_{int}^{\varepsilon,vac}} V^\varepsilon T^{\varepsilon*}\left(\nabla_{x^1} w\cdot\mathbf{n}^1\right)\mathrm{d}s\left(x^\varepsilon\right) + O(\varepsilon).$$

The definition of $T^{\varepsilon*}$ yields

$$\int_{\Omega^\sharp\times\Omega^{1,vac}} T^\varepsilon(\phi^\varepsilon)\Delta_{x^1} w\,\mathrm{d}x^\sharp\mathrm{d}x^1 = \int_{\Omega^\sharp\times\Gamma_{int}^{1,vac}} T^\varepsilon(V^\varepsilon)\nabla_{x^1} w\cdot\mathbf{n}^1\,\mathrm{d}x^\sharp\mathrm{d}s\left(x^1\right) + O(\varepsilon).$$

Passing ε to 0 with Proposition 2.9 we get

$$\int_{\Omega^\sharp\times\Omega^{1,vac}} \phi^0 \Delta_{x^1} w \,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp\times\Gamma^{1,vac}_{int}} V^0 \nabla_{x^1} w \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s\left(x^1\right).$$

Applying Green's formula twice, therefore assuming sufficiently regularity of ϕ^0, combining with conditions satisfied by w and decomposing $\partial\Omega^{1,vac} = \Gamma^{1,vac}_{int} \cup \Gamma^{1,vac}_{per} \cup \Gamma^{1,vac}_{top}$, we obtain

$$\begin{aligned}
&\int_{\Omega^\sharp\times\Omega^{1,vac}} \Delta_{x^1}\phi^0 w \,\mathrm{d}x^\sharp \mathrm{d}x^1 - \int_{\Omega^\sharp\times(\Gamma^{1,vac}_{per}\cup\Gamma^{1,vac}_{top})} \nabla_{x^1}\phi^0 \cdot \mathbf{n}^1 w \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) \\
&+ \int_{\Omega^\sharp\times\Gamma^{1,vac}_{int}} \phi^0 \nabla_{x^1} w \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) = \int_{\Omega^\sharp\times\Gamma^{1,vac}_{int}} V^0 \nabla_{x^1} w \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s\left(x^1\right).
\end{aligned}$$

Choosing w such that $w = 0$ on $\Gamma^{1,vac}_{per} \cup \Gamma^{1,vac}_{top}$ and $\nabla_{x^1} w \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{int}$ yields

$$\Delta_{x^1}\phi^0 = 0 \text{ in } \Omega^{1,vac}.$$

Next, choosing w such that $w = 0$ on $\Gamma^{1,vac}_{per} \cup \Gamma^{1,vac}_{top}$ yields

$$\phi^0 = V^0 \text{ on } \Gamma^{1,vac}_{int}.$$

And then, we choose $w = 0$ on $\Gamma^{1,vac}_{per}$ to find

$$\nabla_{x^1}\phi^0 \cdot \mathbf{n}^1 = 0 \text{ on } \Gamma^{1,vac}_{top}.$$

Finally, with the remaining term we conclude that

$$\nabla_{x^1}\phi^0 \cdot \mathbf{n}^1 \text{ is } \Gamma^{1,vac}_{per} \text{ - antiperiodic.}$$

■

4 Lateral Boundary Layer Model

Due to the periodicity condition in the periodic model of Proposition 3.3, ϕ^0 does not satisfy the nominal boundary conditions on the outer lateral boundary. This leads to introduce the corrector $\phi^\varepsilon_{bl} = \phi^\varepsilon - B^\varepsilon(\phi^0)$ and the corresponding voltage source $v^\varepsilon_{bl} = V^\varepsilon - B^\varepsilon(V^0)$. We investigate the convergence of ϕ^ε_{bl} at the first lateral boundary. The convergence on the other boundaries can be derived in the same way.

4.1 Geometry Notations

Let $\Omega^{\varepsilon,\alpha}_{bl,1}$ be a subdomain of Ω^ε defined as $\Omega^{\varepsilon,\alpha}_{bl,1} = \cup_{c\in\mathcal{I}_{bl,1}}\Omega^\varepsilon_c$ where $\mathcal{I}_{bl,1} := \{c = (c_1, c_2) : c_1 \in \overline{1, n_1}$ and $c_2 \in \overline{1,\alpha}\}$, with $\alpha \in \mathbb{N}^*$ such that $\alpha\varepsilon < L^2_1$, and where L^2_1 is a positive number, see Figure 7. All other notations of subdomains, boundaries and subboundaries, let say $X^{\varepsilon,\alpha,k}_{bl,1,\ell}$, are inherited from those defined for the periodic model through the rule $X^{\varepsilon,\alpha,k}_{bl,1,\ell} = X^{\varepsilon,k}_\ell \cap \overline{\Omega^{\varepsilon,\alpha}_{bl,1}}$. For instance, we shall use $\Omega^{\varepsilon,\alpha,vac}_{bl,1} = \Omega^{\varepsilon,vac} \cap \overline{\Omega^{\varepsilon,\alpha}_{bl,1}}$, $\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int} = \Gamma^{\varepsilon,vac}_{int} \cap \overline{\Omega^{\varepsilon,\alpha}_{bl,1}}$. The same principle is used for the physical domain of each model without explanation. However for each kind of domain and each model there are special cases which are detailed.

Here, there is an additional boundary $\Gamma_{bl,1,\alpha}^{\varepsilon,\alpha,vac} \cup \Gamma_{bl,1,\alpha}^{\varepsilon,\alpha,mec}$ at the end of the boundary layer, see Figure 7, so that $\Gamma_{bl,1,ext}^{\varepsilon,\alpha,vac} = \Gamma_{bl,1,\alpha}^{\varepsilon,\alpha,vac} \cup \Gamma_{bl,1,top}^{\varepsilon,\alpha,vac} \cup \Gamma_{bl,1,lat}^{\varepsilon,\alpha,vac}$.

We next denote the macroscopic domain by $\Omega_{bl,1}^{\sharp} = [0, L_1[$, with a partition $\left\{\Omega_{bl,1c_1}^{\sharp}\right\}_{c_1}$, $\Omega_{bl,1c_1}^{\sharp} = [(c_1 - 1)\varepsilon, c_1\varepsilon[$, $c_1 = 1, ..., n_1$ and denote $x^{\sharp,c_1} = c_1\varepsilon - \varepsilon/2$ as the center of $\Omega_{bl,1c_1}^{\sharp}$.

The finite microscopic domain $\Omega_{bl,1}^1$ is built by $\Omega_{bl,1}^1 = \cup_{\xi=0}^{\alpha-1}(\Omega^1 + (0, 1/2 + \xi, 1/2))$, see Figure 8. We underline that $\Omega_{bl,1}^1$ depends on α even if this is not explicitely written in its notation. The same remark holds true for each model and will not be repeated.

All other notations of subdomains, boundaries and subboundaries, let say $X_{bl,1,\ell}^{1,k}$, are inherited from those defined for the periodic model through the rule $X_{bl,1,\ell}^{1,k} = X_{\ell}^{1,k} \cap \overline{\Omega_{bl,1}^1}$ with some special cases. As shown in Figure 8 the subboundaries $\Gamma_{bl,1,per}^{1,vac}$ and $\Gamma_{bl,1,per}^{1,mech}$ correspond to the parts of $\Gamma_{per}^{1,vac}$ and $\Gamma_{per}^{1,mech}$ which normal vector is collinear to x_2^1. Moreover, the subboundary $\Gamma_{bl,1,\alpha}^{1,vac}$ is to the end of the boundary layer. It results that the boundary $\partial\Omega_{bl,1}^{1,vac}$ of $\Omega_{bl,1}^{1,vac}$ is $\Gamma_{bl,1,int}^{1,vac} \cup \Gamma_{bl,1,per}^{1,vac} \cup \Gamma_{bl,1,top}^{1,vac} \cup \Gamma_{bl,1,0}^{1,vac} \cup \Gamma_{bl,1,\alpha}^{1,vac}$ as Figure 8 shows.

The infinite microscopic domain $\Omega_{bl,1}^{\infty}$ is defined as $\Omega_{bl,1}^{\infty} = \lim_{\alpha\to\infty} \Omega_{bl,1}^1$. Its subdomains, boundary and subboundaries are deduced from those of $\Omega_{bl,1}^1$ by passing to the limit on α.

Remark 4.1 *We use the subscript $i = 1, 2, 3, 4$ for all geometrical notations and operators, the superscript i for all functions to indicate which lateral boundary models they belong to, according to the index in Figure 2. For instance, $\Omega_{bl,1}^{\varepsilon,\alpha}$ and $\Omega_{bl,2}^{\varepsilon,\alpha}$ are the first and the second physical domains, $T_{bl,1}^{\varepsilon}$ and $T_{bl,2}^{\varepsilon}$ are the first and the second boundary layer two-scale transform operators, ϕ_{bl}^1 and ϕ_{bl}^2 are the solutions of the first and the second lateral boundary models.*
When we say "for each α", this means "for all $\alpha \in \mathbb{N}^*$ such that $\alpha\varepsilon < L_1^2$".

Next, we introduce the two-scale transform and its properties for the first lateral model.

4.2 Boundary Layer Two-Scale Transform Operator

As in Section 2.4, Γ^1 is any surface in $\overline{\Omega^1}$ while here $\Gamma_{bl,1}^1 = \cup_{\xi=0}^{\alpha-1}(\Gamma^1 + (0, 1/2 + \xi, 1/2)) \subset \overline{\Omega_{bl,1}^1}$ and $\Gamma_{bl,1}^{\varepsilon,\alpha} = \cup_{c\in\mathcal{I}_{bl,1}} \varepsilon((c_1 - 1/2, c_2 - 1/2, 1/2) + \Gamma^1) \subset \overline{\Omega_{bl,1}^{\varepsilon,\alpha}}$. Then in this section the pair (X^ε, X^1) stands both for $(\Omega_{bl,1}^{\varepsilon,\alpha}, \Omega_{bl,1}^1)$ and for $(\Gamma_{bl,1}^{\varepsilon,\alpha}, \Gamma_{bl,1}^1)$ in the statements. For Section 5.2, we also define $\Gamma_{bl,1}^{\infty} = \lim_{\alpha\to\infty} \Gamma_{bl,1}^1$.

Definition 4.2 *The boundary layer two-scale transform operator $T_{bl,1}^{\varepsilon}$ operating on functions φ with variable in X^ε is defined as*

$$T_{bl,1}^{\varepsilon}(\varphi)(x^\sharp, x^1) = \sum_{c_1} \chi_{\Omega_{bl,1c_1}^{\sharp}}(x^\sharp)\varphi(x^{\sharp,c_1} + \varepsilon x_1^1, \varepsilon x_2^1, \varepsilon x_3^1),$$

for a.e. $x^\sharp \in \Omega_{bl,1}^{\sharp}$, $x^1 \in X^1$.

We introduce the operator $T_{bl,1}^{\varepsilon*}$ defined as

$$T_{bl,1}^{\varepsilon*}(\psi)(x^\varepsilon) = \frac{1}{\varepsilon} \sum_{c_1} \int_{\Omega_{bl,1c_1}^{\sharp}} \psi\left(x^\sharp, \frac{x_1^\varepsilon}{\varepsilon} - (c_1 - \frac{1}{2}), \frac{x_2^\varepsilon}{\varepsilon}, \frac{x_3^\varepsilon}{\varepsilon}\right) \mathrm{d}x^\sharp \chi_{\Omega_{bl,1c_1}^{\sharp}}(x_1^\varepsilon)$$

for all function ψ on $\Omega_{bl,1}^{\sharp} \times X^1$ and for $x^\varepsilon \in X^\varepsilon$.

Property 4.3 *The operator* $T^{\varepsilon*}_{bl,1}$ *is the adjoint of* $T^{\varepsilon}_{bl,1}$ *in the sense*

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha}_{bl,1}} \varphi T^{\varepsilon*}_{bl,1}(\psi)\mathrm{d}x^{\varepsilon} = \int_{\Omega^{\sharp}_{bl,1}\times\Omega^{1,\alpha}_{bl,1}} T^{\varepsilon}_{bl,1}(\varphi)\psi \mathrm{d}x^{\sharp}\mathrm{d}x^{1},$$

for all $\psi \in L^2(\Omega^{\sharp}_{bl,1}\times\Omega^{1}_{bl,1})$ *and* $\varphi \in L^2(\Omega^{\varepsilon,\alpha}_{bl,1})$, *and also in the sense*

$$\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha}_{bl,1}} \varphi T^{\varepsilon*}_{bl,1}(\psi)\,\mathrm{d}s(x^{\varepsilon}) = \int_{\Omega^{\sharp}_{bl,1}\times\Gamma^{1,\alpha}_{bl,1}} T^{\varepsilon}_{bl,1}(\varphi)\psi \mathrm{d}x^{\sharp}\mathrm{d}s(x^{1}),$$

for all $\psi \in L^2(\Omega^{\sharp}_{bl,1}\times\Gamma^{1}_{bl,1}), \varphi \in L^2(\Gamma^{\varepsilon,\alpha}_{bl,1})$.

Definition 4.4 *The operator* $B^{\varepsilon}_{bl,1}$ *is defined as:*

$$B^{\varepsilon}_{bl,1}(\psi)(x^{\varepsilon}) = \psi\left(P(x^{\varepsilon}), \frac{x^{\varepsilon}_1}{\varepsilon} - \frac{1}{2}, \frac{x^{\varepsilon}_2}{\varepsilon}, \frac{x^{\varepsilon}_3}{\varepsilon}\right)$$

for any function ψ *with variables in* $\Omega^{\sharp}_{bl,1}\times X^1$, *where* $P(x^{\varepsilon}) = x^{\varepsilon}_1$.

Proposition 4.5 *For all* ψ *in* $C^1(\Omega^{\sharp}_{bl,1}\times X^1)$, $\Omega^{1}_{bl,1}$ *- periodic in the direction* x^1_1, *then*

$$T^{\varepsilon*}_{bl,1}(\psi)\,(x^{\varepsilon}) = B^{\varepsilon}_{bl,1}(\psi)(x^{\varepsilon}) + O(\varepsilon).$$

Proposition 4.6 *For each* α, *if a function* ψ *with variables in* $\Omega^{\sharp}_1\times\Omega^1$ *respectively in* $\Omega^{\sharp}_1\times\Gamma^1$, *is continuous w.r.t. its first variable and is* Ω^1 *- periodic in the direction* x^1_1, *then*

$$T^{\varepsilon}_{bl,1}(B^{\varepsilon}(\psi))(x^{\sharp},x^1) \to \widetilde{\psi}(x^{\sharp},x^1) \text{ for } (x^{\sharp},x^1) \text{ in } \Omega^{\sharp}_{bl,1}\times\Omega^{1}_{bl,1} \text{ respect. } \Omega^{\sharp}_{bl,1}\times\Gamma^{1}_{bl,1} \text{ when } \varepsilon\to 0,$$

where $\widetilde{\psi}(x^{\sharp},x^1) = \psi\left((x^{\sharp},0),(x^1_1, x^1_2-\frac{1}{2}, x^1_3-\frac{1}{2})\right)$.

Proof. By the definition of $T^{\varepsilon}_{bl,1}$ and B^{ε}, it follows that

$$\begin{aligned} T^{\varepsilon}_{bl,1}(B^{\varepsilon}(\psi))(x^{\sharp},x^1) &= \sum_{c_1}\chi_{\Omega^{\sharp}_{bl,1c_1}}(x^{\sharp})B^{\varepsilon}(\psi)(x^{\sharp,c_1}+\varepsilon x^1_1, \varepsilon x^1_2, \varepsilon x^1_3) \\ &= \sum_{c_1}\chi_{\Omega^{\sharp}_{bl,1c_1}}(x^{\sharp})\psi\left((x^{\sharp,c_1}+\varepsilon x^1_1, \varepsilon x^1_2), (x^1_1, x^1_2-\frac{1}{2}, x^1_3-\frac{1}{2})\right). \end{aligned}$$

Applying the continuity property,

$$\psi\left((x^{\sharp,c}+\varepsilon x^1_1, \varepsilon x^1_2), (x^1_1, x^1_2-\frac{1}{2}, x^1_3-\frac{1}{2})\right) = \psi\left((x^{\sharp},0), (x^1_1, x^1_2-\frac{1}{2}, x^1_3-\frac{1}{2})\right) + o(\varepsilon)$$

where $o(\varepsilon)\to 0$ when $\varepsilon\to 0$. Next, passing ε to 0, we have

$$T^{\varepsilon}_{bl,1}(B^{\varepsilon}(\psi)) \to \psi\left((x^{\sharp},0), ((x^1_1, x^1_2-\frac{1}{2}, x^1_3-\frac{1}{2}))\right)$$

as expected. ■

4.3 Derivation of a Lateral Boundary Model

In this section we assume without repeating it that the following assumptions are fulfilled. It involves the remaining voltage source $V_{bl}^\varepsilon = V^\varepsilon - B^\varepsilon V^0$ and on the corrector $\phi_{bl}^\varepsilon = \phi^\varepsilon - B^\varepsilon \phi^0$ and we recall that by construction $\Omega_{bl,1}^{1,vac}$ depends on α.

Assumption 4.7 *1. For each α, there exist $\phi_{bl}^{1,\alpha}$ in $L^2\left(\Omega_{bl,1}^\sharp, H^1(\Omega_{bl,1}^{1,vac})\right)$, $\Omega_{bl,1}^{1,vac}$-periodic in the direction x_1^1, and $V_{bl}^{1,\alpha}$ in $L^2\left(\Omega_{bl,1}^\sharp \times \Gamma_{bl,1,int}^{1,vac}\right)$ such that $T_{bl,1}^\varepsilon(\phi_{bl}^\varepsilon) \rightharpoonup \phi_{bl}^{1,\alpha}$ weakly in $L^2\left(\Omega_{bl,1}^\sharp \times \Omega_{bl,1}^{1,vac}\right)$ and $T_{bl,1}^\varepsilon(V_{bl}^\varepsilon) \rightharpoonup v_{bl}^{1,\alpha}$ weakly in $L^2\left(\Omega_{bl,1}^\sharp \times \Gamma_{bl,1,int}^{1,vac}\right)$ when $\varepsilon \to 0$.*

2. There exist ϕ_{bl}^1 in $L^2\left(\Omega_{bl,1}^\sharp, H^1(\Omega_{bl,1}^{\infty,vac})\right)$, $\Omega_{bl,1}^{\infty,vac}$-periodic in the direction x_1^1, and V_{bl}^1 in $L^2\left(\Omega_{bl,1}^\sharp \times \Gamma_{bl,1,int}^{\infty,vac}\right)$ such that $\phi_{bl}^{1,\alpha}\chi_{\Omega_{bl,1}^{1,vac}} \rightharpoonup \phi_{bl}^1$ weakly in $L^2\left(\Omega_{bl,1}^\sharp \times \Omega_{bl,1}^{\infty,vac}\right)$ and $V_{bl}^{1,\alpha}\chi_{\Omega_{bl,1}^{1,vac}} \rightharpoonup V_{bl}^1$ weakly in $L^2\left(\Omega_{bl,1}^\sharp \times \Gamma_{bl,1,int}^{\infty,vac}\right)$ when $\alpha \to +\infty$. Moreover ϕ_{bl}^1 and its gradient exponentially decreasing to 0 when $x_2^1 \to +\infty$.

Assumption 4.8 *The limits ϕ^0 and V^0 satisfy the conditions of Proposition 4.6.*

Proposition 4.9 *For each α, when $\varepsilon \to 0$,*

$$T_{bl,1}^\varepsilon \phi^\varepsilon \rightharpoonup \phi_{bl}^{1,\alpha} + \widetilde{\phi^0} \text{ weakly in } L^2\left(\Omega_{bl,1}^\sharp \times \Omega_{bl,1}^{1,vac}\right)$$

and

$$T_{bl,1}^\varepsilon V_{bl}^\varepsilon \rightharpoonup V_{bl}^{1,\alpha} + \widetilde{V^0} \text{ weakly in } L^2\left(\Omega_{bl,1}^\sharp \times \Gamma_{bl,1,int}^{1,vac}\right).$$

Proof. The proof is by passing ε to 0 in $T_{bl,1}^\varepsilon \phi^\varepsilon = T_{bl,1}^\varepsilon(B^\varepsilon \phi^0) + T_{bl,1}^\varepsilon(\phi_{bl}^\varepsilon)$, $T_{bl,1}^\varepsilon V^\varepsilon = T_{bl,1}^\varepsilon(B^\varepsilon V^0) + T_{bl,1}^\varepsilon(V_{bl}^\varepsilon)$ and combining with Proposition 4.6 and Assumptions 4.7 and 4.8. ■

Proposition 4.10 *The limit ϕ_{bl}^1 is solution to*

$$\begin{cases} -\Delta_{x^1}\phi_{bl}^1 = 0 & \text{in } \Omega_{bl,1}^{\infty,vac} \\ \phi_{bl}^1 = V_{bl}^1 & \text{on } \Gamma_{bl,1,int}^{\infty,vac} \\ \nabla_{x^1}\phi_{bl}^1 \cdot \mathbf{n}^1 = 0 & \text{on } \Gamma_{bl,1,top}^{\infty,vac} \\ \nabla_{x^1}\phi_{bl}^1 \cdot \mathbf{n}^1 & \text{is } \Gamma_{bl,1,per}^{\infty,vac} \text{ - antiperiodic} \\ \nabla_{x^1}\phi_{bl}^1 \cdot \mathbf{n}^1 = -\nabla\widetilde{\phi^0} \cdot \mathbf{n}^1 & \text{on } \Gamma_{bl,1,0}^{\infty,vac} \\ \phi_{bl}^1 & \text{is } \Gamma_{bl,1,per}^{\infty,vac} \text{ - periodic.} \end{cases}$$

Proof. The proof starts by finding the very weak form satisfied by the limit $\phi_{bl}^{1,\alpha}$ and then to pass to the limit on $\alpha \to \infty$ to find the very weak form satisfied by ϕ_{bl}^1. The derivation of the corresponding strong form follows. Let us begin with α fixed and replace v^ε in (2.2) by a smooth function v_{bl}^ε in $\Omega_{bl,1}^{\varepsilon,\alpha,vac}$ vanishing out of $\Omega_{bl,1}^{\varepsilon,\alpha,vac}$ and s.t. $v_{bl}^\varepsilon = 0$ on $\Gamma_{bl,1,int}^{\varepsilon,\alpha,vac}$. This yields

$$\int_{\Omega_{bl,1}^{\varepsilon,\alpha,vac}} \phi^\varepsilon \Delta_{x^\varepsilon} v_{bl}^\varepsilon \, dx^\varepsilon = \int_{\Gamma_{bl,1,int}^{\varepsilon,\alpha,vac}} V^\varepsilon \nabla_{x^\varepsilon} v_{bl}^\varepsilon \cdot \mathbf{n}^\varepsilon \, ds\,(x^\varepsilon) + \int_{\Gamma_{bl,1,ext}^{\varepsilon,\alpha,vac}} \phi^\varepsilon \nabla_{x^\varepsilon} v_{bl}^\varepsilon \cdot \mathbf{n}^\varepsilon \, ds\,(x^\varepsilon).$$

Taking a function w in $C^\infty(\Omega^\sharp_{bl,1}\times\overline{\Omega^{1,vac}_{bl,1}})$, $\Omega^{1,vac}_{bl,1}$ - periodic in the direction x^1_1 satisfying $w=0$ on $\Gamma^{1,vac}_{bl,1,int}\cup\Gamma^{1,vac}_{bl,1,\alpha}$ and $\nabla_{x^1}w\cdot\mathbf{n}^1=0$ on $\Gamma^{1,vac}_{bl,1,per}\cup\Gamma^{1,vac}_{bl,1,top}\cup\Gamma^{1,vac}_{bl,1,0}\cup\Gamma^{1,vac}_{bl,1,\alpha}$. We observe that $B^\varepsilon_{bl,1}(w)=0$ on $\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}$, then replacing v^ε_{bl} by $B^\varepsilon_{bl,1}(w)$, we get

$$\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon\Delta_{x^\varepsilon}B^\varepsilon_{bl,1}(w)\,\mathrm{d}x^\varepsilon=\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}}V^\varepsilon\nabla_{x^\varepsilon}B^\varepsilon_{bl,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon)+\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,ext}}\phi^\varepsilon\nabla_{x^\varepsilon}B^\varepsilon_{bl,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon)\,. \tag{4.1}$$

A direct computation shows that

$$\begin{aligned}\frac{\partial B^\varepsilon_{bl,1}w}{\partial x^\varepsilon_i}&=B^\varepsilon_{bl,1}\left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial w}{\partial x^\sharp}+\frac{1}{\varepsilon}\frac{\partial w}{\partial x^1_i}\right),\\ \frac{\partial}{\partial x^\varepsilon_i}\frac{\partial B^\varepsilon_{bl,1}w}{\partial x^\varepsilon_i}&=B^\varepsilon_{bl,1}\left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^\sharp}+\chi_{\mathcal{I}^\sharp}(i)\frac{2}{\varepsilon}\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^1_1}+\frac{1}{\varepsilon^2}\frac{\partial}{\partial x^1_i}\frac{\partial w}{\partial x^1_i}\right),\end{aligned}$$

for $i\in\mathcal{I}=\{1,2,3\}$ and with $\mathcal{I}^\sharp=\{1\}$. Then, the $l.h.s$ of (4.1) becomes

$$\begin{aligned}l.h.s&=\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon B^\varepsilon\left(\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^\sharp}+\frac{2}{\varepsilon}\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^1_1}+\frac{1}{\varepsilon^2}\Delta_{x^1}w\right)\mathrm{d}x^\varepsilon\\&=\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon B^\varepsilon\left(\Delta_{x^1}w\right)\mathrm{d}x^\varepsilon+O(\varepsilon),\end{aligned} \tag{4.2}$$

where

$$O(\varepsilon)=\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon B^\varepsilon(\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^\sharp})\,\mathrm{d}x^\varepsilon+\frac{2}{\varepsilon}\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon B^\varepsilon\left(\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^1_1}\right)\mathrm{d}x^\varepsilon.$$

The $r.h.s$ of (4.1) becomes

$$\begin{aligned}r.h.s=&\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}}V^\varepsilon\left[B^\varepsilon\left(\frac{\partial w}{\partial x^\sharp}\right)n^\varepsilon_1+\frac{1}{\varepsilon}B^\varepsilon\left(\nabla_{x^1}w\right)\cdot\mathbf{n}^\varepsilon\right]\mathrm{d}s\,(x^\varepsilon)\\&+\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,ext}}\phi^\varepsilon\left[B^\varepsilon\left(\frac{\partial w}{\partial x^\sharp}\right)n^\varepsilon_1+\frac{1}{\varepsilon}B^\varepsilon\left(\nabla_{x^1}w\right)\cdot\mathbf{n}^\varepsilon\right]\mathrm{d}s\,(x^\varepsilon)\,.\end{aligned}$$

Decomposing $\Gamma^{\varepsilon,\alpha,vac}_{bl,1,ext}$ into $\Gamma^{\varepsilon,\alpha,vac}_{bl,1,ext}=\Gamma^{\varepsilon,\alpha,vac}_{bl,1,\alpha}\cup\Gamma^{\varepsilon,\alpha,vac}_{bl,1,top}\cup\Gamma^{\varepsilon,\alpha,vac}_{bl,1,lat}$ and combining with $\nabla_{x^1}w\cdot\mathbf{n}^1=0$ on $\Gamma^{1,vac}_{bl,1,per}\cup\Gamma^{1,vac}_{bl,1,top}\cup\Gamma^{1,vac}_{bl,1,0}\cup\Gamma^{1,vac}_{bl,1,\alpha}$ yields $B^\varepsilon\left(\nabla_{x^1}w\right)\cdot\mathbf{n}^\varepsilon=0$ on $\Gamma^{\varepsilon,\alpha,vac}_{bl,1,ext}$, then

$$r.h.s=\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}}V^\varepsilon B^\varepsilon\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\mathrm{d}s\,(x^\varepsilon)+O(\varepsilon), \tag{4.3}$$

where

$$O(\varepsilon)=\int_{\partial\Omega^{\varepsilon,\alpha,vac}}\phi^\varepsilon B^\varepsilon\left(\frac{\partial w}{\partial x^\sharp}\right)n^\varepsilon_1\,\mathrm{d}s(x^\varepsilon).$$

From (4.2) and (4.3),

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon B^\varepsilon_{bl,1}\left(\Delta_{x^1}w\right)\mathrm{d}x^\varepsilon=\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}}V^\varepsilon B^\varepsilon_{bl,1}\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\mathrm{d}s\,(x^\varepsilon)+O(\varepsilon), \tag{4.4}$$

replacing $B^\varepsilon_{bl,1}$ by $T^{\varepsilon*}_{bl,1}$ using Proposition 4.5, Equality (4.4) becomes

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha,vac}_{bl,1}}\phi^\varepsilon T^{\varepsilon*}_{bl,1}\left(\Delta_{x^1}w\right)\mathrm{d}x^\varepsilon=\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{bl,1,int}}V^\varepsilon T^{\varepsilon*}_{bl,1}\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\mathrm{d}s\,(x^\varepsilon)+O(\varepsilon),$$

By the definition of $T^{\varepsilon *}_{bl,1}$, we have

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{1,\alpha,vac}_{bl,1}} T^\varepsilon_{bl,1}(\phi^\varepsilon)\Delta_{x^1} w \mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,\alpha,vac}_{bl,1,int}} T^\varepsilon_{bl,1}(V^\varepsilon)\nabla_{x^1} w\cdot \mathbf{n}^1\,\mathrm{d}x^\sharp \mathrm{d}s\left(x^1\right) + O(\varepsilon).$$

Passing ε to 0, combined with Proposition 4.9,

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{1,vac}_{bl,1}} (\phi^{1,\alpha}_{bl} + \widetilde{\phi^0})\Delta_{x^1} w \mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,int}} (V^{1,\alpha}_{bl} + \widetilde{V^0})\nabla_{x^1} w\cdot \mathbf{n}^1\,\mathrm{d}x^\sharp \mathrm{d}s(x^1), \tag{4.5}$$

for each α.

Now we pass to the limit in α. Equation (4.5) still holds if w is taken on the form of $\tau_\alpha v$, where $(\tau_\alpha)_{\alpha\in[\alpha_0,+\infty[}$ is a family of smooth truncation functions with compact support in $\Omega^\sharp_{bl,1}\times\Omega^{\infty,vac}_{bl,1}$ such that $\tau_\alpha v\to v$ for all $v\in H^2(\Omega^\sharp_{bl,1}\times\overline{\Omega^{\infty,vac}_{bl,1}})$, and $v\in C^\infty(\Omega^\sharp_{bl,1}\times\overline{\Omega^{\infty,vac}_{bl,1}})\cap H^2(\Omega^\sharp_{bl,1}\times\overline{\Omega^{\infty,vac}_{bl,1}})$ is $\Omega^{\infty,vac}_{bl,1}$- periodic in the direction x^1_1, $v=0$ on $\Gamma^{\infty,vac}_{bl,1,int}$, $\nabla_{x^1}v\cdot\mathbf{n}^1=0$ on $\Gamma^{\infty,vac}_{bl,1,per}\cup\Gamma^{\infty,vac}_{bl,1,top}\cup\Gamma^{\infty,vac}_{bl,1,0}$ as well as $|v|$, $|\nabla_{x^1}v|$, and $|\Delta_{x^1}v|$ exponentially decrease to 0 when $x^1_2\to+\infty$. Thus,

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{\infty,vac}_{bl,1}} (\phi^{1,\alpha}_{bl}+\widetilde{\phi^0})\chi_{\Omega^{1,vac}_{bl,1}}\Delta_{x^1}(\tau_\alpha v)\mathrm{d}x^\sharp\mathrm{d}x^1 = \int_{\Omega^\sharp_{bl,1}\times\Gamma^{\infty,vac}_{bl,1,int}} (V^{1,\alpha}_{bl}+\widetilde{V^0})\chi_{\Omega^{1,vac}_{bl,1}}\nabla_{x^1}(\tau_\alpha v)\cdot\mathbf{n}^1\,\mathrm{d}x^\sharp\mathrm{d}s(x^1).$$

Then, passing α to $+\infty$, by Assumption 4.7, we get

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{\infty,vac}_{bl,1}} (\phi^1_{bl} + \widetilde{\phi^0})\Delta_{x^1} v\,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{bl,1}\times\Gamma^{\infty,vac}_{bl,1,int}} (V^1_{bl} + \widetilde{V^0})\nabla_{x^1} v\cdot \mathbf{n}^1\,\mathrm{d}x^\sharp \mathrm{d}s(x^1).$$

To carry out the interpretation of this very weak formulation, we consider that v is vanishing out of a bounded domain which is taken as $\Omega^\sharp_{bl,1}\times\Omega^{1,vac}_{bl,1}$ to avoid new notations. Then

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{1,vac}_{bl,1}} (\phi^1_{bl} + \widetilde{\phi^0})\Delta_{x^1} v\,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,int}} (V^1_{bl} + \widetilde{V^0})\nabla_{x^1} v\cdot \mathbf{n}^1\,\mathrm{d}x^\sharp \mathrm{d}s(x^1),$$

for each α. Applying Green's formula twice, decomposing $\partial\Omega^{1,vac}_{bl,1}$ as $\Gamma^{1,vac}_{bl,1,int}\cup\Gamma^{1,vac}_{bl,1,per}\cup\Gamma^{1,vac}_{bl,1,top}\cup\Gamma^{1,vac}_{bl,1,0}\cup\Gamma^{1,vac}_{bl,1,\alpha}$ using the conditions satisfied by v and $\Delta_{x^1}\widetilde{\phi^0}=0$ in $\Omega^{1,vac}_{bl,1}$, $\widetilde{\phi^0}=\widetilde{V^0}$ on $\Gamma^{1,vac}_{bl,1,int}$, $\nabla_{x^1}\widetilde{\phi^0}\cdot\ \mathbf{n}^1=0$ on $\Gamma^{1,vac}_{bl,1,top}$, $\nabla_{x^1}\widetilde{\phi^0}\cdot\ \mathbf{n}^1$ is $\Gamma^{1,vac}_{bl,1,per}$-antiperiodic resulting from Proposition 3.3,

$$\begin{aligned}
&\int_{\Omega^\sharp_{bl,1}\times\Omega^{1,vac}_{bl,1}} \Delta_{x^1}\phi^1_{bl} v\,\mathrm{d}x^\sharp\mathrm{d}x^1 + \int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,int}} \phi^1_{bl}\nabla_{x^1} v\cdot\mathbf{n}^1\,\mathrm{d}x^\sharp\mathrm{d}s(x^1)\\
&-\int_{\Omega^\sharp_{bl,1}\times\left(\Gamma^{1,vac}_{bl,1,top}\cup\Gamma^{1,vac}_{bl,1,per}\right)} \nabla_{x^1}\phi^1_{bl}\cdot\mathbf{n}^1 v\,\mathrm{d}x^\sharp\mathrm{d}s(x^1) + \int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,0}} \nabla_{x^1}\left(\phi^1_{bl}+\widetilde{\phi^0}\right)\cdot\mathbf{n}^1 v\,\mathrm{d}x^\sharp\mathrm{d}s(x^1)\\
&=\int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,int}} V^1_{bl}\nabla_{x^1} v\cdot\mathbf{n}^1\,\mathrm{d}x^\sharp\mathrm{d}s\left(x^1\right).
\end{aligned}$$

Posing $v=0$ on $\Gamma^{1,vac}_{bl,1,0}\cup\Gamma^{1,vac}_{bl,1,top}\cup\Gamma^{1,vac}_{bl,1,per}$ and $\nabla_{x^1}v\cdot\mathbf{n}^1=0$ on $\Gamma^{1,vac}_{bl,1,int}$, yields

$$\int_{\Omega^\sharp_{bl,1}\times\Omega^{1,vac}_{bl,1}} \Delta_{x^1}(\phi^1_{bl}) v\,\mathrm{d}x^\sharp\mathrm{d}x^1 = 0$$

and then

$$\Delta_{x^1}\phi^1_{bl} = 0 \quad \text{in } \Omega^{1,vac}_{bl,1}.$$

Next, for $v = 0$ on $\Gamma^{1,vac}_{bl,1,0} \cup \Gamma^{1,vac}_{bl,1,top} \cup \Gamma^{1,vac}_{bl,1,per}$,

$$\int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,int}} (\phi^1_{bl} - V^1_{bl})\nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) = 0,$$

then

$$\phi^1_{bl} = V^1_{bl} \text{ on } \Gamma^{1,vac}_{bl,1,int}.$$

For $v = 0$ on $\Gamma^{1,vac}_{bl,1,0} \cup \Gamma^{1,vac}_{bl,1,per}$,

$$\int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,top}} \nabla_{x^1}\phi^1_{bl} \cdot \mathbf{n}^1 v \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) = 0,$$

then

$$\nabla_{x^1}\phi^1_{bl} \cdot \mathbf{n}^1 = 0 \text{ on } \Gamma^{1,vac}_{bl,1,top}.$$

For $v = 0$ on $\Gamma^{1,vac}_{bl,1,per}$

$$\int_{\Omega^\sharp_{bl,1}\times\Gamma^{1,vac}_{bl,1,0}} \nabla_{x^1}\left(\phi^1_{bl} + \widetilde{\phi}^0\right) \cdot \mathbf{n}^1 v \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) = 0,$$

then

$$\nabla_{x^1}\phi^1_{bl} \cdot \mathbf{n}^1 = -\nabla_{x^1}\widetilde{\phi}^0 \cdot \mathbf{n}^1 \text{on } \Gamma^{1,vac}_{bl,1,0}.$$

Last, we get

$$\nabla_{x^1}\phi^1_{bl} \cdot \mathbf{n}^1 \text{ is } \Gamma^{1,vac}_{bl,1,per} \text{ - antiperiodic.}$$

Since these equations hold true for any α then they hold in the infinite domain and the proof is complete. ■

5 Exterior Edge Model

We assume that all lateral boundary models are already derived and identified by the index $i = 1, 2, 3, 4$ of the lateral boundaries, see Figure 2. We consider the contributions of two lateral boundary models corresponding to the indices $i = 1$ and $i = 2$ at the first exterior edge. Obviously, the sum of contributions is not continuous at this edge, and then it leads to propose an edge corrector to overcome this problem. We introduce terms $\phi^\varepsilon_{exe} = \phi^\varepsilon - \left(B^\varepsilon\phi^0 + B^\varepsilon_{bl,1}\phi^1_{bl} + B^\varepsilon_{bl,2}\phi^2_{bl}\right)$ and $v^\varepsilon_{exe} = V^\varepsilon - \left(B^\varepsilon V^0 + B^\varepsilon_{bl,1}v^1_{bl} + B^\varepsilon_{bl,2}V^2_{bl}\right)$, where we recall that ϕ^0 is the solution to the periodic model while ϕ^1_{bl} and ϕ^2_{bl} are the solutions of the first and second lateral boundary problems near the first exterior edge, $B^\varepsilon_{bl,1}$ and $B^\varepsilon_{bl,2}$ are the smooth approximation operators of the first and second adjoint boundary layer two-scale transform operator $T^{\varepsilon*}_{bl,1}$ and $T^{\varepsilon*}_{bl,2}$, and v^1_{bl} and V^2_{bl} are the weak limits of $v^{1,\alpha}_{bl}$ and $v^{2,\alpha}_{bl}$ when $\alpha \to \infty$ which themselves are the weak limits of $T^\varepsilon_{bl,1}(v^\varepsilon_{bl})$ in $L^2\left(\Omega^\sharp_{bl,1} \times \Gamma^{1,vac}_{bl,1,int}\right)$, resp. of $T^\varepsilon_{bl,2}(v^\varepsilon_{bl})$ in $L^2\left(\Omega^\sharp_{bl,2} \times \Gamma^{1,\alpha,vac}_{bl,2,int}\right)$.

5.1 Geometry Notations

Let $\Omega^{\varepsilon,\alpha}_{exe,1} = \cup_{c\in\mathcal{I}_{exe,1}}\Omega^\varepsilon_c$ be a subdomain of Ω^ε where $\mathcal{I}_{exe,1} := \{c = (c_1, c_2) : c_1, c_2 \in \overline{1,\alpha}\}$ with $\alpha\varepsilon < \min\{L^1_1, L^2_1\}$, see Figure 9. The manner to construct its subdomains, boundary and subboundaries follows this of the periodic model. Here the special case is $\Gamma^{\varepsilon,\alpha,vac}_{exe,1,ext} = \Gamma^{\varepsilon,\alpha,vac}_{exe,1,\alpha} \cup \Gamma^{\varepsilon,\alpha,vac}_{exe,1,top} \cup \Gamma^{\varepsilon,\alpha,vac}_{exe,1,lat}$ where $\Gamma^{\varepsilon,\alpha,vac}_{exe,1,\alpha}$ are to the ends x_1 or $x_2 = \alpha\varepsilon$ of the boundary layer $\Omega^{\varepsilon,\alpha}_{exe,1}$.

We introduce the finite microscopic domain $\Omega^1_{exe,1}$ defined by $\Omega^1_{exe,1} = \cup^{\alpha-1}_{\xi,\eta=0}(\Omega^1 + (\xi + 1/2, \eta + 1/2, 1/2))$, see Figure 10. Here the periodic boundaries are replaced by $\Gamma^{1,vac}_{exe,1,bl1}$, $\Gamma^{1,vac}_{exe,1,bl2}$ located to the first and second lateral boundaries and by $\Gamma^{1,vac}_{exe,1,\alpha}$ to the ends x^1_1 or $x^1_2 = \alpha$. Thus $\partial\Omega^{1,vac}_{exe,1} = \Gamma^{1,vac}_{exe,1,int} \cup \Gamma^{1,vac}_{exe,1,top} \cup \Gamma^{1,vac}_{exe,1,bl1} \cup \Gamma^{1,vac}_{exe,1,bl2} \cup \Gamma^{1,vac}_{exe,1,\alpha}$.

The infinite microscopic domain $\Omega^\infty_{exe,1}$ and its related sets are defined as the limits of $\Omega^1_{exe,1}$ and related when α tends to infinity.

5.2 Exterior Edge Boundary Layer Two-Scale Operator

We still consider any surface Γ^1 in $\overline{\Omega^1}$, $\Gamma^1_{exe,1} = \cup^{\alpha-1}_{\xi,\eta=0}(\Omega^1 + (\xi + 1/2, \eta + 1/2, 1/2)) \subset \overline{\Omega^1_{exe,1}}$ and $\Gamma^{\varepsilon,\alpha}_{exe,1} = \cup_{c\in\mathcal{I}_{exe,1}}\varepsilon((c_1 - 1/2, c_2 - 1/2, 1/2) + \Gamma^1) \subset \overline{\Omega^{\varepsilon,\alpha}_{exe,1}}$. Then in this section the pair (X^ε, X^1) stands both for $(\Omega^{\varepsilon,\alpha}_{exe,1}, \Omega^1_{exe,1})$ and for $(\Gamma^{\varepsilon,\alpha}_{exe,1}, \Gamma^1_{exe,1})$.

We introduce the dilation operator $T^\varepsilon_{exe,1}$ for the first exterior edge model.

Definition 5.1 *For any α, the operator $T^\varepsilon_{exe,1}$ operating on any function φ with variable in $\overline{\Omega^{\varepsilon,\alpha}_{exe,1}}$ is defined by*

$$T^\varepsilon_{exe,1}(\varphi)(x^1) = \varphi(\varepsilon x^1) \text{ for } x^1 \in \overline{\Omega^1_{exe,1}}.$$

Here the operator $T^{\varepsilon*}_{exe,1} = (T^\varepsilon_{exe,1})^{-1}$ i.e.

$$T^{\varepsilon*}_{exe,1}(\psi)\,(x^\varepsilon) = \psi(\frac{x^\varepsilon}{\varepsilon}).$$

Property 5.2 *The operator $T^{\varepsilon*}_{exe,1}$ is the adjoint of $T^\varepsilon_{exe,1}$ in the sense*

$$\frac{1}{\varepsilon^3}\int_{\Omega^{\varepsilon,\alpha}_{exe,1}} \varphi T^{\varepsilon*}_{exe,1}(\psi)\,\mathrm{d}x^\varepsilon = \int_{\Omega^1_{exe,1}} T^\varepsilon_{exe,1}(\varphi)\psi\ \mathrm{d}x^1,$$

for all $\varphi \in L^2(\Omega^{\varepsilon,\alpha}_{exe,1}), \psi \in L^2(\Omega^1_{exe,1})$, and in the sense

$$\frac{1}{\varepsilon^2}\int_{\Gamma^{\varepsilon,\alpha}_{exe,1}} \varphi T^{\varepsilon*}_{exe,1}(\psi)\,\mathrm{d}s(x^\varepsilon) = \int_{\Gamma^1_{exe,1}} T^\varepsilon_{exe,1}(\varphi)\psi\,\mathrm{d}s(x^1),$$

for all $\varphi \in L^2(\Gamma^{\varepsilon,\alpha}_{exe,1}), \psi \in L^2(\Gamma^1_{exe,1})$.

In this edge case, the operator $T^{\varepsilon*}_{exe,1}$ and its approximation $B^\varepsilon_{exe,1}$ are identical. However both will be used in the model proof to follow the algorithm of Section 2.5.

Proposition 5.3 *Let B^ε, $B^\varepsilon_{bl,1}$, $B^\varepsilon_{bl,2}$ be the smooth approximation operators of the adjoints of T^ε, $T^\varepsilon_{bl,1}$, $T^\varepsilon_{bl,2}$ respectively.*

1. *For each α, if a function ψ with variables in $\Omega_1^\sharp \times \Omega^1$ respectively in $\Omega_1^\sharp \times \Gamma^1$ is continuous w.r.t. its first variable and is Ω^1 - periodic in the directions x_1^1, x_2^1 then*

$$T^\varepsilon_{exe,1}(B^\varepsilon \psi)(x^1) \to \widetilde{\psi}(x^1) \text{ for } x^1 \text{ in } \Omega^1_{exe,1} \text{ respect. in } \Gamma^1_{exe,1} \text{ when } \varepsilon \to 0,$$

where $\tilde{\psi}(x^1) = \psi(0, x^1 - 1/2)$.

2. *If a function ψ with variables in $\Omega^\sharp_{bl,1} \times \Omega^\infty_{bl,1}$, respectively in $\Omega^\sharp_{bl,1} \times \Gamma^\infty_{bl,1}$, is continuous w.r.t. its first variable in $\Omega^\sharp_{bl,1}$ and is $\Omega^\infty_{bl,1}$ - periodic in the direction x_1^1 then*

$$T^\varepsilon_{exe,1}(B^\varepsilon_{bl,1}\psi)(x^1) \to \widetilde{\psi}(x^1) \text{ for } x^1 \text{ in } \Omega^1_{exe,1}, \text{ respect. in } \Gamma^1_{exe,1}, \text{ when } \varepsilon \to 0,$$

where $\tilde{\psi}(x^1) = \psi(0, (x_1^1 - 1/2, x_2^1, x_3^1))$.

3. *If a function ψ with variables in $\Omega^\sharp_{bl,2} \times \Omega^\infty_{bl,2}$, respectively in $\Omega^\sharp_{bl,2} \times \Gamma^\infty_{bl,2}$, is continuous w.r.t. its first variable in $\Omega^\sharp_{bl,2}$ and is $\Omega^\infty_{bl,2}$ - periodic in the direction x_2^1 then*

$$T^\varepsilon_{exe,1}(B^\varepsilon_{bl,2}\psi)(x^1) \to \widetilde{\psi}(x^1) \text{ for } x^1 \in \Omega^1_{exe,1}, \text{ respect. in } \Gamma^1_{exe,1}, \text{ when } \varepsilon \to 0,$$

where $\tilde{\psi}(x^1) = \psi(0, (x_1^1, x_2^1 - 1/2, x_3^1))$.

5.3 Derivation of an Exterior Edge Model

Let us recall that $\phi^\varepsilon_{exe} = \phi^\varepsilon - \big(B^\varepsilon \phi^0 + B^\varepsilon_{bl,1}\phi^1_{bl} + B^\varepsilon_{bl,2}\phi^2_{bl}\big)$ and $V^\varepsilon_{exe} = V^\varepsilon - \big(B^\varepsilon V^0 + B^\varepsilon_{bl,1} V^1_{bl} + B^\varepsilon_{bl,2} V^2_{bl}\big)$. In this section we assume that the following assumptions are satisfied.

Assumption 5.4 1. *For each α, there exist $\phi^{1,\alpha}_{exe}$ in $L^2(\Omega^{1,vac}_{exe,1})$ and $V^{1,\alpha}_{exe}$ in $L^2(\Gamma^{1,vac}_{exe,1,int})$ such that $T^\varepsilon_{exe,1}(\phi^\varepsilon_{exe}) \rightharpoonup \phi^{1,\alpha}_{exe}$ weakly in $L^2(\Omega^{1,vac}_{exe,1})$ and $T^\varepsilon_{exe,1}(V^\varepsilon_{exe}) \rightharpoonup V^{1,\alpha}_{exe}$ weakly in $L^2(\Gamma^{1,vac}_{exe,1,int})$ when $\varepsilon \to 0$.*

2. *Assume that there exist ϕ^1_{exe} in $H^1(\Omega^{\infty,vac}_{exe,1})$ with ϕ^1_{exe} and its gradient converging exponentially fast to zero when $x_1^1 + x_2^1 \to \infty$, and V^1_{exe} in $L^2(\Gamma^{\infty,vac}_{exe,1,int})$ such that the extensions by zero $\phi^{1,\alpha}_{exe}\chi_{\Omega^{1,vac}_{exe,1}} \rightharpoonup \phi^1_{exe}$ weakly in $L^2(\Omega^{\infty,vac}_{exe,1})$ and $V^{1,\alpha}_{exe}\chi_{\Omega^{1,vac}_{exe,1}} \rightharpoonup V^1_{exe}$ weakly in $L^2(\Gamma^{\infty,vac}_{exe,1,int})$ when $\alpha \to +\infty$.*

The following proposition results from using Proposition 5.3.

Assumption 5.5 *The limits ϕ^0, V^0 satisfy the assumption of Proposition 5.3.1 and similarly, ϕ^1_{bl}, V^1_{bl} and ϕ^2_{bl}, V^2_{bl} satisfy Proposition 5.3.2 and 5.3.3.*

Proposition 5.6 *When $\varepsilon \to 0$,*

$$T^\varepsilon_{exe,1}(\phi^\varepsilon) \rightharpoonup \phi^{1,\alpha}_{exe} + \widetilde{\phi^0} + \widetilde{\phi^1_{bl}} + \widetilde{\phi^2_{bl}}$$

weakly in $L^2(\Omega^{1,vac}_{exe,1})$ and

$$T^\varepsilon_{exe,1}(V^\varepsilon) \rightharpoonup V^{1,\alpha}_{exe} + \widetilde{V^0} + \widetilde{V^1_{bl}} + \widetilde{V^2_{bl}}$$

weakly in $L^2(\Gamma^{1,vac}_{exe,1,int})$, where $\widetilde{\varphi^0}(x^1) = \varphi^0(0, x^1 - 1/2)$, $\widetilde{\varphi^1_{bl}}(x^1) = \varphi^1_{bl}(0, (x_1^1 - 1/2, x_2^1, x_3^1))$ and $\widetilde{\varphi^2_{bl}}(x^1) = \varphi^2_{bl}(0, (x_1^1, x_2^1 - 1/2, x_3^1))$ and with similar expressions for the voltage sources.

Proposition 5.7 *The limit ϕ^1_{exe} satisfies*

$$\begin{cases} \Delta_{x^1}\phi^1_{exe} = 0 & in \quad \Omega^{\infty,vac}_{exe,1} \\ \phi^1_{exe} = V^1_{exe} & on \quad \Gamma^{\infty,vac}_{exe,1,int} \\ \nabla_{x^1}\phi^1_{exe}\cdot\mathbf{n}^1 = 0 & on \quad \Gamma^{\infty,vac}_{exe,1,top} \\ \nabla_{x^1}\phi^1_{exe}\cdot\mathbf{n}^1 = -\nabla_{x^1}\widetilde{\phi^2_{bl}}\cdot\mathbf{n}^1 & on \quad \Gamma^{\infty,vac}_{exe,1,bl1} \\ \nabla_{x^1}\phi^1_{exe}\cdot\mathbf{n}^1 = -\nabla_{x^1}\widetilde{\phi^1_{bl}}\cdot\mathbf{n}^1 & on \quad \Gamma^{\infty,vac}_{exe,1,bl2}. \end{cases}$$

Proof. The outline of the proof runs as the previous ones. Firstly, we take a fixed α and replace v^ε by a smooth function v^ε_{exe} in (2.2) s.t. v^ε_{exe} is defined in $\Omega^{\varepsilon,\alpha,vac}_{exe,1}$, $v^\varepsilon_{exe} = 0$ on $\Gamma^{\varepsilon,\alpha,vac}_{exe,1,int}$ and vanishes out of $\Omega^{\varepsilon,\alpha}_{exe,1}$, then

$$\int_{\Omega^{\varepsilon,\alpha,vac}_{exe,1}} \phi^\varepsilon \Delta_{x^\varepsilon} v^\varepsilon_{exe}\,\mathrm{d}x^\varepsilon = \int_{\Gamma^{\varepsilon,\alpha,vac}_{exe,1,int}} V^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{exe}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right) + \int_{\Gamma^{\varepsilon,\alpha,vac}_{exe,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{exe}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right).$$

After that, we substitute v^ε_{exe} by $\varepsilon^{-1}B^\varepsilon_{exe,1}(w)$ where w is in $C^\infty(\overline{\Omega^{1,vac}_{exe,1}})$, $w = 0$ on $\Gamma^{1,vac}_{exe,1,int}\cup\Gamma^{1,vac}_{exe,1,\alpha}$ and $\nabla_{x^1}w\cdot\mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{exe,1,top}\cup\Gamma^{1,vac}_{exe,1,\alpha}\cup\Gamma^{1,vac}_{exe,1,bl1}\cup\Gamma^{1,vac}_{exe,1,bl2}$. Hence,

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\Omega^{\varepsilon,\alpha,vac}_{exe,1}} \phi^\varepsilon \Delta_{x^\varepsilon} B^\varepsilon_{exe,1}(w)\,\mathrm{d}x^\varepsilon &= \frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{exe,1,int}} V^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{exe,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right) \\ &+ \frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{exe,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{exe,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right). \end{aligned}$$

We check at once that,

$$\frac{\partial B^\varepsilon_{exe,1}w}{\partial x^\varepsilon_i} = \frac{1}{\varepsilon}B^\varepsilon_{exe,1}\left(\frac{\partial w}{\partial x^1_i}\right) \text{ and } \frac{\partial}{\partial x^\varepsilon_i}\frac{\partial B^\varepsilon_{exe,1}w}{\partial x^\varepsilon_i} = \frac{1}{\varepsilon^2}B^\varepsilon_{exe,1}\left(\frac{\partial}{\partial x^1_i}\frac{\partial w}{\partial x^1_i}\right),$$

for all $i = 1,2,3$, and if follows that $B^\varepsilon_{exe,1}\left(\nabla_{x^1}w\right)\cdot\mathbf{n}^\varepsilon = 0$ on $\Gamma^{\varepsilon,\alpha,vac}_{exe,1,ext}$, then

$$\frac{1}{\varepsilon^3}\int_{\Omega^{\varepsilon,\alpha,vac}_{exe,1}} \phi^\varepsilon B^\varepsilon_{exe,1}\left(\Delta_{x^1}w\right)\,\mathrm{d}x^\varepsilon = \frac{1}{\varepsilon^2}\int_{\Gamma^{\varepsilon,\alpha,vac}_{exe,1,int}} V^\varepsilon B^\varepsilon_{exe,1}\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\,\mathrm{d}s\left(x^\varepsilon\right).$$

Approximating $B^\varepsilon_{exe,1}$ by $T^{\varepsilon*}_{exe,1}$ and combining with the definition of $T^{\varepsilon*}_{exe,1}$,

$$\int_{\Omega^{1,vac}_{exe,1}} T^\varepsilon_{exe,1}(\phi^\varepsilon)\Delta_{x^1}w\,\mathrm{d}x^1 = \int_{\Gamma^{1,vac}_{exe,1,int}} T^\varepsilon_{exe,1}(V^\varepsilon)\nabla_{x^1}w\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right).$$

Passing ε to 0, by Proposition 5.6, it follows that

$$\int_{\Omega^{1,vac}_{exe,1}} (\phi^{1,\alpha}_{exe} + \widetilde{\phi^0} + \widetilde{\phi^1_{bl}} + \widetilde{\phi^2_{bl}})\Delta_{x^1}w\,\mathrm{d}x^1 = \int_{\Gamma^{1,vac}_{exe,1,int}} (V^{1,\alpha}_{exe} + \widetilde{V^0} + \widetilde{V^1_{bl}} + \widetilde{V^2_{bl}})\nabla_{x^1}w\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right).$$

We now replace w by $\tau_\alpha v$, where τ_α is a smooth truncation function with compact support in $\Omega^{1,vac}_{exe,1}$ and $v \in C^\infty(\overline{\Omega^{\infty,vac}_{exe,1}})\cap H^2(\overline{\Omega^{\infty,vac}_{exe,1}})$ satisfying $v = 0$ on $\Gamma^{\infty,vac}_{exe,1,int}$, $\nabla_{x^1}v\cdot\mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{exe,1,top}\cup\Gamma^{1,vac}_{exe,1,bl1}\cup\Gamma^{1,vac}_{exe,1,bl2}$, $|v|$, $|\nabla_{x^1}v|$ and $|\Delta_{x^1}v|$ converge exponentially fast to zero when $x^1_1 + x^1_2 \to \infty$, $\tau_\alpha v \to v$ in $H^2(\overline{\Omega^{\infty,vac}_{exe,1}})$ when $\alpha \to \infty$. We obtain

$$\int_{\Omega^{\infty,vac}_{exe,1}} (\phi^{1,\alpha}_{exe}+\widetilde{\phi^0}+\widetilde{\phi^1_{bl}}+\widetilde{\phi^2_{bl}})\chi_{\Omega^{1,vac}_{exe,1}}\Delta_{x^1}(\tau_\alpha v)\,\mathrm{d}x^1 = \int_{\Gamma^{\infty,vac}_{exe,1,int}} (V^{1,\alpha}_{exe}+\widetilde{V^0}+\widetilde{V^1_{bl}}+\widetilde{V^2_{bl}})\chi_{\Omega^{1,vac}_{exe,1}}\nabla_{x^1}\tau_\alpha v\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right).$$

Passing α to $+\infty$, by Assumption 5.4, we get

$$\int_{\Omega^{\infty,vac}_{exe,1}} (\phi^1_{exe} + \widetilde{\phi}^0 + \widetilde{\phi^1_{bl}} + \widetilde{\phi^2_{bl}})\Delta_{x^1} v \, \mathrm{d}x^1 = \int_{\Gamma^{\infty,vac}_{exe,1,int}} (V^1_{exe} + \widetilde{V^0} + \widetilde{V^1_{bl}} + \widetilde{V^2_{bl}})\nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right).$$

Now, we choose v vanishing out of $\Omega^{1,vac}_{exe,1}$ for a given α,

$$\int_{\Omega^{1,vac}_{exe,1}} (\phi^1_{exe} + \widetilde{\phi}^0 + \widetilde{\phi^1_{bl}} + \widetilde{\phi^2_{bl}})\Delta_{x^1} v \, \mathrm{d}x^1 = \int_{\Gamma^{1,vac}_{exe,1,int}} (V^1_{exe} + \widetilde{V^0} + \widetilde{V^1_{bl}} + \widetilde{V^2_{bl}})\nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right).$$

Applying Green's formula twice and decomposing $\partial\Omega^{1,vac}_{exe,1} = \Gamma^{1,vac}_{exe,1,int} \cup \Gamma^{1,vac}_{exe,1,top} \cup \Gamma^{1,vac}_{exe,1,bl1} \cup \Gamma^{1,vac}_{exe,1,bl2} \cup \Gamma^{1,vac}_{exe,1,\alpha}$, combining with conditions satisfied by v, the results from Proposition 3.3 and Proposition 4.10 $\Delta_{x^1}\widetilde{\phi}^0 = \Delta_{x^1}\widetilde{\phi^1_{bl}} = \Delta_{x^1}\widetilde{\phi^2_{bl}} = 0$ in $\Omega^{1,vac}_{exe,1}$, $\widetilde{\phi}^0 = \widetilde{V^0}$, $\widetilde{\phi^1_{bl}} = \widetilde{V^1_{bl}}$, $\widetilde{\phi^2_{bl}} = \widetilde{V^2_{bl}}$ on $\Gamma^{1,vac}_{exe,1,int}$, $\nabla_{x^1}\widetilde{\phi}^0 \cdot \mathbf{n}^1 = \nabla_{x^1}\widetilde{\phi^1_{bl}} \cdot \mathbf{n}^1 = \nabla_{x^1}\widetilde{\phi^2_{bl}} \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{exe,1,top}$, $\nabla_{x^1}(\widetilde{\phi}^0 + \widetilde{\phi^1_{bl}}) \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{exe,1,bl1}$, $\nabla_{x^1}(\widetilde{\phi}^0 + \widetilde{\phi^2_{bl}}) \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{exe,1,bl2}$, we deduce that

$$\begin{aligned}
&\int_{\Omega^{1,vac}_{exe,1}} \Delta_{x^1}(\phi^1_{exe}) v \, \mathrm{d}x^1 - \int_{\Gamma^{1,vac}_{exe,1,top}} \nabla_{x^1}\phi^1_{exe} \cdot \mathbf{n}^1 v \, \mathrm{d}s(x^1) \\
&- \int_{\Gamma^{1,vac}_{exe,1,bl1}} \nabla_{x^1}(\phi^1_{exe} + \widetilde{\phi^2_{bl}}) \cdot \mathbf{n}^1 v \, \mathrm{d}s(x^1) - \int_{\Gamma^{1,vac}_{exe,1,bl2}} \nabla_{x^1}(\phi^1_{exe} + \widetilde{\phi^1_{bl}}) \cdot \mathbf{n}^1 v \, \mathrm{d}s(x^1) \\
&+ \int_{\Gamma^{1,vac}_{exe,1,int}} \phi^1_{exe} \nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s(x^1) = \int_{\Gamma^{1,vac}_{exe,1,int}} V^1_{exe} \nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s(x^1).
\end{aligned}$$

The rest of the proof runs as the previous proofs. ■

6 Interface Model

As the asymptotic voltage source V^0 may exhibit a discontinuity at the interface between two zones, the solution ϕ^0 in Proposition 3.3 inherit of this lack of regularity. This section introduces an interface corrector to deal with this problem starting from the terms $\phi^\varepsilon_{bl} = \phi^\varepsilon - B^\varepsilon(\phi^0)$ and $v^\varepsilon_{bl} = V^\varepsilon - B^\varepsilon(V^0)$.

6.1 Geometry Notations

Let $\Omega^{\varepsilon,\alpha}_{in,1}$ be a subdomain of Ω^ε defined as $\Omega^{\varepsilon,\alpha}_{in,1} = \cup_{c\in\mathcal{I}_{in,1}}\Omega^\varepsilon_c$, where $\mathcal{I}_{in,1} := \{c = (c_1, c_2) : c_1 = \overline{i_1, j_1}$ and $c_2 \in \overline{i_2 - \alpha, i_2 + \alpha}$, $2 \le i, j \le n_1\}$ and $\alpha \in \mathbb{Z}^+$, see Figure 11. The domain $\Omega^{\varepsilon,\alpha}_{in,1}$ is decomposed by two subdomains $\Omega^{\varepsilon,\alpha+}_{in,1}$ and $\Omega^{\varepsilon,\alpha-}_{in,1}$, written as $\Omega^{\varepsilon,\alpha\pm}_{in,1}$ for short, which are subdomains of Ω^ε_2 and Ω^ε_1. The interface $\Gamma^{\varepsilon,\alpha}_{in,1,\text{interf}}$ between $\Omega^{\varepsilon,\alpha+}_{in,1}$ and $\Omega^{\varepsilon,\alpha-}_{in,1}$ is a subboundary of $\Gamma^\varepsilon_{\text{interf}}$. The complementary part of the boundary of $\Omega^{\varepsilon,\alpha\pm}_{in,1}$ is $\Gamma^{\varepsilon,\alpha\pm}_{in,1} = \partial\Omega^{\varepsilon,\alpha\pm}_{in,1} \setminus \Gamma^{\varepsilon,\alpha}_{in,1,\text{interf}}$. All the other notations are then derived from $\Omega^{\varepsilon,\alpha}_{in,1}$, $\Omega^{\varepsilon,\alpha\pm}_{in,1}$, $\Gamma^{\varepsilon,\alpha\pm}_{in,1}$ and $\Gamma^\varepsilon_{\text{interf}}$ with the exceptions $\Gamma^{\varepsilon,\alpha,vac\pm}_{in,1,ext} = \Gamma^{\varepsilon,\alpha,vac\pm}_{in,1,\alpha} \cup \Gamma^{\varepsilon,\alpha,vac\pm}_{in,1,top} \cup \Gamma^{\varepsilon,\alpha,vac\pm}_{in,1,lat}$.

The macroscopic domain $\Omega^\sharp_{in,1} = [L^1_1, L^1_2)$ is built as the partition $\left\{\Omega^\sharp_{in,1c_1} = [c_1\varepsilon, (c_1+1)\varepsilon)\right\}_{c_1=\overline{i_1,j_1-1}}$, with i_1, j_1 s.t. $L^1_1 = i_1\varepsilon$, $L^1_2 = j_1\varepsilon$, and $x^{\sharp,c_1} = c_1\varepsilon + \varepsilon/2$ is the center of $\Omega^\sharp_{in,1c_1}$.

The bounded microscopic domain $\Omega^1_{in,1}$ as in Figure 12 is the union of two subdomains $\Omega^{1+}_{in,1}$ and $\Omega^{1-}_{in,1}$, s.t. $\Omega^{1\pm}_{in,1} = \cup_{\eta=\overline{1,\alpha}}(\Omega^1 + (0, \pm(\eta - 1/2), 1/2))$, with interface $\Gamma^1_{in,1,\text{ interf}}$. The notation system built for the physical domain is transposed to the microscopic domain.

For all regular function v defined in $\Omega^1_{in,1}$, we denote v^+ and v^- the restriction of v in $\Omega^{1+}_{in,1}$ and $\Omega^{1-}_{in,1}$, and $[[v]] = v^+ - v^-$ the jump of v at the interface $\Gamma^1_{in,1,\text{ interf}}$.

The infinite microscopic domain $\Omega^\infty_{in,1}$ and its boundaries are defined as the limit over α of $\Omega^1_{in,1}$ and of its boundaries.

6.2 Interface Boundary Layer Two-Scale Transform Operator

We again consider any surface Γ^1 in $\overline{\Omega^1}$, $\Gamma^1_{in,1} = \cup_{\sigma\in\{+,-\}} \cup_{\eta=\overline{1,\alpha}} (\Gamma^1 + (0, \sigma(\eta - 1/2), 1/2)) \subset \overline{\Omega^1_{in,1}}$ and $\Gamma^{\varepsilon,\alpha}_{in,1} = \cup_{c\in\mathcal{I}_{in,1}}\varepsilon((c_1 - 1/2, c_2 - 1/2, 1/2) + \Gamma^1) \subset \overline{\Omega^{\varepsilon,\alpha}_{in,1}}$. In this section the pair (X^ε, X^1) stands both for $(\Omega^{\varepsilon,\alpha}_{in,1}, \Omega^1_{in,1})$ and for $(\Gamma^{\varepsilon,\alpha}_{in,1}, \Gamma^1_{in,1})$. In Section 7.2 we also use $\Omega^{\infty\pm}_{in,1}$ and $\Gamma^{\infty\pm}_{in,1}$ the limits over α of $\Omega^{1\pm}_{in,1}$ and of $\Gamma^{1\pm}_{in,1} = \cup_{\eta=\overline{1,\alpha}}(\Gamma^1 + (0, \pm(\eta - 1/2), 1/2))$.

Let us introduce the interface boundary layer two-scale transform $T^\varepsilon_{in,1}$.

Definition 6.1 *The interface boundary layer two-scale transform $T^\varepsilon_{in,1}$ operating on functions φ with variables in X^ε is defined by*

$$T^\varepsilon_{in,1}(\varphi)(x^\sharp, x^1) = \sum_{c_1} \chi_{\Omega^\sharp_{in,1c_1}}(x^\sharp)\varphi(x^{\sharp,c_1} + \varepsilon x^1_1, L^2_1 + \varepsilon x^1_2, \varepsilon x^1_3),$$

for a.e. $x^\sharp \in \overline{\Omega^\sharp_{in,1}}$, $x^1 \in X^1$, $L^2_1 = i_2\varepsilon$ and $i_2 \in \mathbb{Z}^+$.

Let us introduce the operator $T^{\varepsilon*}_{in,1}$ defined by

$$T^{\varepsilon*}_{in,1}(\psi)\,(x^\varepsilon) = \frac{1}{\varepsilon}\sum_{c_1}\int_{\Omega^\sharp_{in,1c_1}} \psi\left(x^\sharp, \frac{x^\varepsilon_1 - x^{\sharp,c_1}}{\varepsilon}, \frac{x^\varepsilon_2 - L^2_1}{\varepsilon}, \frac{x^\varepsilon_3}{\varepsilon}\right) \mathrm{d}x^\sharp \chi_{\Omega^\sharp_{in,1c_1}}(x^\varepsilon_1)\,,$$

for all functions ψ with variables in $\overline{\Omega^\sharp_{in,1}} \times X^1$ and all $x^\varepsilon \in X^\varepsilon$.

Property 6.2 *The operator $T^{\varepsilon*}_{in,1}$ is the adjoint of $T^\varepsilon_{in,1}$ in the sense*

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha}_{in,1}} \varphi T^{\varepsilon*}_{in,1}(\psi)\mathrm{d}x^\varepsilon = \int_{\Omega^\sharp_{in,1}\times\Omega^1_{in,1}} T^\varepsilon_{in,1}(\varphi)\psi \mathrm{d}x^\sharp \mathrm{d}x^1,$$

for all $\psi \in L^2(\Omega^\sharp_{in,1} \times \Omega^1_{in,1})$, $\varphi \in L^2(\Omega^{\varepsilon,\alpha}_{in,1})$, and in the sense

$$\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha}_{in,1}} \varphi T^{\varepsilon*}_{in,1}(\psi)\mathrm{d}s(x^\varepsilon) = \int_{\Omega^\sharp_{in,1}\times\Gamma^1_{in,1}} T^\varepsilon_{in,1}(\varphi)\psi\, \mathrm{d}x^\sharp \mathrm{d}s(x^1),$$

for all $\psi \in L^2(\Omega^\sharp_{in,1} \times \Gamma^1_{in,1})$, $\varphi \in L^2(\Gamma^{\varepsilon,\alpha}_{in,1})$.

Definition 6.3 *The operator $B^\varepsilon_{in,1}$ is defined by*

$$B^\varepsilon_{in,1}(\psi)(x^\varepsilon) = \psi\left(P(x^\varepsilon), \frac{x^\varepsilon_1}{\varepsilon} - \frac{1}{2}, \frac{x^\varepsilon_2}{\varepsilon}, \frac{x^\varepsilon_3}{\varepsilon}\right),$$

for any function ψ with variables in $\Omega^\sharp_{in,1} \times X^1$ and all $x^\varepsilon \in X^\varepsilon$, where $P(x^\varepsilon) = x^\varepsilon_1$.

Proposition 6.4 *For every ψ in $C^1(\Omega^\sharp_{in,1}\times X^1)$ and $\Omega^1_{in,1}$ - periodic in the directions x^1_1 and x^1_2, then for all $x^\varepsilon\in X^\varepsilon$,*
$$T^{\varepsilon*}_{in,1}(\psi)\,(x^\varepsilon)=B^\varepsilon_{in,1}(\psi)(x^\varepsilon)+O(\varepsilon).$$

Proposition 6.5 *If ψ is a function with variables in $(\Omega^\sharp_1\cup\Omega^\sharp_2)\times\Omega^1$, respectively in $(\Omega^\sharp_1\cup\Omega^\sharp_2)\times\Gamma^1$, is Ω^1 - periodic in the directions x^1_1, x^1_2 and is continuous w.r.t. its first variable in a vicinity of the interface,*

$$T^\varepsilon_{in,1}(B^\varepsilon(\psi))(x^\sharp,x^1)\to\widetilde{\psi}(x^\sharp,x^1)\text{ for }(x^\sharp,x^1)\text{ in }\Omega^\sharp_{in,1}\times\Omega^1_{in,1}\text{ respect. in }\Omega^\sharp_{in,1}\times\Gamma^1_{in,1}\text{ when }\varepsilon\to0,$$

where $\widetilde{\psi}(x^\sharp,x^1)=\psi\left((x^\sharp,L^2_1),(x^1_1,x^1_2-\frac{1}{2},x^1_3-\frac{1}{2})\right)$.

Proof. By the definitions of $T^\varepsilon_{in,1}$ and B^ε, we obtain

$$\begin{aligned}T^\varepsilon_{in,1}(B^\varepsilon(\psi))(x^\sharp,x^1)&=\sum_{c_1}\chi_{\Omega^\sharp_{in,1c_1}}(x^\sharp)B^\varepsilon(\psi)(x^{\sharp,c_1}+\varepsilon x^1_1,L^2_1+\varepsilon x^1_2,\varepsilon x^1_3)\\&=\sum_{c_1}\chi_{\Omega^\sharp_{in,1c_1}}(x^\sharp)\psi\left((x^{\sharp,c_1}+\varepsilon x^1_1,L^2_1+\varepsilon x^1_2),(x^1_1,x^1_2-\frac{1}{2},x^1_3-\frac{1}{2})\right).\end{aligned}$$

By the continuity property,

$$\psi\left((x^{\sharp,c_1}+\varepsilon x^1_1,L^2_1+\varepsilon x^1_2),(x^1_1+c_1,x^1_2-\frac{1}{2},x^1_3-\frac{1}{2})\right)=\psi\left((x^\sharp,L^2_1),(x^1_1,x^1_2-\frac{1}{2},x^1_3-\frac{1}{2})\right)+o(\varepsilon),$$

for $x^\sharp$ in each $\Omega^\sharp_{in,1,c_1}$. Passing ε to 0, then

$$T^\varepsilon_{in,1}(B^\varepsilon(\psi))\to\psi\left((x^\sharp,L^2_1),((x^1_1,x^1_2-\frac{1}{2},x^1_3-\frac{1}{2}))\right).$$

■

6.3 Derivation of an Interface Model

Let us recall the expressions of the remaining voltage source $V^\varepsilon_{bl}=V^\varepsilon-B^\varepsilon(V^0)$ and the corrector $\phi^\varepsilon_{bl}=\phi^\varepsilon-B^\varepsilon(\phi^0)$. Now we assume that the following assumptions are satisfied.

Assumption 6.6

1. *For each α, there exist $\phi^{1,\alpha}_{in}\in L^2(\Omega^\sharp_{in,1},H^1(\Omega^{1,vac}_{in,1}))$, $\Omega^{1,vac}_{in,1}$ - periodic in the direction x^1_1 and $V^1_{in}\in L^2(\Omega^\sharp_{in,1}\times\Gamma^{1,vac}_{in,1,int})$ such that $T^\varepsilon_{in,1}(\phi^\varepsilon_{bl})\rightharpoonup\phi^{1,\alpha}_{in,1}$ weakly in $L^2(\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1})$ and $T^\varepsilon_{in,1}(V^\varepsilon_{bl})\rightharpoonup V^{1,\alpha}_{in}$ weakly in $L^2(\Omega^\sharp_{in,1}\times\Gamma^{1,vac}_{in,1,int})$.*
2. *There exist $\phi^1_{in}\in L^2(\Omega^\sharp_{in,1},H^1(\Omega^{\infty,vac}_{in,1}))$, $\Omega^{\infty,vac}_{in,1}$ - periodic in the direction x^1_1 and $V^1_{in}\in L^2(\Omega^\sharp_{in,1}\times\Gamma^{\infty,vac}_{in,1,int})$ such that the extensions by zero $\phi^{1,\alpha}_{in}\chi_{\Omega^{1,vac}_{in,1}}\rightharpoonup\phi^1_{in,1}$ weakly in $L^2(\Omega^\sharp_{in,1}\times\Omega^{\infty,vac}_{in,1})$ and $V^{1,\alpha}_{in}\chi_{\Omega^{1,vac}_{in,1}}\rightharpoonup V^1_{in}$ weakly in $L^2(\Omega^\sharp_{in,1}\times\Gamma^{\infty,vac}_{in,1,int})$. Moreover ϕ^1_{in} and it gradient exponentially decrease to 0 when $|x^1_2|\to+\infty$.*

Assumption 6.7 *The limits ϕ^0 and V^0 satisfy the condition of Proposition 6.5.*

Proposition 6.8 *When $\varepsilon \to 0$ then*

$$T^\varepsilon_{in,1}(\phi^\varepsilon) \rightharpoonup \phi^{1,\alpha}_{in} + \widetilde{\phi}^0 \text{ weakly in } L^2(\Omega^\sharp_{in,1} \times \Omega^{1,vac}_{in,1})$$

and

$$T^\varepsilon_{in,1}(V^\varepsilon) \rightharpoonup V^{1,\alpha}_{in} + \widetilde{V^0} \text{ weakly in } L^2(\Omega^\sharp_{in,1} \times \Gamma^{1,vac}_{in,1,int}),$$

where $\widetilde{\psi}(x^\sharp, x^1) = \psi\left((x^\sharp, L^2_1), (x^1_1, x^1_2 - \frac{1}{2}, x^1_3 - \frac{1}{2})\right)$.

Proposition 6.9 *The limit ϕ^1_{in} is a solution to*

$$\begin{cases} -\Delta_{x^1}\phi^1_{in} = 0 & \text{in } \Omega^{\infty,vac}_{in,1} \\ \phi^1_{in} = V^1_{in} & \text{on } \Gamma^{\infty,vac}_{in,1,int} \\ \nabla_{x^1}\phi^1_{in}\cdot\mathbf{n}^1 = 0 & \text{on } \Gamma^{\infty,vac}_{in,1,top} \\ \nabla_{x^1}\phi^1_{in}\cdot\mathbf{n}^1 & \text{is } \Gamma^{\infty,vac}_{in,1,per} \text{ - antiperiodic} \\ \left[\left[\nabla_{x^1}\phi^1_{in}\right]\right]\cdot\mathbf{n}^1 = -\left[\left[\nabla_{x^1}\widetilde{\phi}^0\right]\right]\cdot\mathbf{n}^1 & \text{on } \Gamma^{\infty,vac}_{in,1,interf} \\ \left[\left[\phi^1_{in}\right]\right] = -\left[\left[\widetilde{\phi}^0\right]\right] & \text{on } \Gamma^{\infty,vac}_{in,1,interf} \\ \phi^1_{in} & \text{is } \Gamma^{\infty,vac}_{in,1,per} \text{ - periodic.} \end{cases}$$

Proof. Only some key steps are detailed. We replace v^ε by a smooth function v^ε_{in} in (2.2), where v^ε_{in} is defined in $\Omega^{\varepsilon,\alpha,vac}_{in,1}$, $v^\varepsilon_{in} = 0$ on $\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}$ and vanishes out of $\Omega^{\varepsilon,\alpha,vac}_{in,1}$.

$$\int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon \Delta_{x^\varepsilon} v^\varepsilon_{in}\,\mathrm{d}x^\varepsilon = \int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}} V^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{in}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon) + \int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{in}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon).$$

Then, we substitute v^ε_{in} by $B^\varepsilon_{in,1}(w)$, where w is in $C^\infty(\Omega^\sharp_{in,1}\times\overline{\Omega^{1,vac}_{in,1}})$, $\Omega^{1,vac}_{in,1}$ - periodic in the directions x^1_1, x^1_2, $w = 0$ on $\Gamma^{1,vac\pm}_{in,1,int}\cup\Gamma^{1,vac\pm}_{in,1,\alpha}$ and $\nabla_{x^1}w\cdot\ \mathbf{n}^1 = 0$ on $\Gamma^{1,vac,\pm}_{in,1,top}\cup\Gamma^{1,vac\pm}_{in,1,per}\cup\Gamma^{1,vac\pm}_{in,1,\alpha}$, we get

$$\int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon \Delta_{x^\varepsilon} B^\varepsilon_{in,1}(w)\,\mathrm{d}x^\varepsilon = \int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}} V^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{in,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon) + \int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{in,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\,(x^\varepsilon).$$

As for the other cases,

$$\frac{\partial B^\varepsilon_{in,1}w}{\partial x^\varepsilon_i} = B^\varepsilon_{in,1}\left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial w}{\partial x^\sharp} + \frac{1}{\varepsilon}\frac{\partial w}{\partial x^1_i}\right),$$
$$\frac{\partial}{\partial x^\varepsilon_i}\frac{\partial B^\varepsilon_{in,1}w}{\partial x^\varepsilon_i} = B^\varepsilon_{in,1}\left(\chi_{\mathcal{I}^\sharp}(i)\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^\sharp} + \chi_{\mathcal{I}^\sharp}(i)\frac{2}{\varepsilon}\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^1_1} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial x^1_i}\frac{\partial w}{\partial x^1_i}\right),$$

for all $i \in \mathcal{I} = \{1,2,3\}$ where $\mathcal{I}^\sharp = \{1\}$.

We check that $B^\varepsilon_{in,1}(\nabla_{x^1}w)\cdot\mathbf{n}^\varepsilon = 0$ on $\Gamma^{\varepsilon,\alpha,vac}_{in,1,ext}$ and a calculation reveals that

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon B^\varepsilon_{in,1}(\Delta_{x^1}w)\,\mathrm{d}x^\varepsilon = \frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}} V^\varepsilon B^\varepsilon_{in,1}\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\,\mathrm{d}s\,(x^\varepsilon) + O(\varepsilon),$$

where

$$\begin{aligned} O(\varepsilon) \;=\; & \int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon B^\varepsilon_{in,1}\left(\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^\sharp}\right)\mathrm{d}x^\varepsilon + \frac{2}{\varepsilon}\int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon B^\varepsilon_{in,1}\left(\frac{\partial}{\partial x^\sharp}\frac{\partial w}{\partial x^1_1}\right)\mathrm{d}x^\varepsilon \\ & -\int_{\partial\Omega^{\varepsilon,\alpha,vac}_{in}} \phi^\varepsilon B^\varepsilon_{in,1}\left(\frac{\partial w}{\partial x^\sharp}\right) n^\varepsilon_1\,\mathrm{d}s(x^\varepsilon). \end{aligned}$$

Thanks to Proposition 6.4, we have

$$\frac{1}{\varepsilon^2}\int_{\Omega^{\varepsilon,\alpha,vac}_{in,1}} \phi^\varepsilon T^{\varepsilon*}_{in,1}\left(\Delta_{x^1} w\right) \mathrm{d}x^\varepsilon = \frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}} V^\varepsilon T^{\varepsilon*}_{in,1}\left(\nabla_{x^1} w \cdot \mathbf{n}^1\right) \mathrm{d}s\left(x^\varepsilon\right) + O(\varepsilon). \tag{6.1}$$

By the definition of $T^{\varepsilon*}_{in,1}$, it follows that

$$\int_{\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1}} T^\varepsilon_{in,1}(\phi^\varepsilon)\Delta_{x^1} w \ \ \mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{in,1}\times\Gamma^{\varepsilon,\alpha,vac}_{in,1,int}} T^\varepsilon_{in,1}(V^\varepsilon)\nabla_{x^1} w \cdot \ \ \mathbf{n}^1 \mathrm{d}x^\sharp \,\mathrm{d}s\left(x^1\right) + O(\varepsilon).$$

Passing ε to 0, combined with Proposition 6.8, we obtain

$$\int_{\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1}} (\phi^{1,\alpha}_{in} + \widetilde{\phi}^0)\Delta_{x^1} w \,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{in}\times\Gamma^{1,vac}_{in,1,int}} (V^{1,\alpha}_{in} + \widetilde{V^0})\nabla_{x^1} w \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s\left(x^1\right).$$

for each α.

It follows that the above equality still holds if w is taken on the form of $\tau_\alpha v$, where τ_α is a smooth truncation function with compact support $\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1}$ and $v \in C^\infty(\Omega^\sharp_{in,1}\times\overline{\Omega^{\infty,vac}_{in,1}}) \cap H^2(\Omega^\sharp_{in,1}\times\overline{\Omega^{\infty,vac}_{in,1}})$, $\Omega^{\infty,vac}_{in,1}$ - periodic in the directions x^1_1, x^1_2, $v = 0$ on $\Gamma^{\infty,vac\pm}_{in,1,int}$, $\nabla_{x^1} v\cdot \ \mathbf{n}^1 = 0$ on $\Gamma^{\infty,vac,\pm}_{in,1,top}\cup\Gamma^{\infty,vac\pm}_{in,1,per}$, $|v|$, $|\nabla_{x^1} v|$, and $|\Delta_{x^1} v|$ exponentially decrease to 0 when $|x^1_2| \to +\infty$, and $\tau_\alpha v \to v$ in $H^2(\Omega^\sharp_{in,1}\times\overline{\Omega^{\infty,vac}_{in,1}})$ when α tends to infinity. Then

$$\int_{\Omega^\sharp_{in,1}\times\Omega^{\infty,vac}_{in,1}} (\phi^{1,\alpha}_{in} + \widetilde{\phi}^0)\chi_{\Omega^{1,vac}_{in,1}}\Delta_{x^1} w \,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{in,1}\times\Gamma^{\infty,vac}_{in,1,int}} (V^{1,\alpha}_{in} + \widetilde{V^0})\chi_{\Omega^{1,vac}_{in,1}}\nabla_{x^1} w \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s(x^1).$$

Passing α to $+\infty$, by Assumption 4.7 , we get

$$\int_{\Omega^\sharp_{in,1}\times\Omega^{\infty,vac}_{in,1}} (\phi^1_{in} + \widetilde{\phi}^0)\Delta_{x^1} v \,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{in,1}\times\Gamma^{\infty,vac}_{in,1,int}} (V^1_{in} + \widetilde{V^0})\chi_{\Omega^{1,vac}_{in,1}}\nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s(x^1).$$

Now, we choose v vanishing out of $\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1}$ for a given α,

$$\int_{\Omega^\sharp_{in,1}\times\Omega^{1,vac}_{in,1}} (\phi^1_{in} + \widetilde{\phi}^0)\Delta_{x^1} v \,\mathrm{d}x^\sharp \mathrm{d}x^1 = \int_{\Omega^\sharp_{in,1}\times\Gamma^{1,vac}_{in,1,int}} (V^1_{in} + \widetilde{V^0})\nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}x^\sharp \mathrm{d}s(x^1).$$

Applying Green's formula twice, then

$$\begin{aligned}
&\sum_\pm \int_{\Omega^\sharp_{in,1}\times\Omega^{1,vac\pm}_{in,1}} \Delta_{x^1}(\phi^{1\pm}_{in} + \widetilde{\phi^0}^\pm)v \quad \mathrm{d}x^\sharp \mathrm{d}x^1 \\
&-\sum_\pm \int_{\Omega^\sharp_{in,1}\times\partial\Omega^{1,vac\pm}_{in,1}} \nabla_{x^1}(\phi^{1\pm}_{in} + \widetilde{\phi^0}^\pm)\cdot\mathbf{n}^{1\pm} v \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) \\
&+\sum_\pm \int_{\Omega^\sharp_{in,1}\times\partial\Omega^{1,vac\pm}_{in,1}} (\phi^{1\pm}_{in} + \widetilde{\phi^0}^\pm)\nabla_{x^1} v\cdot\mathbf{n}^{1\pm} \,\mathrm{d}x^\sharp \mathrm{d}s(x^1) \\
&= \sum_\pm \int_{\Omega^\sharp_{in,1}\times\Gamma^{1,vac\pm}_{in,1,int}} (v^{1\pm}_{in} + \widetilde{V^0}^\pm)\nabla_{x^1} v\cdot\mathbf{n}^{1\pm} \,\mathrm{d}x^\sharp \mathrm{d}s\left(x^1\right).
\end{aligned}$$

Decomposing $\Omega^{1,vac}_{in,1}$ into two parts $\Omega^{1,vac\pm}_{in,1}$ with their boundaries $\partial\Omega^{1,vac,\pm}_{in,1} = \Gamma^{1,vac,\pm}_{in,1,int}\cup\Gamma^{1,vac,\pm}_{in,1,top}\cup\Gamma^{1,vac,\pm}_{bl,1,per}\cup\Gamma^{1,vac\pm}_{in,1,\alpha}\cup\Gamma^{1,vac}_{in,1,\mathrm{interf}}$, combining with the results of Proposition 3.3, $\Delta_{x^1}\widetilde{\phi^0}^\pm = 0$ in $\Omega^{1,vac\pm}_{in,1}$,

$\widetilde{\phi^0}^{\pm} = \widetilde{V^0}^{\pm}$ on $\Gamma^{1,vac\pm}_{in,1,int}$, $\nabla_{x^1}\widetilde{\phi^0}^{\pm}\cdot\mathbf{n}^{1\pm} = 0$ on $\Gamma^{1,vac\pm}_{in,1,top}$, $\nabla_{x^1}\widetilde{\phi^0}^{\pm}\cdot\mathbf{n}^{1\pm}$ is $\Gamma^{1,vac,\pm}_{bl,1,per}$ - antiperiodic, and from the conditions satisfied by v it remains

$$
\begin{aligned}
&\sum_{\pm}\int_{\Omega^{\sharp}_{in,1}\times\Omega^{1,vac\pm}_{in,1}} \Delta_{x^1}(\phi^{1\pm}_{in}) v \,\mathrm{d}x^{\sharp}\mathrm{d}x^1 \\
&-\sum_{\pm}\int_{\Omega^{\sharp}_{in,1}\times(\Gamma^{1,vac\pm}_{in,1,top}\cup\Gamma^{1,vac\pm}_{in,1,per})} \nabla_{x^1}\phi^{1\pm}_{in}\cdot\mathbf{n}^{1\pm} v\,\mathrm{d}x^{\sharp}\mathrm{d}s(x^1) \\
&-\int_{\Omega^{\sharp}_{in,1}\times\Gamma^{1,vac}_{in,1,\mathrm{interf}}} \left[\nabla_{x^1}\left(\phi^{1+}_{in}+\widetilde{\phi^0}^{+}\right)-\nabla_{x^1}\left(\phi^{1-}_{in}+\widetilde{\phi^0}^{-}\right)\right]\cdot\mathbf{n}^{1+} v\,\mathrm{d}x^{\sharp}\mathrm{d}s(x^1) \\
&+\sum_{\pm}\int_{\Omega^{\sharp}_{in,1}\times\Gamma^{1,vac\pm}_{in,1,int}} \phi^{1\pm}_{in}\nabla_{x^1}v\cdot\mathbf{n}^{1\pm}\,\mathrm{d}x^{\sharp}\mathrm{d}s(x^1) \\
&+\int_{\Omega^{\sharp}_{in,1}\times\Gamma^{1,vac}_{in,1,\mathrm{interf}}} \left[(\phi^{1+}_{in}+\widetilde{\phi^0}^{+})-\left(\phi^{1-}_{in}+\widetilde{\phi^0}^{-}\right)\right]\nabla_{x^1}v\cdot\mathbf{n}^{1+}\,\mathrm{d}x^{\sharp}\mathrm{d}s(x^1) \\
&=\sum_{\pm}\int_{\Omega^{\sharp}_{in,1}\times\Gamma^{1,vac\pm}_{in,1,int}} v^{1\pm}_{in}\nabla_{x^1}v\cdot\mathbf{n}^{1\pm}\,\mathrm{d}x^{\sharp}\mathrm{d}s\left(x^1\right).
\end{aligned}
$$

The rest of proof runs as the previous proofs. ■

7 Internal Edge Model

We assume that all interface models are yet built with the index $i = 1, 2, 3, 4$ as in Figure 2. We consider the contributions of two interface models $i = 1$ and $i = 2$ at the first internal edge zone, see Figure 13. Since the sum of contributions is not continuous at this edge, we introduce an internal edge corrector to overcome the lack of continuity. Here, the corrector and the remaining voltage source are

$$
\begin{aligned}
\phi^{\varepsilon}_{ine} &= \phi^{\varepsilon} - B^{\varepsilon}\phi^0 - B^{\varepsilon}_{in,2}\phi^{2-}_{in}\chi_{\Omega^{\varepsilon,\alpha,vac,2}_{ine,1}} - (B^{\varepsilon}_{in,1}\phi^{1+}_{in} + B^{\varepsilon}_{in,2}\phi^{2+}_{in})\chi_{\Omega^{\varepsilon,\alpha,vac,3}_{ine,1}} - B^{\varepsilon}_{in,1}\phi^{1-}_{in}\chi_{\Omega^{\varepsilon,\alpha,vac,4}_{ine,1}}, \\
V^{\varepsilon}_{ine} &= V^{\varepsilon} - B^{\varepsilon}V^0 - B^{\varepsilon}_{in,2}V^{2-}_{in}\chi_{\Omega^{\varepsilon,\alpha,vac,2}_{ine,1}} - (B^{\varepsilon}_{in,1}V^{1+}_{in} + B^{\varepsilon}_{in,2}V^{2+}_{in})\chi_{\Omega^{\varepsilon,\alpha,vac,3}_{ine,1}} - B^{\varepsilon}_{in,1}V^{1-}_{in}\chi_{\Omega^{\varepsilon,\alpha,vac,4}_{ine,1}},
\end{aligned}
$$

where ϕ^0 is the solution of the periodic model, $\phi^{1\pm}_{in}$ and $\phi^{2\pm}_{in}$ are the solutions of the first and second interface models in the interface zones near the first internal edge zone, $B^{\varepsilon}_{in,1}$ and $B^{\varepsilon}_{in,2}$ are the smooth approximation operators of the first and second adjoint interface two-scale operators $T^{\varepsilon*}_{in,1}$ and $T^{\varepsilon*}_{in,2}$, $V^{1\pm}_{in}$ and $V^{2\pm}_{in}$ are the weak limits of $V^{1,\alpha\pm}_{in}\chi_{\Omega^{1,vac\pm}_{in,1}}$ in $L^2(\Omega^{\sharp}_{in,1}\times\Gamma^{\infty,vac\pm}_{in,1,int})$ and of $V^{2,\alpha\pm}_{in}\chi_{\Omega^{1,vac\pm}_{in,2}}$ in $L^2(\Omega^{\sharp}_{in,2}\times\Gamma^{\infty,vac\pm}_{in,2,int})$ when α tends to $+\infty$, $V^{1,\alpha\pm}_{in}$ and $V^{2,\alpha\pm}_{in}$ are the weak limits of $T^{\varepsilon}_{in,1}(V^{\varepsilon}_{in})$ in $L^2\left(\Omega^{\sharp}_{in,1}\times\Gamma^{1,vac\pm}_{in,1,int}\right)$ and of $T^{\varepsilon}_{in,2}(V^{\varepsilon}_{in})$ in $L^2\left(\Omega^{\sharp}_{in,2}\times\Gamma^{1,vac\pm}_{in,2,int}\right)$ when ε tends to 0. The domains $\Omega^{\varepsilon,\alpha,vac,i}_{ine,1}$ is introduced in the next section.

7.1 Geometry Notations

The whole internal edge boundary layer domain $\Omega^{\varepsilon,\alpha}_{ine,1}$, which subscript $ine,1$ refers to the first internal edge, is a subdomain of $\Omega^{\varepsilon}_1\cup\Omega^{\varepsilon}_2$ defined as $\Omega^{\varepsilon,\alpha}_{ine,1} = \cup_{c\in\mathcal{I}_{ine,1}}\Omega^{\varepsilon}_c$. Here $\mathcal{I}_{ine,1}$ is a set of multi-indices $c = (c_1, c_2) : c_1 \in \overline{i_1-\alpha, i_1+\alpha-1}$, and $c_2 \in \overline{i_2-\alpha, i_2+\alpha-1}$, i_1, i_2 being such that $\Omega^{\varepsilon}_{(i_1,i_2)}$ is the first internal edge cell, see Figure 14.

The domain $\Omega^{\varepsilon,\alpha}_{ine,1}$ is decomposed into four nonoverlapping subdomains $\Omega^{\varepsilon,\alpha,i}_{ine,1} = \cup_{c\in\mathcal{I}^i_{ine,1}}\Omega^\varepsilon_c$ with the multi-index sets $\mathcal{I}^i_{ine,1}$

$$
\begin{aligned}
\mathcal{I}^1_{ine,1} &= \left\{(c_1,c_2) : c_1 \in \overline{i_1-\alpha, i_1-1},\ c_2 \in \overline{i_2-\alpha, i_2-1}\right\},\\
\mathcal{I}^2_{ine,1} &= \left\{(c_1,c_2) : c_1 \in \overline{i_1-\alpha, i_1-1},\ c_2 \in \overline{i_2, i_2+\alpha-1}\right\},\\
\mathcal{I}^3_{ine,1} &= \left\{(c_1,c_2) : c_1 \in \overline{i_1, i_1+\alpha-1},\ c_2 \in \overline{i_2, i_2+\alpha-1}\right\},\\
\mathcal{I}^4_{ine,1} &= \left\{(c_1,c_2) : c_1 \in \overline{i_1, i_1+\alpha-1},\ c_2 \in \overline{i_2-\alpha, i_2-1}\right\}.
\end{aligned}
$$

We observe that $\Omega^{\varepsilon,\alpha,i}_{ine,1}$ is a subdomain of Ω^ε_1 for $i = 1,2,4$ and of Ω^ε_2 for $i = 3$. For the sake of concision, interface numbering is with indices modulo 4, e.g. 5 plays the role of 1 and so on. Precisely, the interface between $\Omega^{\varepsilon,\alpha,i}_{ine,1}$ and $\Omega^{\varepsilon,\alpha,i+1}_{ine,1}$ is noted $\Gamma^{\varepsilon,\alpha}_{ine,1,\text{ interf},i+1}$ for $i = 1,2,3,4$ and $\Gamma^{\varepsilon,\alpha}_{ine,1,\text{interf},5}$ or $\Gamma^{\varepsilon,\alpha}_{ine,1,\text{interf},1}$ for $i = 4$. The whole interface is $\Gamma^{\varepsilon,\alpha}_{ine,1,\text{interf}} = \cup_{i=1}^4\Gamma^{\varepsilon,\alpha}_{ine,1,\text{interf},i}$. The boundary $\partial\Omega^{\varepsilon,\alpha,val,i}_{ine,1}$ of $\Omega^{\varepsilon,\alpha,val,i}_{ine,1}$ is decomposed as $\Gamma^{\varepsilon,\alpha,val,i}_{ine,1,int}\cup\Gamma^{\varepsilon,\alpha,val,i}_{ine,1,ext}\cup\Gamma^{\varepsilon,\alpha}_{ine,1,\text{ interf},i}\cup\Gamma^{\varepsilon,\alpha}_{ine,1,\text{ interf},i+1}$. All the other notations for subdomains, boundaries and subboudaries are derived from these definitions with the exceptions $\Gamma^{\varepsilon,\alpha,vac,i}_{ine,1,ext} = \Gamma^{\varepsilon,\alpha,vac,i}_{ine,1,top} \cup \Gamma^{\varepsilon,\alpha,vac,i}_{ine,1,\alpha}$.

The finite microscopic domain $\Omega^1_{ine,1} = \cup_{i=1}^4\Omega^{1,i}_{ine,1}$ is also parametrized by α, with

$$
\begin{aligned}
\Omega^{1,1}_{ine,1} &= \cup_{\xi,\eta=\overline{0,\alpha-1}}(\Omega^1 + (-\xi-1/2, -\eta-1/2, 1/2)),\\
\Omega^{1,2}_{ine,1} &= \cup_{\xi,\eta=\overline{0,\alpha-1}}(\Omega^1 + (-\xi-1/2, \eta+1/2, 1/2)),\\
\Omega^{1,3}_{ine,1} &= \cup_{\xi,\eta=\overline{0,\alpha-1}}(\Omega^1 + (\xi+1/2, \eta+1/2, 1/2)),\\
\Omega^{1,4}_{ine,1} &= \cup_{\xi,\eta=\overline{0,\alpha-1}}(\Omega^1 + (\xi+1/2, -\eta-1/2, 1/2)),
\end{aligned}
$$

see Figure 15.

The notation system built for the physical domain is transposed to the microscopic domain without the need to detail it. The infinite microscopic domain $\Omega^\infty_{ine,1}$ is defined as the limit of $\Omega^1_{ine,1}$ when α tends to infinity.

Finally, for all regular function v defined in $\Omega^1_{in,1}$, we denote v^i the restriction of v to $\Omega^{1,i}_{in,1}$ and $[[v]]$ stands for a jump of v at the interface defined by the following formula

$$
[[v]] = \begin{cases} v^1 - v^4 & \text{at } \Gamma^{1,}_{ine,1,\text{interf},1} \\ v^1 - v^2 & \text{at } \Gamma^{1,vac}_{ine,1,\text{interf},2} \\ v^3 - v^2 & \text{at } \Gamma^{1,vac}_{ine,1,\text{interf},3} \\ v^3 - v^4 & \text{at } \Gamma^{1,vac}_{ine,1,\text{interf},4}. \end{cases}
$$

7.2 Internal Edge Boundary Layer Two-Scale Operator

We consider any surface Γ^1 in $\overline{\Omega^1}$, $\Gamma^1_{ine,1} = \cup_{\sigma\in\{+,-\}}\cup_{\eta=\overline{1,\alpha}}(\Gamma^1 + (0, \sigma(\eta-1/2), 1/2)) \subset \overline{\Omega^1_{ine,1}}$ and $\Gamma^{\varepsilon,\alpha}_{in,1} = \cup_{c\in\mathcal{I}_{in,1}}\varepsilon((c_1-1/2, c_2-1/2, 1/2)+\Gamma^1) \subset \overline{\Omega^{\varepsilon,\alpha}_{in,1}}$. Then in this section the pair (X^ε, X^1) stands both for $(\Omega^{\varepsilon,\alpha}_{in,1}, \Omega^1_{ine,1})$ and for $(\Gamma^{\varepsilon,\alpha}_{in,1}, \Gamma^1_{ine,1})$. Now we introduce the dilation operator $T^\varepsilon_{ine,1}$ at the first internal edge.

Definition 7.1 *The operator $T^\varepsilon_{ine,1}$ operating on functions φ with variable in X^ε is defined by*

$$
T^\varepsilon_{ine,1}(\varphi)(x^1) = \varphi(\varepsilon x^1_1 + L^1_1, \varepsilon x^1_2 + L^2_1, \varepsilon x^1_3)
$$

for $x^1 \in X^1$ where $L^1_1 = i_1\varepsilon$ and $L^2_1 = i_2\varepsilon$ for some $i_1, i_2 \in \mathbb{Z}^+$.

Here the operator $T^{\varepsilon *}_{ine,1} = (T^{\varepsilon}_{ine,1})^{-1}$ i.e.

$$T^{\varepsilon *}_{ine,1}(\psi)\,(x^{\varepsilon}) = \psi\left(\frac{x_1^{\varepsilon} - L_1^1}{\varepsilon}, \frac{x_2^{\varepsilon} - L_1^2}{\varepsilon}, \frac{x_3^{\varepsilon}}{\varepsilon}\right).$$

Property 7.2 *The operator $T^{\varepsilon *}_{ine,1}$ is the adjoint of $T^{\varepsilon}_{ine,1}$ in the sense*

$$\frac{1}{\varepsilon^3}\int_{\Omega^{\varepsilon,\alpha}_{ine,1}} \varphi T^{\varepsilon *}_{ine,1}(\psi)\,\mathrm{d}x^{\varepsilon} = \int_{\Omega^1_{ine,1}} T^{\varepsilon}_{ine,1}(\varphi)\psi\;\,\mathrm{d}x^1,$$

for all $\psi \in L^2(\Omega^1_{ine,1})$, $\varphi \in L^2(\Omega^{\varepsilon,\alpha}_{ine,1})$ and in the sense

$$\frac{1}{\varepsilon^2}\int_{\Gamma^{\varepsilon,\alpha}_{ine,1}} \varphi T^{\varepsilon *}_{ine,1}(\psi)\,\mathrm{d}s(x^{\varepsilon}) = \int_{\Gamma^1_{ine,1}} T^{\varepsilon}_{ine,1}(\varphi)\psi\,\mathrm{d}s(x^1),$$

for all $\psi \in L^2(\Gamma^1_{ine,1})$, $\varphi \in L^2(\Gamma^{\varepsilon,\alpha}_{ine,1})$.

In this internal edge case, the operator $T^{\varepsilon *}_{ine,1}$ and its approximation $B^{\varepsilon}_{ine,1}$ are identical however both will be used to follow the algorithm of Section 2.5.

Proposition 7.3 *Let B^{ε}, $B^{\varepsilon}_{in,1}$ and $B^{\varepsilon}_{in,2}$ be the smooth approximation operators of $T^{\varepsilon *}$, $T^{\varepsilon *}_{in,1}$ and $T^{\varepsilon *}_{in,2}$, then*

1. *If a function ψ with variables in $(\Omega_1^{\sharp} \cup \Omega_2^{\sharp}) \times \Omega^1$, respectively in $(\Omega_1^{\sharp} \cup \Omega_2^{\sharp}) \times \Gamma^1$, is continuous w.r.t. its first variable and is Ω^1 - periodic in the directions x_1^1, x_2^1 then*

$$T^{\varepsilon}_{ine,1}(B^{\varepsilon}\psi)(x^1) \to \widetilde{\psi}(x^1) \textit{ for } x^1 \textit{ in } \Omega^1_{ine,1}, \textit{ respect. in } \Gamma^1_{ine,1} \textit{ when } \varepsilon \to 0,$$

   where $\widetilde{\psi}(x^1) = \psi((L_1^1, L_1^2), x^1 - 1/2))$.

2. *If a function $\psi^{\pm}$ with variables in $\Omega^{\sharp}_{in,1} \times \Omega^{\infty\pm}_{in,1}$, respectively in $\Omega^{\sharp}_{in,1} \times \Gamma^{\infty\pm}_{in,1}$, is continuous w.r.t. its first variable and is $\Omega^{\infty\pm}_{in,1}$ - periodic in the direction x_1^1 then*

$$\begin{aligned} T^{\varepsilon}_{ine,1}(B^{\varepsilon}_{in,1}\psi^{+})(x^1) &\to \widetilde{\psi^{+}}(x^1) \textit{ for } x^1 \textit{ in } \Omega^{1,3}_{ine,1}, \textit{ respect. in } \Gamma^1_{ine,1} \cap \overline{\Omega^{1,3}_{ine,1}}, \\ \textit{and } T^{\varepsilon}_{ine,1}(B^{\varepsilon}_{in,1}\psi^{-})(x^1) &\to \widetilde{\psi^{-}}(x^1) \textit{ for } x^1 \in \Omega^{1,4}_{ine,1}, \textit{ respect. in } \Gamma^1_{ine,1} \cap \overline{\Omega^{1,4}_{ine,1}}, \end{aligned}$$

   when $\varepsilon \to 0$, where $\widetilde{\psi^{\pm}}(x^1) = \psi^{\pm}(L_1^1, (x_1^1 - 1/2, x_2^1, x_3^1))$.

3. *If a function $\psi^{\pm}$ with variables in $\Omega^{\sharp}_{in,2} \times \Omega^{\infty\pm}_{in,2}$, respectively in $\Omega^{\sharp}_{in,2} \times \Gamma^{\infty\pm}_{in,2}$, continuous w.r.t. its first variable and is $\Omega^{\infty\pm}_{in,2}$ - periodic in the direction x_2^1 then*

$$\begin{aligned} T^{\varepsilon}_{ine,1}(B^{\varepsilon}_{in,2}\psi^{+})(x^1) &\to \widetilde{\psi^{+}}(x^1) \textit{ for } x^1 \in \overline{\Omega^{1,3}_{ine,1}}, \textit{ respect. in } \Gamma^1_{ine,1} \cap \overline{\Omega^{1,3}_{ine,1}}, \\ \textit{and } T^{\varepsilon}_{ine,1}(B^{\varepsilon}_{in,2}\psi^{-})(x^1) &\to \widetilde{\psi^{-}}(x^1) \textit{ for } x^1 \in \overline{\Omega^{1,2}_{ine,1}}, \textit{ respect. in } \Gamma^1_{ine,1} \cap \overline{\Omega^{1,2}_{ine,1}}, \end{aligned}$$

   when $\varepsilon \to 0$, where $\widetilde{\psi^{\pm}}(x^1) = \psi^{\pm}(L_1^2, (x_1^1, x_2^1 - 1/2, x_3^1))$.

7.3 Derivation of an Internal Edge Model

The following assumptions are supposed to be fulfilled in the next propositions.

Assumption 7.4 *We assume that*

1. *For each α, there exist $\phi^{1,\alpha}_{ine}$ in $H^1(\Omega^{1,vac}_{ine,1})$ and $v^{1,\alpha}_{ine}$ in $L^2(\Gamma^{1,vac}_{ine,1,int})$ such that $T^{\varepsilon}_{ine,1}(\phi^{\varepsilon}_{ine}) \rightharpoonup \phi^{1,\alpha}_{ine}$ weakly in $L^2(\Omega^{1,vac}_{ine,1})$ and $T^{\varepsilon}_{ine,1}(v^{\varepsilon}_{ine}) \rightharpoonup v^{1,\alpha}_{ine}$ weakly in $L^2(\Gamma^{1,vac}_{ine,1,int})$.*
2. *There exist ϕ^{1}_{ine} in $H^1(\Omega^{\infty,vac}_{ine,1})$, ϕ^{1}_{ine} and its gradient converge exponentially fast to zero when $|x^1_1| + |x^1_2| \to +\infty$, and v^1_{ine} in $L^2(\Gamma^{\infty,vac}_{ine,1,int})$ such that $\phi^{1,\alpha}_{ine}\chi_{\Omega^{1,vac}_{ine,1}} \rightharpoonup \phi^1_{ine}$ weakly in $L^2(\Omega^{\infty,vac}_{ine,1})$ and $v^{1,\alpha}_{ine}\chi_{\Omega^{1,vac}_{ine,1}} \rightharpoonup v^1_{ine}$ weakly in $L^2(\Gamma^{\infty,vac}_{ine,1,int})$.*

Assumption 7.5 *The limits ϕ^0, V^0 satisfy the assumption of Proposition 7.3.1. Similarly, $\phi^{1\pm}_{in}$, $V^{1\pm}_{in}$ and $\phi^{2\pm}_{in}$, $V^{2\pm}_{in}$ satisfy the assumption of Proposition 7.3.2 and 7.3.3.*

Proposition 7.6 *When $\varepsilon \to 0$,*

$$T^{\varepsilon}_{ine,1}(\phi^{\varepsilon}) \rightharpoonup \phi^{1,\alpha}_{ine} + \widetilde{\phi^0} + \widetilde{\phi^2_{in}}\chi_{\Omega^{1,vac,2}_{ine,1}\cup\Omega^{1,vac,3}_{ine,1}} + \widetilde{\phi^1_{in}}\chi_{\Omega^{1,vac,3}_{ine,1}\cup\Omega^{1,vac,4}_{ine,1}}$$

weakly in $L^2(\Omega^{1,vac}_{ine,1})$ and

$$T^{\varepsilon}_{ine,1}(V^{\varepsilon}) \rightharpoonup V^{1,\alpha}_{ine} + \widetilde{V^0} + \widetilde{V^2_{in}}\chi_{\Omega^{1,vac,2}_{ine,1}\cup\Omega^{1,vac,3}_{ine,1}} + \widetilde{V^1_{in}}\chi_{\Omega^{1,vac,3}_{ine,1}\cup\Omega^{1,vac,4}_{ine,1}}$$

weakly in $L^2(\Gamma^{1,vac}_{ine,1,int})$, where $\widetilde{\phi^0}(x^1) = \phi^0((L^1_1, L^2_1), x^1 - 1/2)$, $\widetilde{\phi^1_{in}}(x^1) = \phi^1_{in}(L^2_1, (x^1_1 - 1/2, x^1_2, x^1_3))$, and $\widetilde{\phi^2_{in}}(x^1) = \phi^2_{in}(L^1_1, (x^1_1, x^1_2 - 1/2, x^1_3))$ and with similar expressions for the voltage sources.

Proposition 7.7 *The limit ϕ^1_{ine} is a solution to*

$$\begin{cases} -\Delta_{x^1}\phi^1_{ine} = 0 & \text{in } \Omega^{\infty,vac}_{ine,1} \\ \phi^1_{ine} = V^1_{ine} & \text{on } \Gamma^{\infty,vac}_{ine,1,int} \\ \nabla_{x^1}\phi^1_{ine}\cdot\mathbf{n}^1 = 0 & \text{on } \Gamma^{\infty,vac}_{ine,1,top} \\ [[\phi^1_{ine}]] = \widetilde{\phi^{1-}_{in}} & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,1} \\ [[\nabla_{x^1}\phi^1_{ine}]]\cdot\mathbf{n}^1 = \nabla_{x^1}\widetilde{\phi^{1-}_{in}}\cdot\mathbf{n}^1 & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,1} \\ [[\phi^1_{ine}]] = \widetilde{\phi^{2-}_{in}} & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,2} \\ [[\nabla_{x^1}\phi^1_{ine}]]\cdot\mathbf{n}^1 = \nabla_{x^1}\widetilde{\phi^{2-}_{in}}\cdot\mathbf{n}^1 & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,2}. \\ [[\phi^1_{ine}]] = -\ \widetilde{\phi^{1+}_{in}} & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,3} \\ [[\nabla_{x^1}\phi^1_{ine}]]\cdot\mathbf{n}^1 = -\nabla_{x^1}\widetilde{\phi^{1+}_{in}}\cdot\mathbf{n}^1 & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,3} \\ [[\phi^1_{ine}]] = -\widetilde{\phi^{2+}_{in}} & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,4} \\ [[\nabla_{x^1}\phi^1_{ine}]]\cdot\mathbf{n}^1 = -\nabla_{x^1}\widetilde{\phi^{2+}_{in}}\cdot\mathbf{n}^1 & \text{on } \Gamma^{\infty,vac}_{ine,1,interf,4}. \end{cases}$$

Proof. The main idea of the proof is the same as for the other models. Firstly, we replace v^ε in (2.2) by a smooth function v^ε_{ine} defined in $\Omega^{\varepsilon,\alpha,vac}_{ine,1}$ and vanishing out of $\Omega^{\varepsilon,\alpha,vac}_{ine,1}$, then

$$\int_{\Omega^{\varepsilon,\alpha,vac}_{ine,1}} \phi^\varepsilon \Delta_{x^\varepsilon} v^\varepsilon_{ine}\,\mathrm{d}x^\varepsilon = \int_{\Gamma^{\varepsilon,\alpha,vac}_{ine,1,int}} V^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{ine}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right) + \int_{\Gamma^{\varepsilon,\alpha,vac}_{ine,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} v^\varepsilon_{ine}\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right).$$

After that, we substitute v^ε_{ine} by $\varepsilon^{-1}B^\varepsilon_{ine,1}(w)$ where w is in $C^\infty(\overline{\Omega^{1,vac}_{exe,1}})$ such that $w=0$ on $\Gamma^{1,vac}_{ine,1,int}\cup\Gamma^{1,vac}_{ine,1,\alpha}$ and $\nabla_{x^1}w\cdot\mathbf{n}^1=0$ on $\Gamma^{1,vac}_{ine,1,top}\cup\Gamma^{1,vac}_{ine,1,\alpha}$, hence

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Omega^{\varepsilon,\alpha,vac}_{ine,1}} \phi^\varepsilon \Delta_{x^\varepsilon} B^\varepsilon_{ine,1}(w)\,\mathrm{d}x^\varepsilon &= \frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{ine,1,int}} V^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{ine,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right)\\ &+\frac{1}{\varepsilon}\int_{\Gamma^{\varepsilon,\alpha,vac}_{ine,1,ext}} \phi^\varepsilon \nabla_{x^\varepsilon} B^\varepsilon_{ine,1}(w)\cdot\mathbf{n}^\varepsilon\,\mathrm{d}s\left(x^\varepsilon\right).\end{aligned}$$

Obviously,

$$\frac{\partial B^\varepsilon_{ine,1}w}{\partial x^\varepsilon_i} = \frac{1}{\varepsilon}B^\varepsilon_{ine,1}\left(\frac{\partial w}{\partial x^1_i}\right) \text{ and } \frac{\partial}{\partial x^\varepsilon_i}\frac{\partial B^\varepsilon_{ine,1}w}{\partial x^\varepsilon_i} = \frac{1}{\varepsilon^2}B^\varepsilon_{ine,1}\left(\frac{\partial}{\partial x^1_i}\frac{\partial w}{\partial x^1_i}\right),$$

for all $i=1,2,3$, and $B^\varepsilon_{ine,1}(\nabla_{x^1}w)\cdot\mathbf{n}^\varepsilon=0$ on $\Gamma^{\varepsilon,\alpha,vac}_{ine,1,ext}$. Thus,

$$\frac{1}{\varepsilon^3}\int_{\Omega^{\varepsilon,\alpha,vac}_{ine,1}} \phi^\varepsilon B^\varepsilon_{ine,1}\left(\Delta_{x^1}w\right)\,\mathrm{d}x^\varepsilon = \frac{1}{\varepsilon^2}\int_{\Gamma^{\varepsilon,\alpha,vac}_{ine,1,int}} V^\varepsilon B^\varepsilon_{ine,1}\left(\nabla_{x^1}w\cdot\mathbf{n}^1\right)\,\mathrm{d}s\left(x^\varepsilon\right).$$

Replacing $B^\varepsilon_{ine,1}$ by $T^{\varepsilon*}_{ine,1}$, then transposing $T^{\varepsilon*}_{ine,1}$ to $T^\varepsilon_{ine,1}$, we have

$$\int_{\Omega^{1,vac}_{ine,1}} T^\varepsilon_{ine,1}(\phi^\varepsilon)\Delta_{x^1}w\,\mathrm{d}x^1 = \int_{\Gamma^{1,vac}_{ine,1,int}} T^\varepsilon_{ine,1}(V^\varepsilon)\nabla_{x^1}w\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right).$$

Decomposing $\Omega^{1,vac}_{ine,1}=\cup_{i=1}^4\Omega^{1,vac,i}_{ine,1}$ and $\Gamma^{1,vac}_{ine,1}=\cup_{i=1}^4\Gamma^{1,vac,i}_{ine,1,int}$ the above equality becomes

$$\textstyle\sum_{i=1}^4 \displaystyle\int_{\Omega^{1,vac,i}_{ine,1}} T^\varepsilon_{ine,1}(\phi^\varepsilon)\Delta_{x^1}w\,\mathrm{d}x^1 = \textstyle\sum_{i=1}^4 \displaystyle\int_{\Gamma^{1,vac,i}_{ine,1,int}} T^\varepsilon_{ine,1}(V^\varepsilon)\nabla_{x^1}w\cdot\mathbf{n}^{1,i}\,\mathrm{d}s\left(x^1\right).$$

Passing ε to 0, and combining with Proposition 7.6, gives

$$\begin{aligned}l.h.s &= \int_{\Omega^{1,vac,1}_{ine,1}}\left(\phi^{1,\alpha,1}_{ine}+\widetilde{\phi^{0-}}\right)\Delta_{x^1}w\,\mathrm{d}x^1 + \int_{\Omega^{1,vac,2}_{ine,1}}\left(\phi^{1,\alpha,2}_{ine}+\widetilde{\phi^{0-}}+\widetilde{\phi^{2-}_{in}}\right)\Delta_{x^1}w\,\mathrm{d}x^1\\ &+\int_{\Omega^{1,vac,3}_{ine,1}}\left(\phi^{1,\alpha,3}_{ine}+\widetilde{\phi^{0+}}+\widetilde{\phi^{1+}_{in}}+\widetilde{\phi^{2+}_{in}}\right)\Delta_{x^1}w\,\mathrm{d}x^1 + \int_{\Omega^{1,vac,4}_{ine,1}}\left(\phi^{1,\alpha,4}_{ine}+\widetilde{\phi^{0-}}+\widetilde{\phi^{1-}_{in}}\right)\Delta_{x^1}w\,\mathrm{d}x^1\end{aligned}$$

and

$$\begin{aligned}r.h.s &= \int_{\Gamma^{1,vac,1}_{ine,1}}\left(V^{1,\alpha,1}_{ine}+\widetilde{V^{0-}}\right)\nabla_{x^1}w\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right) + \int_{\Gamma^{1,vac,2}_{ine,1}}\left(V^{1,\alpha,2}_{ine}+\widetilde{V^{0-}}+\widetilde{V^{2-}_{in}}\right)\nabla_{x^1}w\cdot\mathbf{n}^1\;\mathrm{d}s\left(x^1\right)\\ &+\int_{\Gamma^{1,vac,3}_{ine,1}}\left(V^{1,\alpha,3}_{ine}+\widetilde{V^{0+}}+\widetilde{V^{1+}_{in}}+\widetilde{V^{2+}_{in}}\right)\nabla_{x^1}w\cdot\mathbf{n}^1\,\mathrm{d}s\left(x^1\right)\\ &+\int_{\Gamma^{1,vac,4}_{ine,1}}\left(V^{1,\alpha,4}_{ine}+\widetilde{V^{0-}}+\widetilde{V^{1-}_{in}}\right)\nabla_{x^1}w\cdot\mathbf{n}^1\quad\mathrm{d}s\left(x^1\right).\end{aligned}$$

It follows that these above equalities still hold if w is taken on the form of $\tau_\alpha v$, where $v \in C^\infty(\overline{\Omega^{\infty,vac}_{ine,1}}) \cap H^2(\overline{\Omega^{\infty,vac}_{ine,1}})$, $v = 0$ on $\Gamma^{\infty,vac}_{ine,1,int}$, $v = 0$ on $\Gamma^{\infty,vac}_{ine,1,int}$ and $\nabla_{x^1} v \cdot \ \mathbf{n}^1 = 0$ on $\Gamma^{\infty,vac}_{ine,1,top}$, $|v|$, $|\nabla_{x^1} v|$, and $|\Delta_{x^1} v|$ exponentially decrease to 0 when $|x^1_1| + |x^1_2| \to +\infty$, and τ_α is a smooth truncation function with compact support $\Omega^{1,vac}_{ine,1}$. Then

$$\begin{aligned} l.h.s =& \int_{\Omega^{\infty,vac,1}_{ine,1}} \left(\phi^{1,\alpha,1}_{ine} + \widetilde{\phi^{0-}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \Delta_{x^1} \tau_\alpha v \, \mathrm{d}x^1 + \int_{\Omega^{\infty,vac,2}_{ine,1}} \left(\phi^{1,\alpha,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \Delta_{x^1} \tau_\alpha v \, \mathrm{d}x^1 \\ &+ \int_{\Omega^{\infty,vac,3}_{ine,1}} \left(\phi^{1,\alpha,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \Delta_{x^1} \tau_\alpha v \, \mathrm{d}x^1 \\ &+ \int_{\Omega^{\infty,vac,4}_{ine,1}} \left(\phi^{1,\alpha,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \Delta_{x^1} \tau_\alpha v \, \mathrm{d}x^1, \end{aligned}$$

and

$$\begin{aligned} r.h.s =& \int_{\Gamma^{\infty,vac,1}_{ine,1}} \left(V^{1,\alpha,1}_{ine} + \widetilde{V^{0-}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \nabla_{x^1} \tau_\alpha v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right) \\ &+ \int_{\Gamma^{\infty,vac,2}_{ine,1}} \left(V^{1,\alpha,2}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{2-}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \nabla_{x^1} \tau_\alpha v \cdot \mathbf{n}^1 \ \mathrm{d}s\left(x^1\right) \\ &+ \int_{\Gamma^{\infty,vac,3}_{ine,1}} \left(V^{1,\alpha,3}_{ine} + \widetilde{V^{0+}} + \widetilde{V^{1+}_{in}} + \widetilde{V^{2+}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \nabla_{x^1} \tau_\alpha v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right) \\ &+ \int_{\Gamma^{\infty,vac,4}_{ine,1}} \left(V^{1,\alpha,4}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{1-}_{in}}\right) \chi_{\Omega^{1,vac}_{ine,1}} \nabla_{x^1} \tau_\alpha v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right). \end{aligned}$$

Passing α to $+\infty$, by Assumption 7.4,

$$\begin{aligned} l.h.s =& \int_{\Omega^{\infty,vac,1}_{ine,1}} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}}\right) \Delta_{x^1} v \, \mathrm{d}x^1 + \int_{\Omega^{\infty,vac,2}_{ine,1}} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1 \\ &+ \int_{\Omega^{\infty,vac,3}_{ine,1}} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1 + \int_{\Omega^{\infty,vac,4}_{ine,1}} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1, \end{aligned}$$

and

$$\begin{aligned} r.h.s =& \int_{\Gamma^{\infty,vac,1}_{ine,1}} \left(V^{1,1}_{ine} + \widetilde{V^{0-}}\right) \nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right) + \int_{\Gamma^{\infty,vac,2}_{ine,1}} \left(V^{1,2}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{2-}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^1 \ \mathrm{d}s\left(x^1\right) \\ &+ \int_{\Gamma^{\infty,vac,3}_{ine,1}} \left(V^{1,3}_{ine} + \widetilde{V^{0+}} + \widetilde{V^{1+}_{in}} + \widetilde{V^{2+}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^1 \, \mathrm{d}s\left(x^1\right) \\ &+ \int_{\Gamma^{\infty,vac,4}_{ine,1}} \left(V^{1,4}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{1-}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^1 \quad \mathrm{d}s\left(x^1\right). \end{aligned}$$

Now, we choose v vanishing out of $\Omega^{1,vac}_{ine,1}$ for a given α,

$$\begin{aligned} l.h.s =& \int_{\Omega^{1,vac,1}_{ine,1}} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}}\right) \Delta_{x^1} v \, \mathrm{d}x^1 + \int_{\Omega^{1,vac,2}_{ine,1}} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1 \\ &+ \int_{\Omega^{1,vac,3}_{ine,1}} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1 + \int_{\Omega^{1,vac,4}_{ine,1}} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}}\right) \Delta_{x^1} v \mathrm{d}x^1 \end{aligned}$$

that we note

$$= T_1 + T_2 + T_3 + T_4,$$

and

$$r.h.s = \int_{\Gamma^{1,vac,1}_{ine,1}} \left(V^{1,1}_{ine} + \widetilde{V^{0-}} \right) \nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}s\left(x^1\right) + \int_{\Gamma^{1,vac,2}_{ine,1}} \left(V^{1,2}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{2-}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^1 \;\mathrm{d}s\left(x^1\right)$$
$$+ \int_{\Gamma^{1,vac,3}_{ine,1}} \left(V^{1,3}_{ine} + \widetilde{V^{0+}} + \widetilde{V^{1+}_{in}} + \widetilde{V^{2+}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}s\left(x^1\right)$$
$$+ \int_{\Gamma^{1,vac,4}_{ine,1}} \left(V^{1,4}_{ine} + \widetilde{V^{0-}} + \widetilde{V^{1-}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^1 \,\mathrm{d}s\left(x^1\right).$$

Applying Green's formula twice to each term T_i yields,

$$T_1 = \int_{\Omega^{1,vac,1}_{ine,1}} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}} \right) \Delta_{x^1} v \,\mathrm{d}x^1$$
$$= \int_{\Omega^{1,vac,1}_{ine,1}} \Delta_{x^1} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}} \right) v \,\mathrm{d}x^1 + \int_{\partial\Omega^{1,vac,1}_{ine,1}} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}} \right) \nabla_{x^1} v \cdot \mathbf{n}^{1,1} \mathrm{d}s(x^1)$$
$$- \int_{\partial\Omega^{1,vac,1}_{ine,1}} v \nabla_{x^1} \left(\phi^{1,1}_{ine} + \widetilde{\phi^{0-}} \right) \cdot \mathbf{n}^{1,1} \mathrm{d}s(x^1),$$

$$T_2 = \int_{\Omega^{1,vac,2}_{ine,1}} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}} \right) \Delta_{x^1} v \,\mathrm{d}x^1$$
$$= \int_{\Omega^{1,vac,2}_{ine,1}} \Delta_{x^1} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}} \right) v \,\mathrm{d}x^1 + \int_{\partial\Omega^{1,vac,2}_{ine,1}} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^{1,2} \mathrm{d}s(x^1)$$
$$- \int_{\partial\Omega^{1,vac,2}_{ine,1}} v \nabla_{x^1} \left(\phi^{1,2}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{2-}_{in}} \right) \cdot \mathbf{n}^{1,2} \mathrm{d}s(x^1),$$

$$T_3 = \int_{\Omega^{1,vac,3}_{ine,1}} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}} \right) \Delta_{x^1} v \,\mathrm{d}x^1$$
$$= \int_{\Omega^{1,vac,3}_{ine,1}} \Delta_{x^1} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}} \right) v \,\mathrm{d}x^1 + \int_{\partial\Omega^{1,vac,3}_{ine,1}} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^{1,3} \mathrm{d}s(x^1)$$
$$- \int_{\partial\Omega^{1,vac,3}_{ine,1}} v \nabla_{x^1} \left(\phi^{1,3}_{ine} + \widetilde{\phi^{0+}} + \widetilde{\phi^{1+}_{in}} + \widetilde{\phi^{2+}_{in}} \right) \cdot \mathbf{n}^{1,3} \mathrm{d}s(x^1),$$

and

$$T_4 = \int_{\Omega^{1,vac,4}_{ine,1}} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}} \right) \Delta_{x^1} v \,\mathrm{d}x^1$$
$$= \int_{\Omega^{1,vac,4}_{ine,1}} \Delta_{x^1} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}} \right) v \,\mathrm{d}x^1 + \int_{\partial\Omega^{1,vac,4}_{ine,1}} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}} \right) \nabla_{x^1} v \cdot \mathbf{n}^{1,4} \mathrm{d}s(x^1)$$
$$- \int_{\partial\Omega^{1,vac,4}_{ine,1}} v \nabla_{x^1} \left(\phi^{1,4}_{ine} + \widetilde{\phi^{0-}} + \widetilde{\phi^{1-}_{in}} \right) \cdot \mathbf{n}^{1,4} \mathrm{d}s(x^1).$$

Decomposing each $\partial\Omega^{1,vac,i}_{ine,1} = \Gamma^{1,vac,i}_{ine,1,int} \cup \Gamma^{1,vac,i}_{ine,1,top} \cup \Gamma^{1,vac,i}_{ine,1,\alpha} \cup \Gamma^{1,vac}_{ine,1,\text{interf},i} \cup \Gamma^{1,vac}_{ine,1,\text{interf},i+1}$ for $i = 1, 2, 3, 4$ and combining with the conditions satisfied by v, with the results from Proposition 3.3 and with Proposition 6.9 it follows that $\Delta_{x^1}\widetilde{\phi^{0\pm}} = 0$ in $\Omega^{1,vac}_{ine,1}$, $\Delta_{x^1}\widetilde{\phi^{1+}_{in}} = 0$ in $\Omega^{1,vac,3}_{ine,1}$, $\Delta_{x^1}\widetilde{\phi^{1-}_{in}} = 0$ in $\Omega^{1,vac,4}_{ine,1}$, $\Delta_{x^1}\widetilde{\phi^{2+}_{in}} = 0$ in $\Omega^{1,vac,3}_{ine,1}$, $\Delta_{x^1}\widetilde{\phi^{2-}_{in}} = 0$ in $\Omega^{1,vac,2}_{ine,1}$, $\nabla_{x^1}\widetilde{\phi}^0 \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac}_{ine,1,top}$, $\nabla_{x^1}\widetilde{\phi^{1+}_{in}} \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac,3}_{ine,1,top}$, $\nabla_{x^1}\widetilde{\phi^{1-}_{in}} \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac,4}_{ine,1,top}$, $\nabla_{x^1}\widetilde{\phi^{2+}_{in}} \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac,3}_{ine,1,top}$, $\nabla_{x^1}\widetilde{\phi^{2-}_{in}} \cdot \mathbf{n}^1 = 0$ on $\Gamma^{1,vac,2}_{ine,1,top}$, $\left[\left[\widetilde{\phi}^0 + \widetilde{\phi^1_{in}}\right]\right] = \left[\left[\nabla_{x^1}\widetilde{\phi}^0 + \nabla_{x^1}\widetilde{\phi^1_{in}}\right]\right] \cdot \mathbf{n}^{1,3} = 0$ on $\Gamma^{1,vac}_{ine,1,\text{ interf},4}$, $\left[\left[\widetilde{\phi}^0 + \widetilde{\phi^2_{in}}\right]\right] = \left[\left[\nabla_{x^1}\widetilde{\phi}^0 + \nabla_{x^1}\widetilde{\phi^2_{in}}\right]\right] \cdot \mathbf{n}^{1,3} = 0$ on $\Gamma^{1,vac}_{ine,1,\text{ interf},3}$, thus we get

$$
\begin{aligned}
&\sum_{i=1}^{4} \int_{\Omega^{1,vac,i}_{ine,1}} \Delta_{x^1}\left(\phi^{1,i}_{ine}\right) v \, \mathrm{d}x^1 - \sum_{i=1}^{4} \int_{\Gamma^{1,vac,i}_{ine,1,top}} v \nabla_{x^1}\phi^{1,i}_{ine} \cdot \mathbf{n}^{1,i} \mathrm{d}s(x^1) \\
&+ \int_{\Gamma^{1,vac}_{ine,1,\text{interf},1}} \left(\phi^{1,1}_{ine} - \phi^{1,4}_{ine} - \widetilde{\phi^{1-}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^{1,1} - v\left[\nabla_{x^1}(\phi^{1,1}_{ine} - \phi^{1,4}_{ine}) - \nabla_{x^1}\widetilde{\phi^{1-}_{in}}\right] \cdot \mathbf{n}^{1,1} \mathrm{d}s(x^1) \\
&+ \int_{\Gamma^{1,vac}_{ine,1,\text{interf},2}} \left(\phi^{1,1}_{ine} - \phi^{1,2}_{ine} - \widetilde{\phi^{2-}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^{1,1} - v\left[\nabla_{x^1}(\phi^{1,1}_{ine} - \phi^{1,2}_{ine}) - \nabla_{x^1}\widetilde{\phi^{2-}_{in}}\right] \cdot \mathbf{n}^{1,1} \mathrm{d}s(x^1) \\
&+ \int_{\Gamma^{1,vac}_{ine,1,\text{interf},3}} \left(\phi^{1,3}_{ine} - \phi^{1,2}_{ine} + \widetilde{\phi^{1+}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^{1,3} - v\left[\nabla_{x^1}(\phi^{1,3}_{ine} - \phi^{1,2}_{ine}) + \nabla_{x^1}\widetilde{\phi^{1+}_{in}}\right] \cdot \mathbf{n}^{1,3} \mathrm{d}s(x^1) \\
&+ \int_{\Gamma^{1,vac}_{ine,1,\text{interf},4}} \left(\phi^{1,3}_{ine} - \phi^{1,4}_{ine} + \widetilde{\phi^{2+}_{in}}\right) \nabla_{x^1} v \cdot \mathbf{n}^{1,3} - v\left[\nabla_{x^1}(\phi^{1,3}_{ine} - \phi^{1,4}_{ine}) + \nabla_{x^1}\widetilde{\phi^{2+}_{in}}\right] \cdot \mathbf{n}^{1,3} \mathrm{d}s(x^1) \\
&+ \sum_{i=1}^{4} \int_{\Gamma^{1,vac,i}_{ine,1,int}} \phi^{1,i}_{ine} \nabla_{x^1} v \cdot \mathbf{n}^{1,i} \mathrm{d}s(x^1) = \sum_{i=1}^{4} \int_{\Gamma^{1,vac,i}_{ine,1,int}} V^{1,i}_{ine} \nabla_{x^1} v \cdot \mathbf{n}^{1,i} \mathrm{d}s(x^1)
\end{aligned}
$$

The rest of the proof runs similarly as the proofs of the previous models. ■

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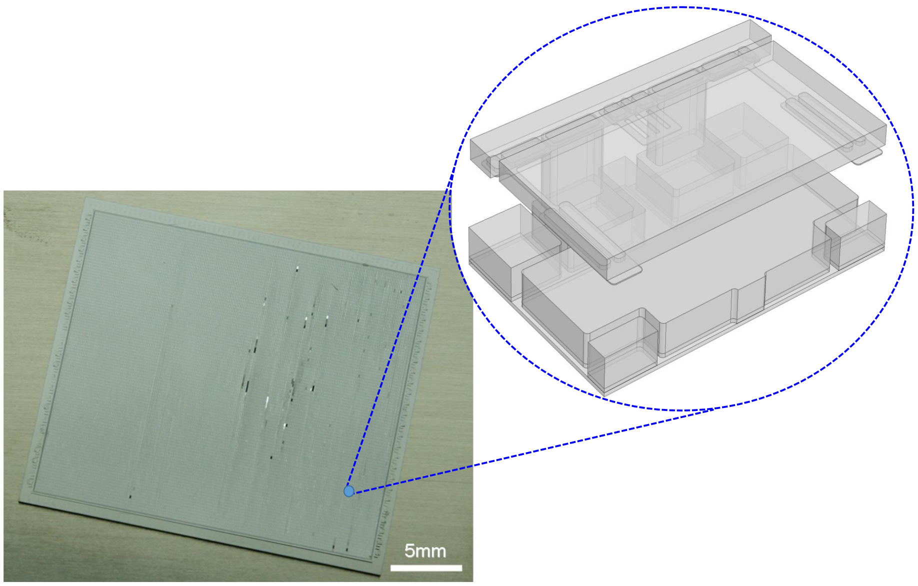

Figure 1: Top view of the MIRA array with 100×200 cells. The zoom represents a single cell.

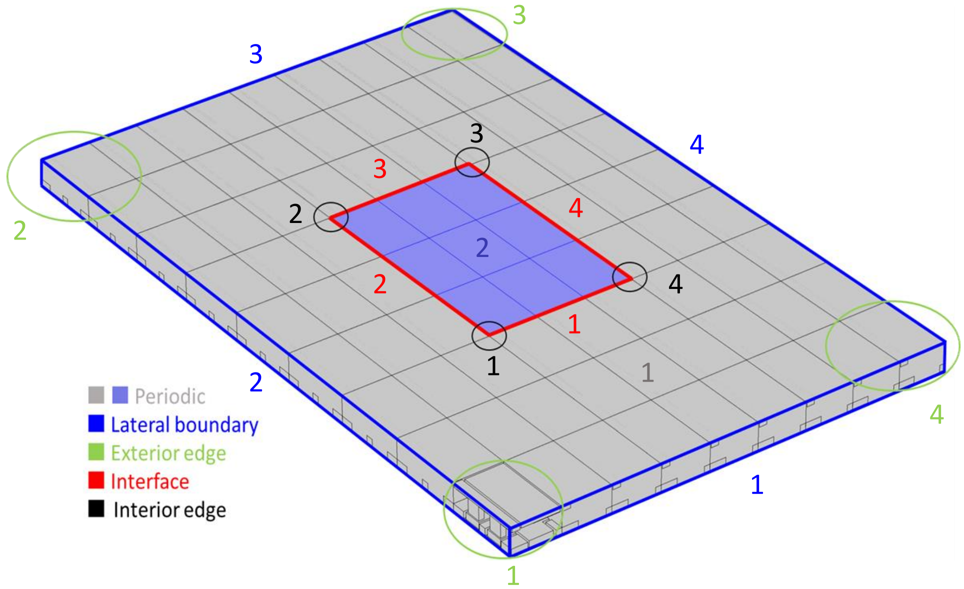

Figure 2: Zones where the asymptotic models are taken into account. The corresponding color numbers indicate the models'index.

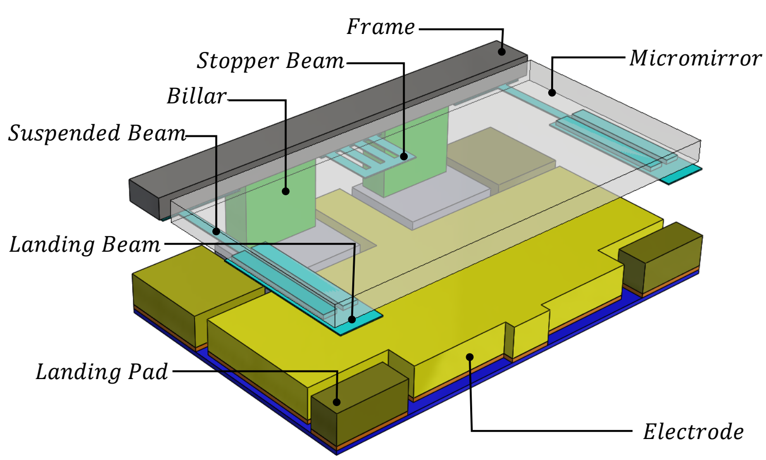

Figure 3: Overview of the components of a MIRA cell.

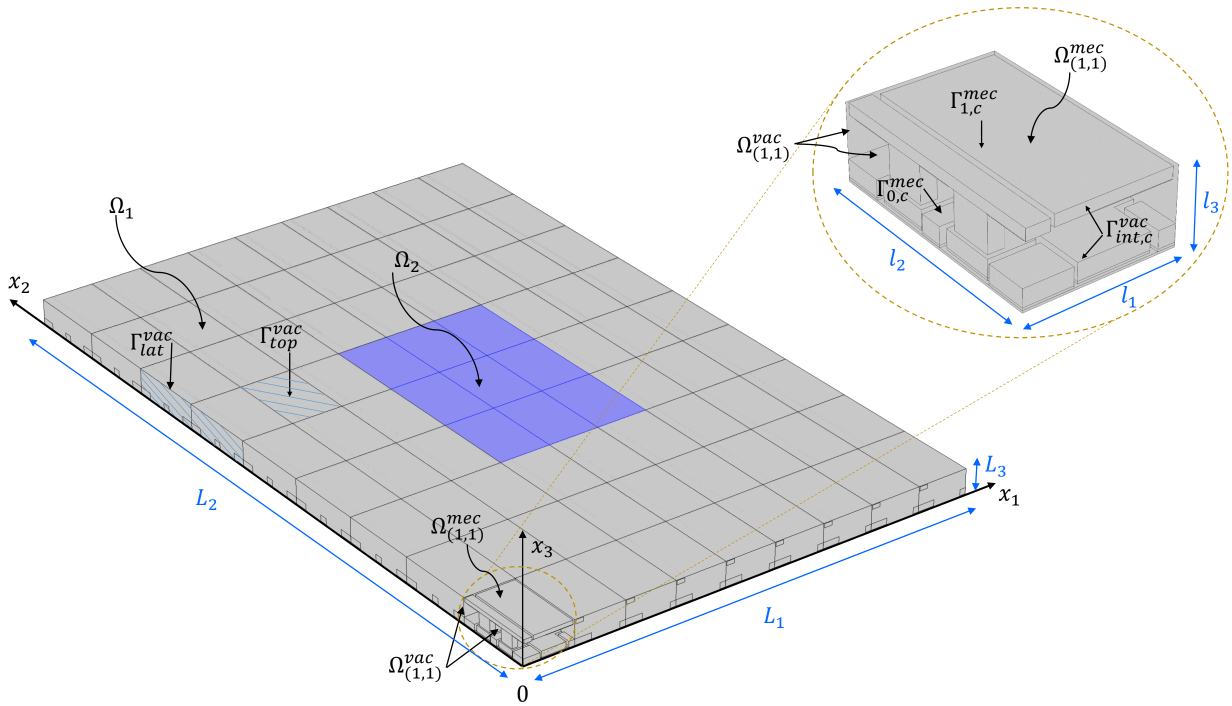

Figure 4: Representation of two zones the external zone Ω_1 and the internal zone Ω_2 with different actuation voltage in the MIRA array. The zoom illustrates one cell $\Omega_{(1,1)}$ of the array with the mechanical structure in $\Omega^{mec}_{(1,1)}$ surrounded by the vacuum in $\Omega^{vac}_{(1,1)}$.

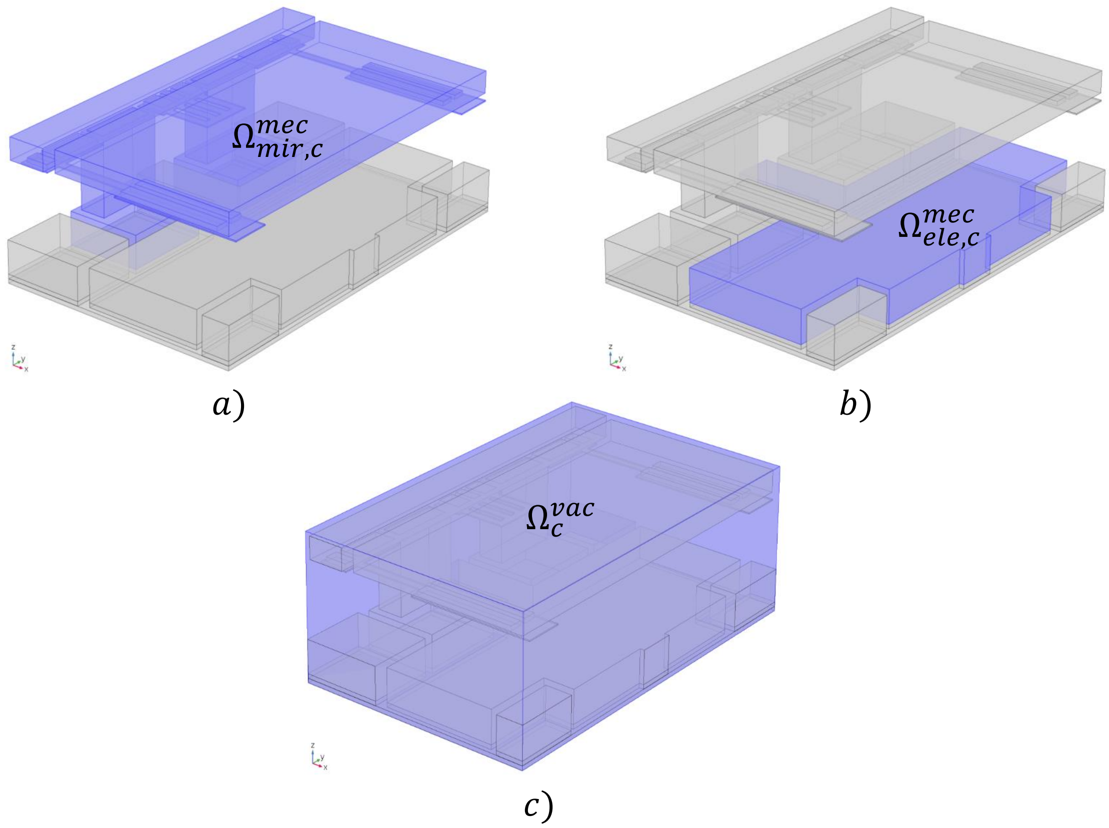

Figure 5: Illustration of the components of the cell Ω_c of the MIRA array. The mechanical part Ω_c^{mec} is made with two components, (a) the mirror part $\Omega_{ele,c}^{mec}$ and (b) the electrode part $\Omega_{mir,c}^{mec}$. Figure (c) represents the vacuum part Ω_c^{vac}.

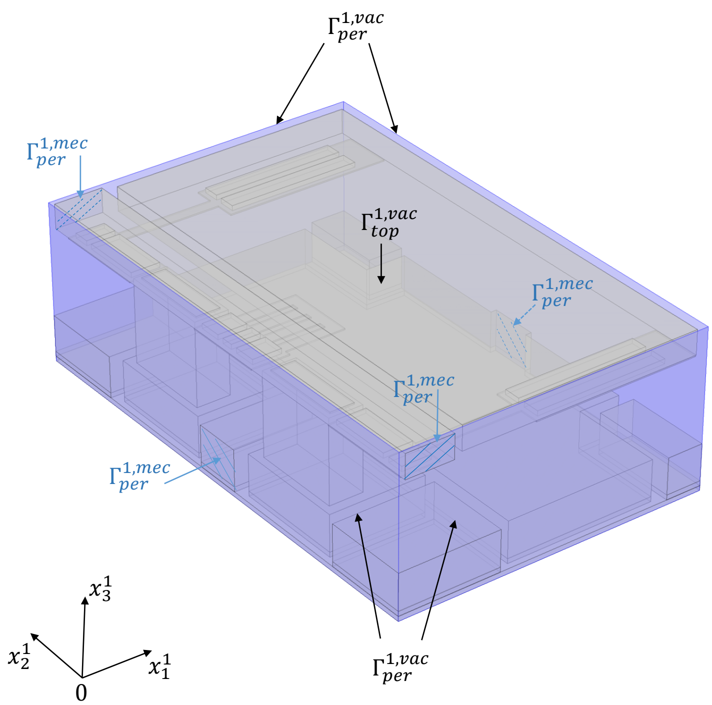

Figure 6: The reference cell $\Omega^1 =]-1/2, 1/2[^3$ made with the mechanical part $\Omega^{1,mec}$ surrounded by vacuum in $\Omega^{1,vac}$.

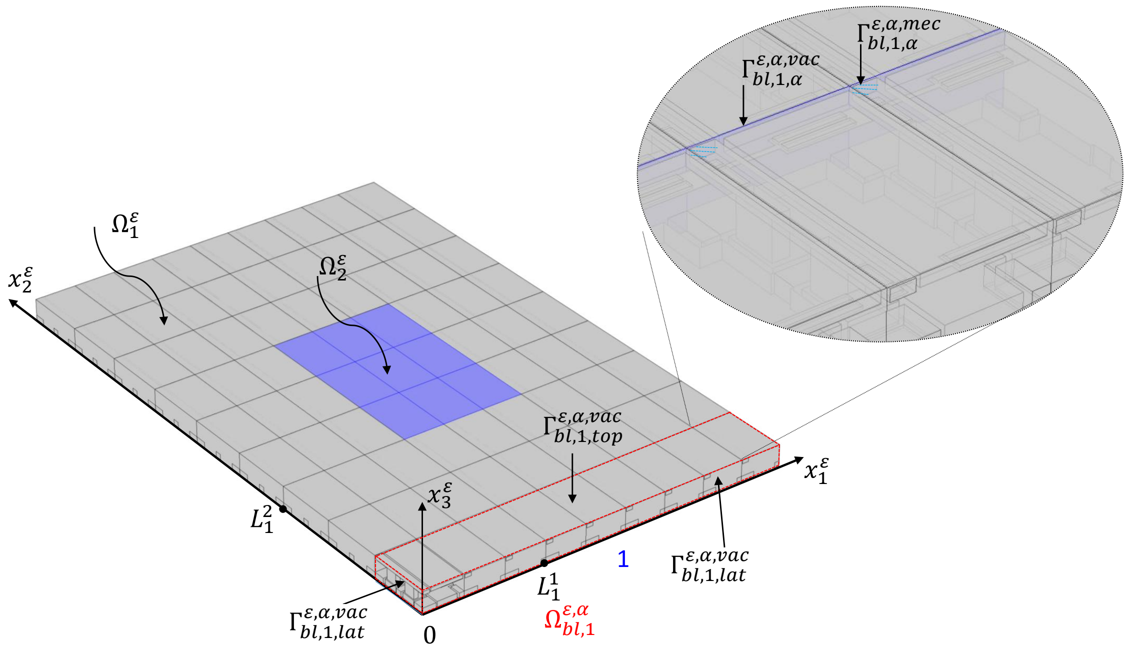

Figure 7: The physical domain $\Omega^{\alpha\varepsilon}_{bl,1}$ for the first lateral boundary model with two subdomains the mechanical body $\Omega^{\alpha\varepsilon,mec}_{bl,1}$ and the vacuum part $\Omega^{\alpha\varepsilon,vac}_{bl,1}$ with $\alpha = 1$. The zoom represents the internal subboundaries of the vacuum and the mechanical part between cells of the external zone.

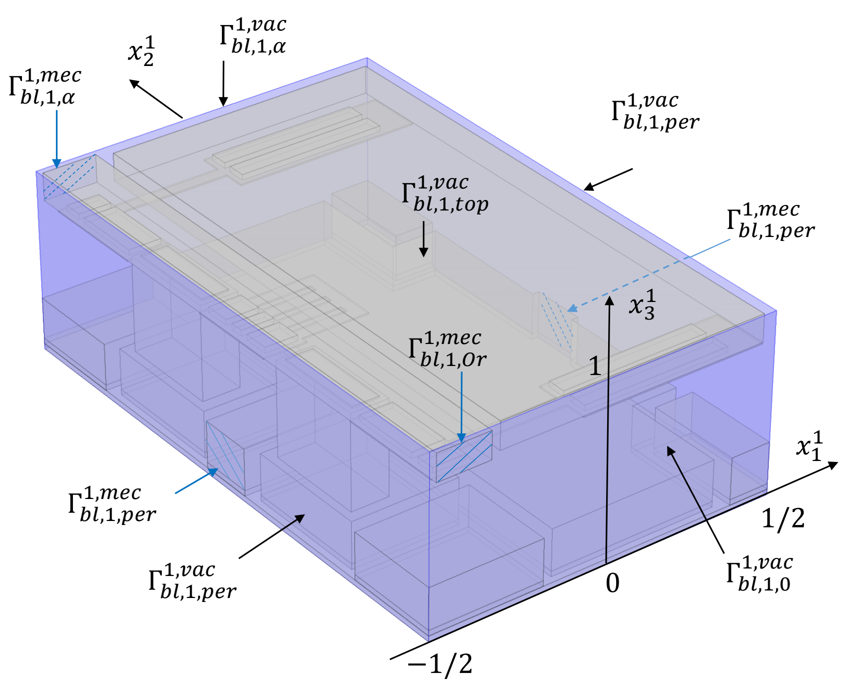

Figure 8: The microscopic domain $\Omega^1_{bl,1}$ with two subdomains $\Omega^{1,mec}_{bl,1}$ and $\Omega^{1,vac}_{bl,1}$ with $\alpha = 1$.

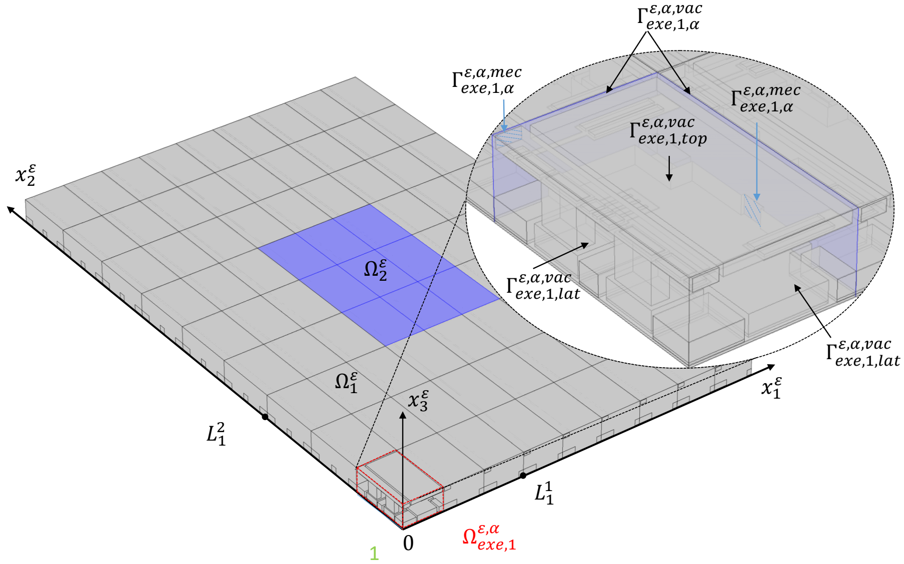

Figure 9: The first exterior edge physical domain $\Omega^{\alpha\varepsilon}_{exe,1}$ including two subdomains $\Omega^{\alpha\varepsilon,vac}_{exe,1}$ and $\Omega^{\alpha\varepsilon,mec}_{exe,1}$ with $\alpha = 1$. The zoom illustrates their boundaries.

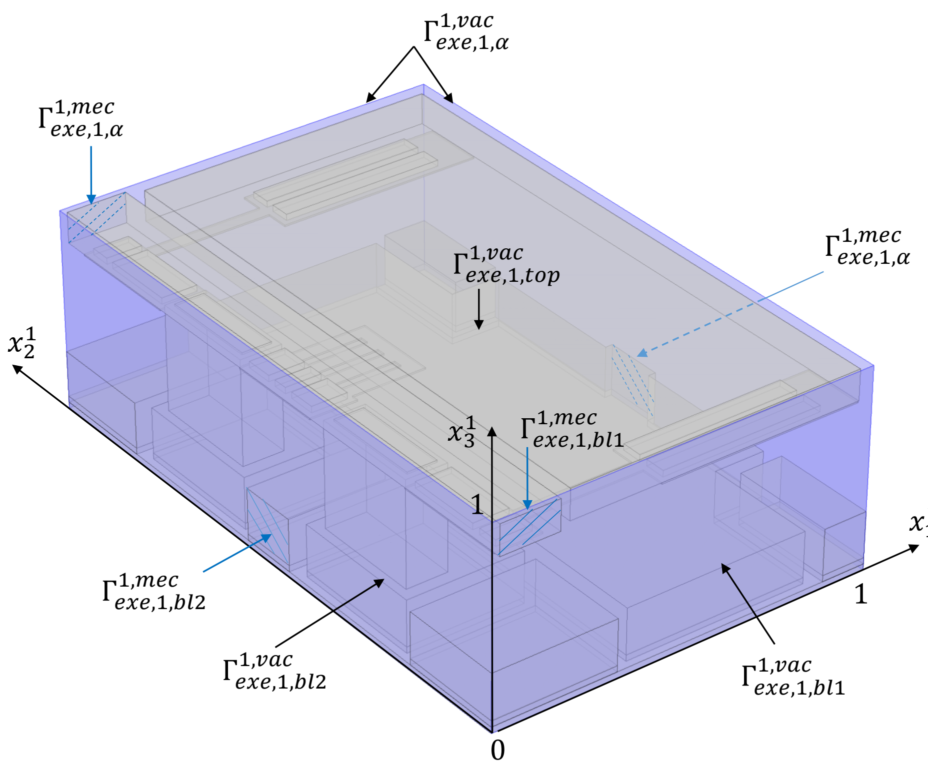

Figure 10: The first exterior edge physical domain $\Omega^{\alpha\varepsilon}_{exe,1}$ with two subdomains $\Omega^{\alpha\varepsilon,vac}_{exe,1}$ and $\Omega^{\alpha\varepsilon,mec}_{exe,1}$.

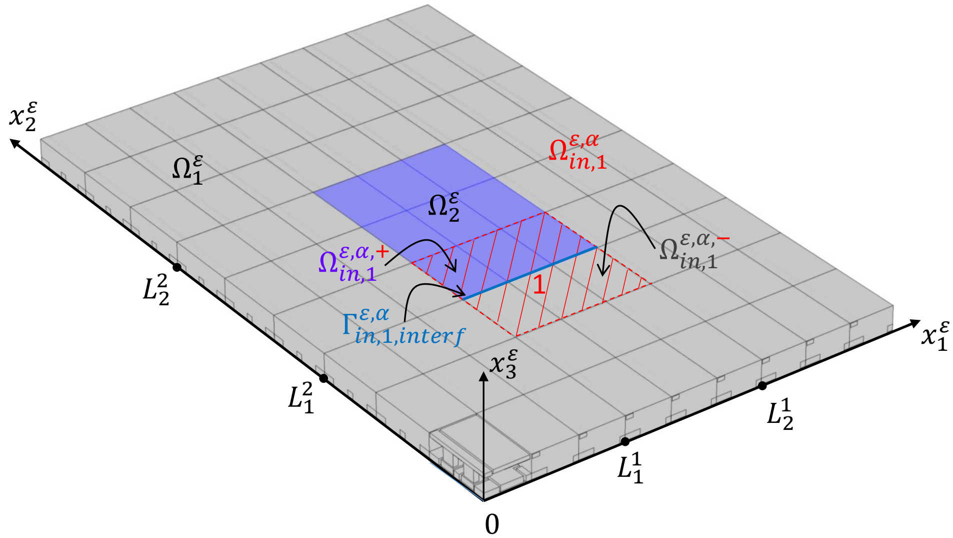

Figure 11: The first interface physical domain $\Omega_{in,1}^{\alpha\varepsilon}$ with two nonoverlapping subdomains $\Omega_{in,1}^{\alpha\varepsilon+}$ and $\Omega_{in,1}^{\alpha\varepsilon-}$, each domain $\Omega_{in,1}^{\alpha\varepsilon\pm}$ is assembled by two parts the vacuum part $\Omega_{in,1}^{\alpha\varepsilon,vac\pm}$ and the mechanical part $\Omega_{in,1}^{\alpha\varepsilon,mec\pm}$, with $\alpha = 1$.

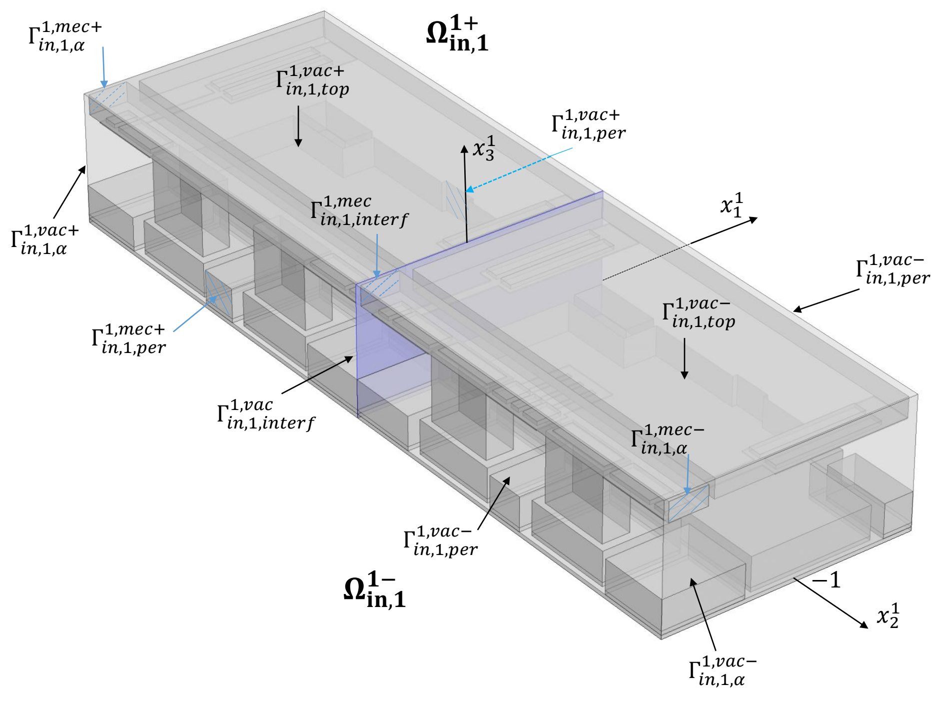

Figure 12: The first interface microscopic domain $\Omega_{in,1}^{1}$ with two nonoverlapping subdomains $\Omega_{in,1}^{1,\pm}$, each of them also involves two parts, the vacuum part $\Omega_{in,1}^{1,vac\pm}$ and the mechanical part $\Omega_{in,1}^{1,mec\pm}$, in the case of $\alpha = 1$.

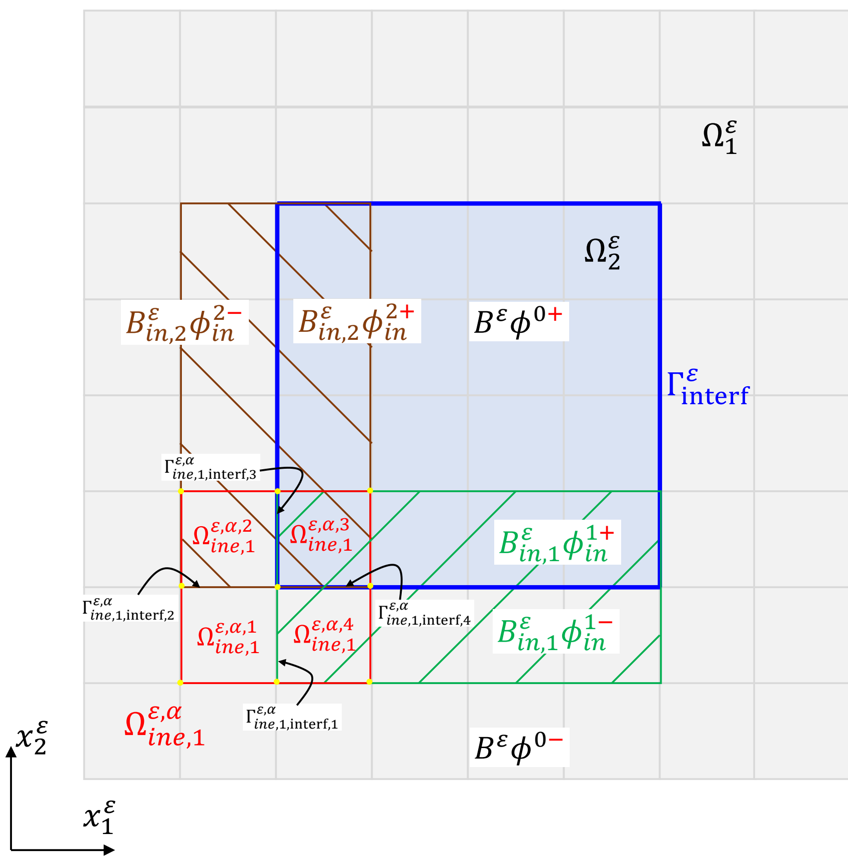

Figure 13: Description of the geometry of the internal edge problem. The green and maroon colors represent the zones of the first and the second interface models. The red region is the zone of the first internal edge model made with four subregions. The electrostatic potential has a different approximation in each of these subregions.

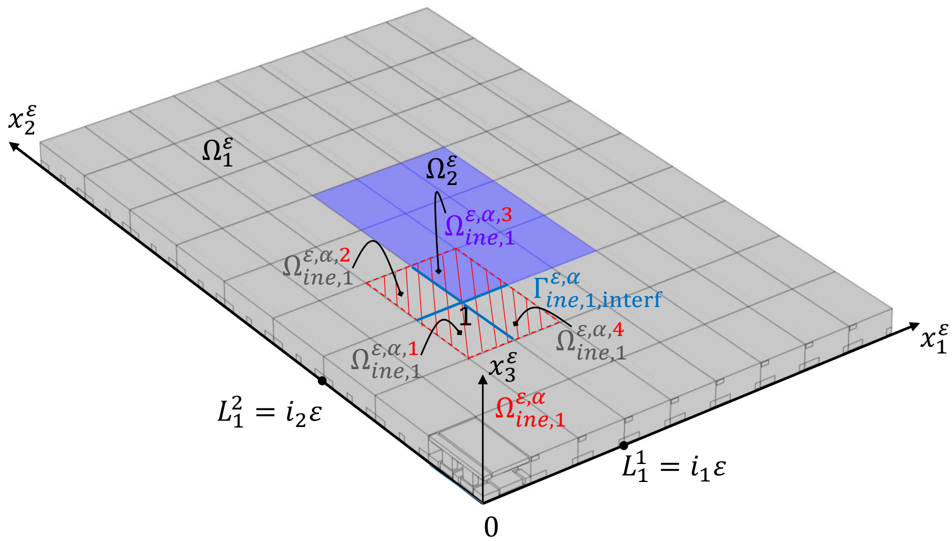

Figure 14: The first internal edge $\Omega_{ine,1}^{\alpha\varepsilon}$ in the physical domain with $\alpha = 1$.

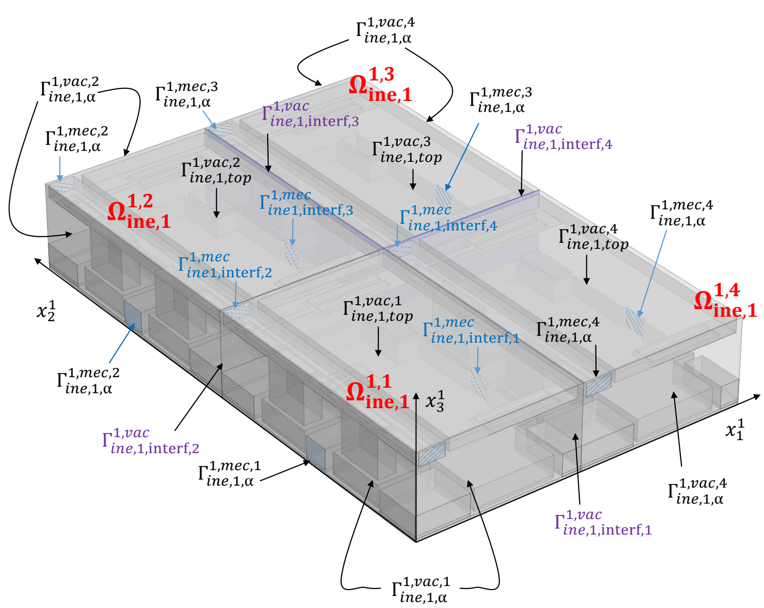

Figure 15: The first internal edge $\Omega_{ine,1}^{1}$ in the microscopic domain with $\alpha = 1$.