Hypergraphs of linear systems over the two-element field and quantum contextuality proofs

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Joint work with Frédéric Holweck[3,4]

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Measurement in a basis

$$|q\rangle = a\,|0\rangle + b\,|1\rangle \begin{cases} \xrightarrow{\;|a|^2\;} |0\rangle \rightsquigarrow +1 \\ \xrightarrow[\;|b|^2\;]{} |1\rangle \rightsquigarrow -1 \end{cases}$$

$|a|^2 + |b|^2 = 1$

encoded by the Pauli matrix $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

eigenvalues	1	-1
eigenvectors	$\|0\rangle$	$\|1\rangle$

Pauli group

Pauli matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

matrix product

.	I	X	Y	Z
I	I	X	Y	Z
X	X	I	iZ	$-iY$
Y	Y	$-iZ$	I	iX
Z	Z	iY	$-iX$	I

Pauli group $\mathcal{P} = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}, .)$
commuting pair $X.I = I.X$
anticommuting pair $Y.Z = iX$ and $Z.Y = -iX$, so $Y.Z = -Z.Y$

Generalized Pauli group

N-qubit Pauli operator	$G_1 G_2 \cdots G_N$, with $G_i \in \{I, X, Y, Z\}$ for $G_1 \otimes G_2 \otimes \cdots \otimes G_N$
generalized Pauli group	$\mathcal{P}_N = (\{1, -1, i, -i\} \times \{I, X, Y, Z\}^N, .)$
commuting pair	$YX.ZZ = (Y.Z)(X.Z) = (iX)(-iY) = XY$ $ZZ.YX = (Z.Y)(Z.X) = (-iX)(iY) = XY$
anticommuting pair	$XY.IZ = (X.I)(Y.Z) = iXX$ $IZ.XY = (I.X)(Z.Y) = -iXX$

Mutually commuting multi-qubit Pauli operators are *compatible observables*

Contextuality : The Kochen-Specker theorem

No non-contextual hidden-variable theory can reproduce the outcomes predicted by quantum physics[1]

Without loss of generality, a non-contextual hidden-variable (NCHV) theory admits the existence of a function $v : \mathcal{P}_N \to \{-1, 1\}$ that determines (as $v(M)$) the result of any measurement with the multi-qubit Pauli observable M (among its two eigenvalues -1 and 1) independently of former measurements, even when they are compatible (commuting)

$$1 \times 1 \times -1 = -1$$

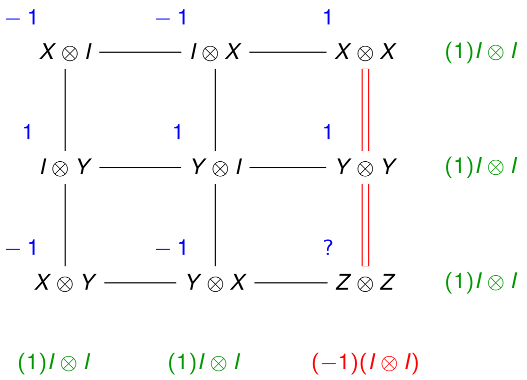
$$\boxed{XY} \text{—} \boxed{YZ} \text{—} \boxed{ZX} = \boxed{-II}$$

[1] Kochen, Simon and Specker, Ernst. "The Problem of Hidden Variables in Quantum Mechanics". *Indiana Univ. Math. J.*. 1968.

The Mermin-Peres magic square

Hypergraph with 9 vertices and 6 (signed) hyperedges (represented as straight lines)

- Each vertex is labelled by a 2-qubit observable
- Each hyperedge/line is a measurement context



This signed hypergraph is *contextual*: no vertex valuation with -1 or $+1$ satisfies all context signs

Linear system of a signed hypergraph

Signed hypergraph

$$\begin{array}{cccccc}
v_1 & & v_2 & & v_3 & \\
X \otimes I & — & I \otimes X & — & X \otimes X & h_1 \\
v_4 \,| & & v_5 \,| & & v_6 \,\| & \\
I \otimes Y & — & Y \otimes I & — & Y \otimes Y & h_2 \\
v_7 \,| & & v_8 \,| & & v_9 \,\| & \\
X \otimes Y & — & Y \otimes X & — & Z \otimes Z & h_3 \\
h_4 & & h_5 & & h_6 &
\end{array}$$

the product of observables on the hyperedge h_i is $(-1)^{E_i} I \otimes I$

$+1 = (-1)^0$

$-1 = (-1)^1$

Linear system

$$H = \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{array} \qquad E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{c} I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ I \otimes I \\ -(I \otimes I) \end{array}$$

The signed hypergraph is contextual iff $\nexists\, x.\ H\,x = E$

Contextuality degree

The *contextuality degree*[1] of a linear system $H\,x = E$ is the minimal Hamming distance $d(H\,x, E)$ between E and a vector $H\,x$, i.e. the **minimal number of unsatisfied equations**

$$d\left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right) = 2$$

Same definition for the associated signed hypergraph (V, H, s), with $V = \{1, 2, \ldots, |V|\}$ for the $|V|$ equations and the *sign function* $s : H \to \{0, 1\}$ such that $s(h) = E_h$

[1] de Boutray, H., Holweck, F., Giorgetti, A., Masson, P.-A., and Saniga, M. "Contextuality degree of quadrics in multi-qubit symplectic polar spaces". *Journal of Physics A: Mathematical and Theoretical.* 2022.

Two other associated signed hypergraphs

Linear system $H\,x = E \quad \leftrightarrow \quad$ signed hypergraph (V, H, s)

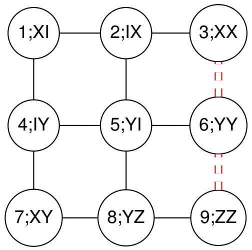
$$V = \{1, 2, \ldots, |V|\} \qquad s : H \to \{0, 1\} \qquad s(h) = E_h$$

For any $x \in \mathbb{F}_2^{|V|}$, two sub-hypergraphs:

$H(x)^{\mathtt{sat}}$ whose hyperedges correspond to the `sat`isfied equations in $H\,x = E$

$H(x)^{\mathtt{uns}}$ whose hyperedges correspond to the `uns`atisfied equations in $H\,x = E$

Example 1: Mermin-Peres signed hypergraphs (1/2)

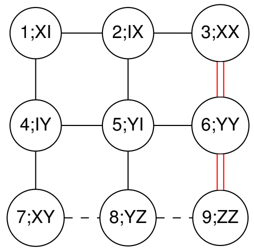


First Mermin-Peres signed hypergraph (V_{MP}, H_{MP}, s_{MP}) with $V_{MP} = \{1, .., 9\}$, $H_{MP} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}\}$ and $s_{MP}(h) = 1$ iff $h = \{3,6,9\}$

Degree 1

For $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $H(x)^{\text{uns}} = \{\{3,6,9\}\}$

Example 1: Mermin-Peres signed hypergraphs (1/2)



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Degree 1

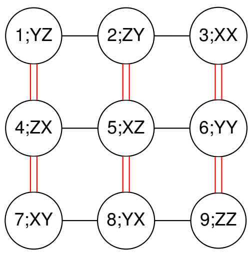
For $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $H(x)^{\text{uns}} = \{\{3,6,9\}\}$

For $x' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^T$, $H(x')^{\text{uns}} = \{\{7,8,9\}\}$

(Q_1) degree = number of negative hyperedges?

(Q_2) Does $[0 \cdots 0]^T$ always minimize the number of unsatisfiable equations?

Example 1: Mermin-Peres signed hypergraphs (2/2)

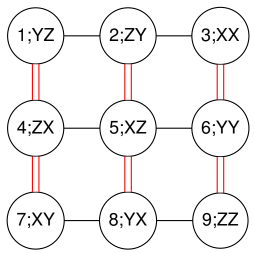


Second Mermin-Peres signed hypergraph, with 3 negative hyperedges

Degree 1

For $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $|H(x)^{\text{uns}}| = 3$

Example 1: Mermin-Peres signed hypergraphs (2/2)



Second Mermin-Peres signed hypergraph, with 3 negative hyperedges

Degree 1

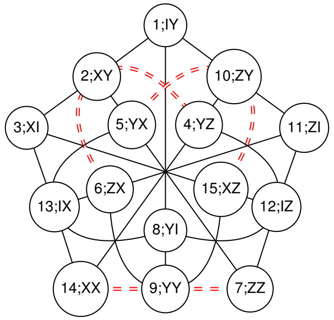
For $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $|H(x)^{\text{uns}}| = 3$

Here, degree < number of negative hyperedges and $[0 \cdots 0]^T$ does not minimize the number of unsatisfiable equations

(Q_3) $\exists^? x_{opt}.\ H(x_{opt})^{\text{uns}} \subseteq H^- = s^{-1}(\{1\})$

Here $x_{opt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T$

Example 2: The doily



(V_2, H_2, s_2) where $V_2 = \{1, \dots, 15\}$, see H_2 and s_2 in the picture

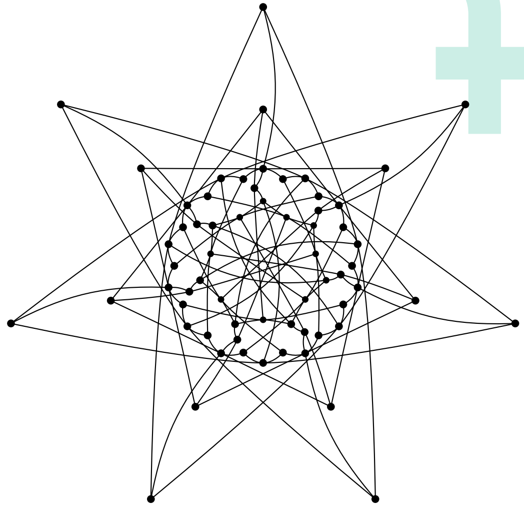
Degree 3: $x_{opt} = [0 \ \cdots \ 0]^T$

$$H(x_{opt})^{\mathrm{uns}} = \{\{2,4,6\}\{5,10,15\}\{7,9,14\}\}$$

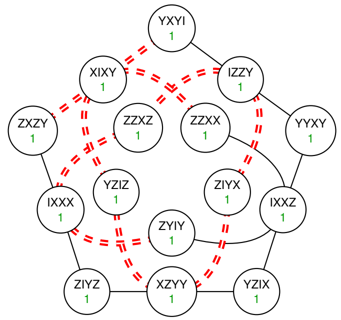
Can be labelled by all (non-identity) two-qubits observables to form the *doily* $W(3,2)$

Example 3: $W(5,2)$

- (V_3, H_3, s_3)
- $|V_3| = 4^3 - 1 = 63$: all the three-qubits (non-identity) observables
- $|H_3| = 315$ hyperedges: all mutually commuting triples over the vertices
- s_3: 90 of them are negative
- $H(x_{opt})^{\mathrm{uns}}$ form a **split Cayley hexagon of order two**
- $|H(x_{opt})^{\mathrm{uns}}| = 63 < 90$



New heuristic for MaxLin2



YXYI	XZYY	ZYIY	ZXZY	IXXZ	ZIYX	ZIYZ	IZZY	ZZXX	YZIX	XIXY	ZZXZ	YYXY	IXXX	YZIZ	Distance
1	1	1	1	0	2	0	1	1	0	2	2	0	1	2	**5**

Randomly negate the value of the assignments where the number of unsatisfied contexts is maximal[1]

[1] Muller, Axel, Saniga, Metod, Giorgetti, Alain, Holweck, Frédéric, and Kelleher, Colm. *A new heuristic approach for contextuality degree estimates and its four- to six-qubit portrayals*. 2024.

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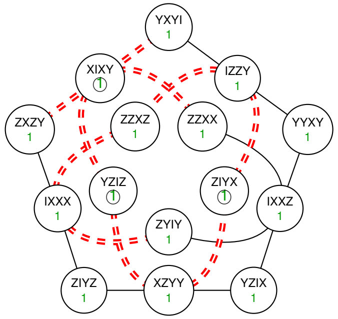

YXYI	XZYY	ZYIY	ZXZY	IXXZ	ZIYX	ZIYZ	IZZY	ZZXX	YZIX	XIXY	ZZXZ	YYXY	IXXX	YZIZ	Distance
1	1	1	1	0	2	0	1	1	0	2	2	0	1	2	**5**

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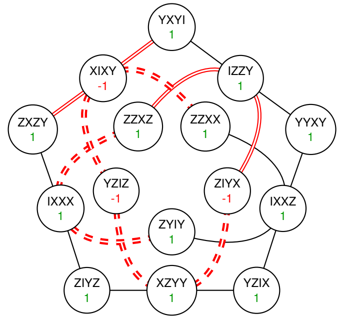

YXYI	XZYY	ZYIY	ZXZY	IXXZ	ZIYX	ZIYZ	IZZY	ZZXX	YZIX	XIXY	ZZXZ	YYXY	IXXX	YZIZ	Distance
1	1	1	1	0	2	0	1	1	0	2	2	0	1	2	**5**
0	1	1	0	0	1	0	0	1	0	1	1	0	1	2	**3**

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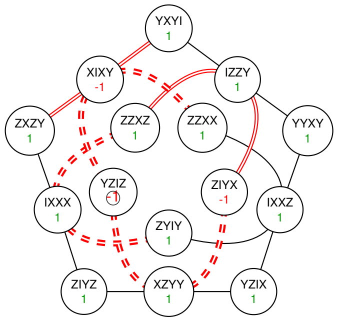

YXYI	XZYY	ZYIY	ZXZY	IXXZ	ZIYX	ZIYZ	IZZY	ZZXX	YZIX	XIXY	ZZXZ	YYXY	IXXX	YZIZ	Distance
1	1	1	1	0	2	0	1	1	0	2	2	0	1	2	**5**
0	1	1	0	0	1	0	0	1	0	1	1	0	1	2	**3**

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[1] Muller, Axel, Saniga, Metod, Giorgetti, Alain, Holweck, Frédéric, and Kelleher, Colm. *A new heuristic approach for contextuality degree estimates and its four- to six-qubit portrayals.* 2024.

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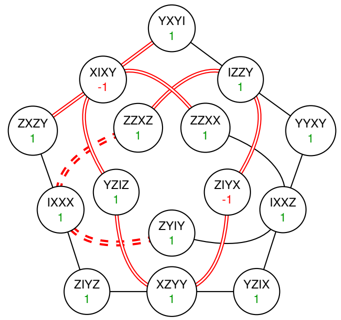


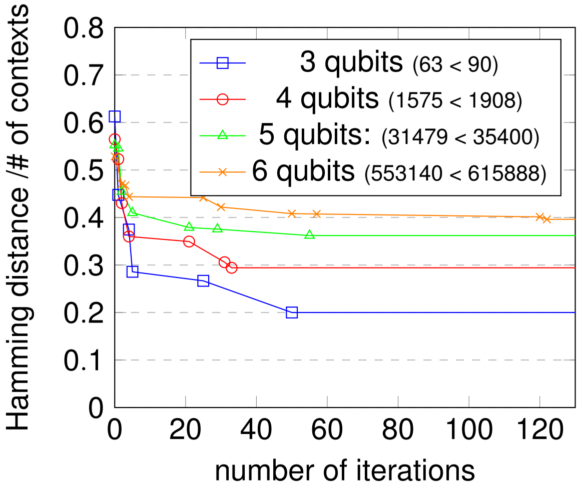
YXYI	XZYY	ZYIY	ZXZY	IXXZ	ZIYX	ZIYZ	IZZY	ZZXX	YZIX	XIXY	ZZXZ	YYXY	IXXX	YZIZ	Distance
1	1	1	1	0	2	0	1	1	0	2	2	0	1	2	**5**
0	1	1	0	0	1	0	0	1	0	1	1	0	1	2	**3**
0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	**1**

Randomly negate the value of the assignments where the number of unsatisfied contexts is maximal[1]

1 is the lowest possible bound for a two-spread

[1] Muller, Axel, Saniga, Metod, Giorgetti, Alain, Holweck, Frédéric, and Kelleher, Colm. *A new heuristic approach for contextuality degree estimates and its four- to six-qubit portrayals.* 2024.

Results with the stochastic local search heuristic



Minimal Hamming distances (y-axis) computed by the heuristic method divided by the total number of contexts, over number of iterations (x-axis), on an i7-12700H running on 20 threads, with adjusted scales for each number $3 \leq N \leq 6$ of qubits.

Conclusion

Summary

- Quantum contextuality linked to solving linear systems modulo 2
- Unsatisfied equations form sub-hypergraphs with high symmetry
- Efficient stochastic local search heuristic to estimate the contextuality degree[1]

Perspectives

- Take advantage of symmetries
- Provide more (larger) examples

[1] Muller, Axel, Saniga, Metod, Giorgetti, Alain, Holweck, Frédéric, and Kelleher, Colm. *A new heuristic approach for contextuality degree estimates and its four- to six-qubit portrayals*. 2024.

Questions?

Communications

- Axel Muller et al. (Oct. 2022). "Multi-Qubit Doilies: Enumeration for All Ranks and Classification for Ranks Four and Five". In: *Journal of Computational Science* 64. ISSN: 1877-7503. DOI: `10.1016/j.jocs.2022.101853`
- Axel Muller et al. (2024b). "New and improved bounds on the contextuality degree of multi-qubit configurations". In: *Mathematical Structures in Computer Science* 34.4, pp. 322–343. DOI: `10.1017/S0960129524000057`
- Axel Muller et al. (2024a). *A new heuristic approach for contextuality degree estimates and its four- to six-qubit portrayals*. Under submission. arXiv: `2407.02928 [quant-ph]`. URL: `https://arxiv.org/abs/2407.02928`
- Axel Muller and Alain Giorgetti (2024). *An abstract structure determines the contextuality degree of observable-based Kochen-Specker proofs*. arXiv: `2410.14463 [quant-ph]`. URL: `https://arxiv.org/abs/2410.14463`

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