# Energy-shaping and entropy-assignment boundary control of the heat equation


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### Abstract

This paper shows a finite-dimensional controller design for the boundary control of the heat equation on a 1D spatial domain. The controller exponentially stabilizes the plant at the desired equilibrium profile. The controller is defined using irreversible port-Hamiltonian systems formulation, and it is motivated by passivity-based control techniques developed for port-Hamiltonian systems defined on 1D spatial domains. The boundary controller is designed to have an exponentially stabilizing energy-shaping and entropy-assignment effect. It works with an actuation at one boundary and a reflective boundary condition at the other. The controller can handle situations where measurements are available at only one or both boundaries. The paper characterizes the existence of structural invariant functions to shape the closed-loop energy and assign the required closed-loop entropy. The design approach is illustrated through numerical simulations.



*Keywords:* Irreversible port-Hamiltonian systems, heat equation, boundary control, passivity-based control.


## 1. Introduction

Irreversible port-Hamiltonian systems (IPHS) formulations have been proposed for the description of thermodynamic systems in [1–4] as an extension of PHS [5–7] for irreversible thermodynamic systems. In this formulation, thermodynamic driving forces are described by locally defined pseudo-brackets, which are used to modulate the geometric structure of the system to express not only the first law (conservation of energy) but also the second law of Thermodynamics (irreversible entropy creation). The heat equation corresponds to a particular class of IPHS in which only heat transport occurs. The control of the heat equation has been widely studied in the literature. The parabolic form of the one-dimensional (1D) heat equation is commonly used as a benchmark for control design methods for distributed-parameter systems, such as the ones based on null controllability [8], adaptive control [9], flatness [10], backstepping [11] and non-negativity control constraints [12], among others [13, 14]. In these works, the state variable is commonly given as a deviation variable, i.e., the difference between the temperature profile of the process and the target equilibrium profile, designing the control laws to guarantee the convergence to zero of the system. When designing controllers for such systems, at least one boundary condition is assumed to be zero or constant throughout. This assumption influences the control design process, requiring the system trajectories and equilibrium profiles to meet these boundary conditions. On the other hand, many design methods [9, 11] rely on transformations that allow the rewrite of the system dynamics in a more suitable formulation to design the control laws. The transformed system generally lacks physical interpretation, requiring an inverse transformation for the physical interpretation of both design and control parameters. In [15] a boundary controller for the heat equation was derived using the IPHS formulation and as Lyapunov function the availability function [16, 17], however the stability proof of the controller relies on the explicit calculation of the closed-loop trajectories.

This work focuses on designing a controller to stabilize the heat equation on a 1D spatial domain. The controller uses energy-shaping and entropy assignment techniques to stabilize the desired equilibrium profile exponentially. The system has actuation at one boundary and a reflective boundary condition at the other boundary, and we employ the IPHS formulation to describe the system dynamics. These techniques are motivated by the passivity-based control methods developed for one-dimensional PHS [18–20]. The existence of structural invariant functions is characterized by shaping the closed-loop energy and assigning the required closed-loop entropy. The proposed boundary controller encompasses the cases in which measurements are available at only one or both boundaries.

The paper is organized as follows. Section 2 presents a brief recall on boundary controlled (BC)-IPHS and the BC-IPHS formulation of the heat equation. Section 3 gives the proposed boundary controller. Section 4 shows numerical simulations, and Section 5 presents some final remarks and comments on future work.


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## 2. Irreversible port-Hamiltonian formulation of the heat equation

Boundary-controlled PHS (BC-PHS) were introduced in [21] and generalized to dissipative PHS in [22, 23]. This class of formulations, which has its roots in mechanical (and electrical) engineering, arises naturally in the modeling of multi-physical systems and has been proven to be very useful for the study of well-posedness, stabilization, control design, and spatial approximation [24]. A dissipative PHS defined on a 1D spatial domain $\zeta \in [0, L]$ is given by

$$\begin{aligned}\partial_t \mathbf{x} &= (P_0 + P_1 \partial_\zeta)\, \delta_{\mathbf{x}} H \\ &\quad - (G_0 S_0 G_0^\top - G_1 \partial_\zeta S_1 G_1^\top \partial_\zeta) \delta_{\mathbf{x}} H, \\ \mathbf{u}(t) &= \tilde{W}_B \begin{bmatrix} \delta_{\mathbf{x}} H|_L \\ \delta_{\mathbf{x}} H|_0 \end{bmatrix}, \quad \mathbf{y}(t) = \tilde{W}_C \begin{bmatrix} \delta_{\mathbf{x}} H|_L \\ \delta_{\mathbf{x}} H|_0 \end{bmatrix}\end{aligned} \tag{1}$$

with the extensive variables as state $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$, $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$, $P_0 = -P_0^\top \in \mathbb{R}^{n\times n}$, $G_0 = G_0^\top \in \mathbb{R}^{n\times m}$, $G_1 \in \mathbb{R}^{n\times m}$, $S_0 = S_0^\top > 0$ , and $S_1 = S_1^\top > 0 \in \mathbb{R}^{m\times m}$ with $m \leq n$. We refer to $\delta_{\mathbf{x}}\mathcal{H}$ and $\delta_s \mathcal{H}$ as the variational derivatives of the total energy concerning $\mathbf{x}$ and $s$, respectively, as defined in [5, 6]. In (1), the matrices $P_0$ and $P_1$ are related to the interconnection between energy-storing elements, while matrices $G_0$ and $G_1$ to the interconnection between energy storing and energy dissipating elements. The matrices $S_0$ and $S_1$ contain the parameters of the energy-dissipating phenomena, such as resistance or damping coefficients. The matrices $\tilde{W}_B$ and $\tilde{W}_C$ are of appropriate size and parametrized [22, 23] such that the energy balance is

$$\begin{aligned}\dot{H} &= \int_0^L \delta_{\mathbf{x}} H^\top \left(G_1 \partial_\zeta S_1 G_1^\top \partial_\zeta - G_0 S_0 G_0^\top\right) \delta_{\mathbf{x}} H \, \mathrm{d}\zeta + \mathbf{y}^\top \mathbf{u} \\ &\leq \mathbf{y}^\top \mathbf{u}\end{aligned}$$

If $S_i = 0$, then (1) is energy preserving or reversible. If $S_i \neq 0$, the system is dissipative, meaning that (mechanical or/and electrical) energy is being transformed into heat by some dissipative phenomena, such as mechanical friction or the Joule effect.

**Remark 1.** *Even if defined for dissipative electromechanical systems, formulation (1) can be used to represent the heat equation. In this case, with only the thermal domain, the entropy function is used in place of the state variable $\mathbf{x}$ and $P_0 = P_1 = G_0 = 0$.*

IPHS on finite-dimensional domains were defined [1] as an extension of PHS to represent not only the energy balance but also the entropy balance, essential in thermodynamic systems. The extension of this framework for infinite-dimensional systems defined on 1D spatial domains was initially proposed in [2] for a class of diffusion processes and generalized for a large class of thermodynamic systems in [4].

### *2.1. Boundary controlled IPHS*

The state variables of a BC-IPHS are given by $n + 1$ extensive variables composed of $\mathbf{x}$ and the entropy per unit length $s$. According to [4], the total energy functional of the system is defined as

$$\mathcal{H}(t) = \int_0^L h(\mathbf{x}(\zeta, t), s(\zeta, t)) \mathrm{d}\zeta \tag{2}$$

where $h(\mathbf{x}, s)$ is the total energy per unit length (thermal, electric, magnetic, mechanical, etc), such that, $\partial_s h = T$ with $T = T(\zeta, t)$ denoting the temperature. The total entropy functional is given by

$$\mathcal{S}(t) = \int_0^L s(\zeta, t) \mathrm{d}\zeta. \tag{3}$$

Unlike PHS, the structure matrices of IPHS depend explicitly on the total energy's variational derivative (co-energy variables). This allows us to guarantee that both the first and second laws of Thermodynamics are satisfied as a structural property. The dependence of the structure matrices on the co-energy variables is characterized by a set of nonlinear modulating functions that are defined by the thermodynamic driving forces and the physical parameters of the irreversible thermodynamic phenomena that are present in the system. The following locally defined pseudo-brackets can express the thermodynamic driving forces

$$\begin{aligned}\{\mathcal{Z}|\mathcal{G}|\mathcal{W}\} &= \begin{bmatrix} \delta_{\mathbf{x}} \mathcal{Z} \\ \delta_s \mathcal{Z} \end{bmatrix}^\top \begin{bmatrix} 0 & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathbf{x}} \mathcal{W} \\ \delta_s \mathcal{W} \end{bmatrix} \\ \{\mathcal{Z}|\mathcal{W}\} &= (\delta_s \mathcal{Z}) \partial_\zeta \left(\delta_s \mathcal{W}\right)\end{aligned}$$

for some smooth functionals $\mathcal{Z}$ and $\mathcal{W}$, where $\mathcal{G}^*$ is the formal adjoint of the differential operator $\mathcal{G}$.

**Definition 1.** *[4] A BC-IPHS undergoing $m$ irreversible processes is defined by the PDE*

$$\begin{aligned}\begin{bmatrix} \partial_t \mathbf{x} \\ \partial_t s \end{bmatrix} &= \begin{bmatrix} P_0 & G_0 \mathbf{R}_0(\mathbf{x}) \\ -\mathbf{R}_0(\mathbf{x})^\top G_0^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\mathbf{x}} \\ e_s \end{bmatrix} + \\ &\begin{bmatrix} P_1 \partial_\zeta(\cdot) & \partial_\zeta \left(G_1 \mathbf{R}_1(\mathbf{x}) \cdot\right) \\ -\mathbf{R}_1(\mathbf{x})^\top G_1^\top \partial_\zeta(\cdot) & r_s(\mathbf{x}) \partial_\zeta(\cdot) + \partial_\zeta(r_s(\mathbf{x}) \cdot) \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\mathbf{x}} \\ e_s \end{bmatrix}\end{aligned} \tag{4}$$

*where $\mathbf{e}_{\mathbf{x}}(\zeta, t) = \delta_{\mathbf{x}}\mathcal{H}$ and $e_s(\zeta, t) = \delta_s \mathcal{H}$ are the co-state variables, $\mathbf{R}_0(\mathbf{x}), \mathbf{R}_1(\mathbf{x}) \in \mathbb{R}^{m\times 1}$ and $r_s(\mathbf{x}) \in \mathbb{R}$ stand for the vectors of modulating functions with*

$$\begin{aligned}R_{0,i} &= \gamma_{0,i}(\mathbf{x}, \mathbf{e}_{\mathbf{x}}, e_s) \{\mathcal{S}|G_0(:, i)|\mathcal{H}\}, \\ R_{1,i} &= \gamma_{1,i}(\mathbf{x}, \mathbf{e}_{\mathbf{x}}, e_s) \{\mathcal{S}|G_1(:, i)\partial_\zeta|\mathcal{H}\}\end{aligned}$$

*and*

$$r_s = \gamma_s(\mathbf{x}, \mathbf{e}_{\mathbf{x}}, e_s) \{\mathcal{S}|\mathcal{H}\}$$

*with $\gamma_{k,i}(\mathbf{x}, \mathbf{e}_{\mathbf{x}}, e_s), \gamma_s(\mathbf{x}, \mathbf{e}_{\mathbf{x}}, e_s) : \mathbb{R}^{2n+1} \to \mathbb{R}$, $\gamma_{k,i}, \gamma_s \geq 0$, $k = \{0, 1\}$. The boundary inputs and outputs are given by*

$$\mathbf{u}(t) = W_B \begin{bmatrix} \tilde{\mathbf{e}}|_L \\ \tilde{\mathbf{e}}|_0 \end{bmatrix} \quad \text{and} \quad \mathbf{y}(t) = W_C \begin{bmatrix} \tilde{\mathbf{e}}|_L \\ \tilde{\mathbf{e}}|_0 \end{bmatrix} \tag{5}$$

*where the boundary port variables are* $\tilde{\mathbf{e}}(\zeta,t) = \begin{bmatrix} \mathbf{e}_{\mathbf{x}} \\ \mathbf{R}(\mathbf{x})e_s \end{bmatrix}$ *with* $\mathbf{R}(\mathbf{x}) = \begin{bmatrix} 1 & \mathbf{R}_1^\top(\mathbf{x}) & r_s(\mathbf{x}) \end{bmatrix}^\top \in \mathbb{R}^{m+2}$*, and* $W_B$ *and* $W_C$ *are matrices of appropriate size (their precise parametrization can be found in [4])such that the total energy balance*

$$\dot{\mathcal{H}} = \mathbf{y}^\top \mathbf{u} \tag{6}$$

*and the total entropy balance*

$$\dot{\mathcal{S}} = \int_0^L \sigma_s \mathrm{d}\zeta + \mathbf{u}_s^\top \mathbf{y}_s \tag{7}$$

*are satisfied, where* $\mathbf{u}_s$ *and* $\mathbf{y}_s$ *are entropy conjugated inputs/outputs, respectively, and* $\sigma_s \geq 0$ *is the total internal entropy production [4].*

Let us comment on the Definition 1. The matrices $P_0$, $P_1$, $G_0$, and $G_1$ have the same physical interpretation as for PHS. The main feature of (4) is the definition of the vector of modulating functions $\mathbf{R}_0$ and $\mathbf{R}_1$. The elements of these vectors are modulating functions defined by each of the irreversible thermodynamic phenomena of the system. For instance, if the entropy generated by an electric current through a resistance is considered, then the corresponding modulating function will be defined by the resistance's constitutive relation, characterized by the positive function $\gamma$, and by the thermodynamic driving force, which produces the entropy, which in this case is the electrical current and which is precisely given by a pseudo-bracket $\{\mathcal{S}|\mathcal{G}|\mathcal{H}\}$, where $\mathcal{G}$ is the operator which characterizes how the resistance is interconnected with the energy storing elements of the system. The input and output vectors (5) are defined by the boundary values of the co-energy variables. However, similar to dissipative BC-PHS [22], the mechanical energy transformed into entropy at the boundaries has to be taken into account, the reason why the boundary port variables are completed with the vector $\mathbf{R}(\mathbf{x})e_s$. It is interesting to notice in the second structure matrix of (4) the diagonal element $r_s(\mathbf{x})\partial_\zeta + \partial_\zeta(r_s(\mathbf{x}))$. This element does not characterize entropy production due to the conversion of mechanical energy into entropy but rather the entropy produced by pure heat transport. In this case, $r_s$ is defined by the function $\gamma_s$, which characterizes the thermodynamic parameters of a constitutive relation like Fourier's law, and the temperature, which is precisely defined by the pseudo-bracket $\{\mathcal{S}|\mathcal{H}\}$. This term models the heat equation. For further details and examples of IPHS on finite and infinite dimensional domains and the general characterization of $W_B$ and $W_C$, we refer the reader to [1, 4]

### 2.2. BC-IPHS formulation of the heat equation

Consider the conservation law of the internal energy per unit length $u = u(\zeta,t)$ defined on interval $[0,L]$, as follows

$$\partial_t u = -\partial_\zeta q, \quad \forall \zeta \in [0,L] \tag{8}$$

where $q = q(\zeta,t)$ denotes the heat flux. Using the Fourier's law $q = -k\partial_\zeta T$, where $k$ is the heat conduction coefficient of the medium and $T = T(\zeta,t)$ denotes the temperature, and the calorimetric law $\mathrm{d}u = c_v \mathrm{d}T$, with $c_v$ the heat capacity per unit length, we rewrite (8) as

$$c_v \partial_t T = \partial_\zeta \left(k \partial_\zeta T\right), \tag{9}$$

which corresponds to the standard form of the heat equation [5]. This formulation is useful for simulations since it is well-known and easy to discretize when $c_v$ is time-invariant [25]. However, from the first and second thermodynamic laws perspective, the information must be derived implicitly since (9) does not retain the structure of a conservation law. On the other hand, defining the entropy flux $q_s = q_s(\zeta,t)$ as $q_s = q/T$, and using Gibbs' relation $\mathrm{d}u = T\mathrm{d}s$, the heat equation can be expressed through the entropy balance, i.e.,

$$\begin{aligned} T\partial_t s &= -\,\partial_\zeta \left(T q_s\right) \\ \partial_t s &= -\,\frac{q_s}{T}\partial_\zeta T - \partial_\zeta q_s \end{aligned} \tag{10}$$

where $-\partial_\zeta q_s$ refers to the entropy diffusion through the media, and $-\frac{q_s}{T}\partial_\zeta T = \frac{k}{T^2}\left(\partial_\zeta T\right)^2 \geq 0$ describes the entropy production per unit length due to the heat flux. Moreover, considering the total energy and total entropy,

$$\mathcal{H} = \int_0^L u \,\mathrm{d}\zeta \quad \text{and} \quad \mathcal{S} = \int_0^L s \,\mathrm{d}\zeta,$$

respectively, we obtain that $\delta_s\mathcal{H} = \partial_s u = T$ and $\delta_s\mathcal{S} = 1$. Since the only irreversible process is due to the entropy flux, there is only one thermodynamic driving force, which is given by the pseudo-bracket $\{\mathcal{S}|\mathcal{H}\} = \delta_s\mathcal{S}\partial_\zeta(\delta_s\mathcal{H}) = \partial_\zeta T$. Then, defining $r_s = \gamma_s\{\mathcal{S}|\mathcal{H}\}$, with $\gamma_s = \frac{k}{T^2} > 0$, and $e_s = \delta_s\mathcal{H} = T$ we obtain that $q_s = -r_s e_s$ and $-\frac{q_s}{T}\partial_\zeta T = \gamma_s\{\mathcal{S}|\mathcal{H}\}^2$. As a consequence, assuming a reflective boundary condition at $\zeta = 0$ and a boundary control on the entropy flux at $\zeta = L$, i.e.,

$$-q_s|_0 = r_s e_s|_0 = 0, \tag{11}$$

$$-q_s|_L = r_s e_s|_L = \mathbf{u}, \tag{12}$$

the heat equation leads to the BC-IPHS of Definition 1 with $P_0 = P_1 = G_0 = G_1 = 0$ and $\tilde{\mathbf{e}} = \begin{bmatrix} e_s & r_s e_s \end{bmatrix}^\top$, i.e.,

$$\begin{aligned} \partial_t s &= r_s \partial_\zeta e_s + \partial_\zeta \left(r_s e_s\right) \\ \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} &= W_B \begin{bmatrix} \tilde{\mathbf{e}}|_L \\ \tilde{\mathbf{e}}|_0 \end{bmatrix} = \begin{bmatrix} -q_s|_L \\ -q_s|_0 \end{bmatrix}, \\ \mathbf{y} &= W_C \begin{bmatrix} \tilde{\mathbf{e}}|_L \\ \tilde{\mathbf{e}}|_0 \end{bmatrix} = \begin{bmatrix} T|_L \\ -T|_0 \end{bmatrix} \end{aligned} \tag{13}$$

where $W_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $W_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$.

The BC-IPHS (13) expresses explicitly the first and second laws of Thermodynamics. The temperature $T$ is the power-conjugated variable minus the entropy flux, $-q_s$, i.e., $\mathbf{u}$ represents the entropy exchange with the external environment and the product $[\mathbf{u}\ 0]^\top \mathbf{y}$ the power supplied to the system. As a consequence, the total entropy and total energy balance are

$$\begin{aligned}\dot{\mathcal{S}} &= \int_0^L \partial_t s \, \mathrm{d}\zeta = \int_0^L r_s \partial_\zeta (e_s) \, \mathrm{d}\zeta + r_s e_s|_0^L \\ &= \int_0^L \underbrace{\gamma_s \{\mathcal{S}|\mathcal{H}\}^2}_{\sigma_s \geq 0} \mathrm{d}\zeta + \mathbf{u}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\dot{\mathcal{H}} &= \int_0^L e_s \partial_t s \, \mathrm{d}\zeta = \int_0^L \left(e_s r_s \partial_\zeta \left(e_s\right) + e_s \partial_\zeta \left(r_s e_s\right)\right) \mathrm{d}\zeta \\ &= \left(e_s r_s e_s\right)|_0^L = -T q_s|_0^L = \mathbf{y}^\top \begin{bmatrix}\mathbf{u} \\ 0\end{bmatrix}\end{aligned} \tag{15}$$

which shows that the BC-IPHS (13) is conservative and satisfies the first and second laws of Thermodynamics.

On the other hand, considering an isotropic assumption ($k$ constant), the heat equation reaches the dynamic equilibrium for linear temperature profiles [26]. In this case in particular, we say that $T$ is a solution of (13) if $T \in \mathbb{T}$ where $\mathbb{T} := \{f(\zeta) \in H^2([0, L], \mathbb{R}) | \partial_\zeta f|_0 = 0\}$, i.e., $T^*$ is an equilibrium profile of (13) if $T^* \in \mathbb{T}^*$ where $\mathbb{T}^* := \{f(\zeta) \in \mathbb{T} | \partial_\zeta \left(k \partial_\zeta f\right) = 0 \text{ in } (0, L)\}$.

## 3. Boundary control by interconnection

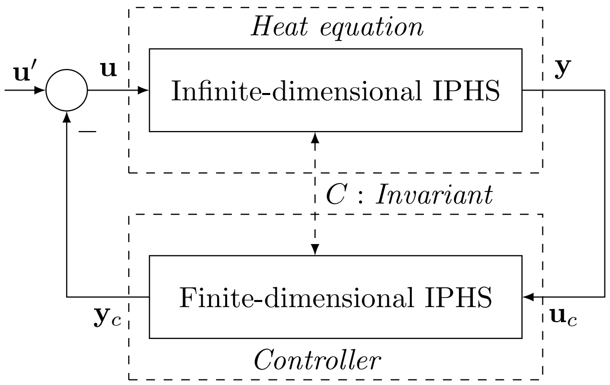



Figure 1: CbI of the heat equation

This section presents a boundary controller designed as the interconnection of the BC-IPHS formulation of the heat equation with a nonlinear finite-dimensional controller. The design extends the control by interconnection (CbI) for BC-PHS [20] to BC-IPHS. The objective is to characterize the conditions for the existence of closed-loop invariant functions, which are then used to shape the closed-loop energy function and assign the closed-loop entropy. The control scheme is shown in Figure 1. Denoting by $x_c \in \mathbb{R}$ the controller state and by $H_c(x_c) \in \mathbb{R}$ its energy function, the nonlinear controller is given by

$$\dot{x}_c = 0 e_c + G_c(x_c, \mathbf{u}_c)\mathbf{u}_c, \qquad \mathbf{y}_c = G_c^\top(x_c, \mathbf{u}_c) e_c, \tag{16}$$

where $e_c = \partial_{x_c} H_c$. We consider the parametrization $G_c(x_c, \mathbf{u}_c)\mathbf{u}_c$ to encompass various types of systems, such as the heat equation [1, Sec. 2.4]. Figure 1 defines the following interconnection rule between the heat equation and the controller

$$\mathbf{u}_c = \mathbf{y}, \qquad \begin{bmatrix}\mathbf{u} \\ 0\end{bmatrix} = -\mathbf{y}_c + \begin{bmatrix}\mathbf{u}' \\ 0\end{bmatrix} \tag{17}$$

Denoting by $\mathcal{C}$ and $\mathcal{B}$ the boundary operators such that the output and input in (13) can be expressed as $\mathbf{y} = \mathcal{C} e_s$ and $\begin{bmatrix}\mathbf{u} \\ 0\end{bmatrix} = \mathcal{B} e_s$, respectively, we obtain the coupled PDE-ODE system that follows

$$\begin{aligned}\underbrace{\begin{bmatrix}\partial_t s \\ \dot{x}_c\end{bmatrix}}_{\dot{\mathbf{x}}_{cl}} &= \underbrace{\begin{bmatrix}r_s \partial_\zeta(\cdot) + \partial_\zeta(r_s \cdot) & 0 \\ G_c(x_c, \mathbf{u}_c)\mathcal{C} & 0\end{bmatrix}}_{\mathcal{J}_{cl}} \underbrace{\begin{bmatrix}e_s \\ e_c\end{bmatrix}}_{\mathbf{e}_{cl}} \\ \begin{bmatrix}\mathbf{u}' \\ 0\end{bmatrix} &= \underbrace{\begin{bmatrix}\mathcal{B} & G_c^\top(x_c, \mathbf{u}_c)\end{bmatrix}}_{W_{B_{cl}}} \mathbf{e}_{cl}\end{aligned} \tag{18}$$

where $\mathbf{e}_{cl} \in \mathbb{E}_{cl}$ denote the co-states of the closed-loop system, with $\mathbb{E}_{cl} = \mathbb{T} \times \mathbb{R}$ the corresponding co-state space and the inner product

$$\left\langle \mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} = \int_0^L f_1^1(\zeta) f_1^2(\zeta) \mathrm{d}\zeta + f_2^1 f_2^2$$

for all $\mathbf{f}^i = [f_1^i(\zeta) \quad f_2^i]^\top \in \mathbb{E}_{cl}$. The operator $\mathcal{J}_{cl}$ satisfies

$$\begin{aligned}\left\langle \mathbf{f}^1, \mathcal{J}_{cl} \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} &= \left\langle -\mathcal{J}_{cl} \mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} \\ &\quad + [\mathcal{C} f_1^1]^\top W_{B_{cl}} \mathbf{f}^2 + [W_{B_{cl}} \mathbf{f}^1]^\top \mathcal{C} f_1^2.\end{aligned} \tag{19}$$

Setting $W_{B_{cl}} \mathbf{f}^i = 0$, $\forall \mathbf{f}^i \in \mathbb{E}_{cl}$ we obtain that $\left\langle \mathbf{f}^1, \mathcal{J}_{cl} \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}} = \left\langle -\mathcal{J}_{cl} \mathbf{f}^1, \mathbf{f}^2 \right\rangle_{\mathbb{E}_{cl}}$, i.e., $\mathcal{J}_{cl}$ is formally skew-adjoint on the space $\mathbb{E}_{cl}$.

**Definition 2.** *[20, 27] Consider the boundary control system (18) with $\mathbf{u}' = 0$. A function $C : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is an invariant of (18) if $\dot{C} = 0$ along the trajectories (18) for any $\mathbf{e}_{cl}$.*

**Assumption 1.** *The function $C(s, x_c)$ is searched of the particular form*

$$C(s, x_c) = \Gamma x_c + \int_0^L f(s(\zeta)) \mathrm{d}\zeta = \kappa \tag{20}$$

*where $\kappa$ is a constant and $f(s)$ is a continuous function.*

**Proposition 1.** *Consider the BC system (18) with $\mathbf{u}' = 0$. Then (20) is an invariant of (18) if*

$$\langle \mathcal{J}_{cl} \boldsymbol{\epsilon}, \mathbf{e}_{cl} \rangle_{\mathbb{E}_{cl}} = 0 \tag{21}$$

$$\begin{bmatrix}\mathcal{B} & G_c^\top(x_c, \mathbf{u}_c)\end{bmatrix} \boldsymbol{\epsilon} = 0 \tag{22}$$

*where*

$$\boldsymbol{\epsilon} = \begin{bmatrix}\epsilon_s \\ \epsilon_c\end{bmatrix} = \begin{bmatrix}\delta_s C \\ \partial_{x_c} C\end{bmatrix} = \begin{bmatrix}\partial_s f(s) \\ \Gamma\end{bmatrix} \tag{23}$$

*Proof.* The time derivative of (20) is

$$\dot{C} = \Gamma \dot{x}_c + \int_0^L (\partial_s f(s) \partial_t s)\, \mathrm{d}\zeta$$

Using (23), $\dot{C}$ becomes along the trajectories of (18), with $W_{B_{cl}} \mathbf{e}_{cl} = 0$ since $\mathbf{u}' = 0$, as

$$\dot{C} = \langle \boldsymbol{\epsilon}, \mathcal{J}_{cl} \mathbf{e}_{cl} \rangle_{\mathbb{E}_{cl}} \overset{(19)}{=} - \langle \mathcal{J}_{cl} \boldsymbol{\epsilon}, \mathbf{e}_{cl} \rangle_{\mathbb{E}_{cl}} + [W_{B_{cl}} \boldsymbol{\epsilon}]^\top \mathcal{C} e_s$$

Notice that $C$ is an invariant of (18) if and only if $\dot{C} = 0$, i.e., $- \langle \mathcal{J}_{cl} \boldsymbol{\epsilon}, \mathbf{e}_{cl} \rangle_{\mathbb{E}_{cl}} + [W_{B_{cl}} \boldsymbol{\epsilon}]^\top \mathcal{C} e_s = 0$, which is true if conditions (21) and (22) are satisfied. □

**Proposition 2.** *The function $C$ satisfies Proposition 1 if $f(s) = \alpha u(s) + c_1$ where $c_1$ is a function that does not depend on $s$. The state of the control system (16) is then given by the state feedback*

$$x_c = -\frac{\alpha}{\Gamma} \int_0^L u(s) \mathrm{d}\zeta + \frac{\bar{k}}{\Gamma} = -\frac{\alpha}{\Gamma} \mathcal{H}(s) + \frac{\bar{k}}{\Gamma} \tag{24}$$

*with $\bar{k} = \left( k + \int_0^L c_1 \mathrm{d}\zeta \right)$ and the controller energy function is*

$$H_c = \frac{\Gamma}{\alpha} x_c + k_c = -\mathcal{H}(s) + k' \tag{25}$$

*where $k' = \frac{\bar{k}}{\alpha} + k_c$, with $k_c$ a constant. Furthermore, the energy function of the closed-loop system is constant and equal to $k'$.*

*Proof.* Considering $\mathbf{u}' = 0$ in (18), i.e., $\mathcal{B} e_s = -G_c^\top e_c$, from condition (21),

$$\begin{aligned}
0 &= \langle \mathcal{J}_{cl} \boldsymbol{\epsilon}, \mathbf{e}_{cl} \rangle_{\mathbb{E}_{cl}} \\
&= \int_0^L (r_s \partial_\zeta \epsilon_s + \partial_\zeta (r_s \epsilon_s))\, e_s \mathrm{d}\zeta + (G_c \mathcal{C} \epsilon_s)\, e_c \\
&= \int_0^L (r_s \partial_\zeta \epsilon_s + \partial_\zeta (r_s \epsilon_s))\, e_s \mathrm{d}\zeta - (\mathcal{C} \epsilon_s)^\top \mathcal{B} e_s \\
&= \int_0^L (r_s \partial_\zeta \epsilon_s + \partial_\zeta (r_s \epsilon_s))\, e_s \mathrm{d}\zeta - \int_0^L \partial_\zeta (\epsilon_s (r_s e_s))\, \mathrm{d}\zeta \\
&= - \int_0^L (\epsilon_s \partial_\zeta (r_s e_s) - e_s \partial_\zeta (r_s \epsilon_s))\, \mathrm{d}\zeta
\end{aligned}$$

which is satisfied for all $e_s$ if and only if $\epsilon_s \partial_\zeta (r_s e_s) - e_s \partial_\zeta (r_s \epsilon_s) = 0$. The only nontrivial solution is given by

$$\epsilon_s = \partial_s f(s) = \alpha e_s \quad \text{with} \quad \partial_\zeta \alpha = 0 \tag{26}$$

Since $e_s = \delta_s \mathcal{H} = \partial_s u(s) = T(\zeta, t)$ implies $\partial_s f(s) = \alpha \partial_s u(s)$ the function $C$ satisfies Proposition 1 if $f(s) = \alpha u(s) + c_1$ with $c_1$ independent of $s$. Then from (20) the state of the controller (16) is given by the feedback (24). From condition (22) $\mathcal{B} \epsilon_s = -G_c^\top (x_c, \mathbf{u}_c) \epsilon_c$ and by (26) it is obtained that $\alpha \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix} = -G_c^\top (x_c, \mathbf{u}_c) \Gamma$. The interconnection rule (17) becomes

$$-\alpha \mathbf{y}_c = -G_c^\top (x_c, \mathbf{u}_c) \Gamma, \qquad \mathbf{y}_c = G_c^\top (x_c, \mathbf{u}_c) \frac{\Gamma}{\alpha}$$

and comparing terms with (16) we have that $e_c = \partial_{x_c} H_c = \frac{\Gamma}{\alpha}$, implying that the energy of the controller is (25). The energy of the closed-loop system is thus given by

$$H_{cl} = \mathcal{H} + H_c = \int_0^L u(s) \mathrm{d}\zeta + \frac{\Gamma}{\alpha} x_c = k'.$$

Which is constant. □

Although the energy $H_{cl}$ of the closed-loop system is constant, the state of the controller provides a measure of the internal energy of the heat equation. For instance, setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = 0$ the state of the controller is

$$x_c(t) = \int_0^L u(\zeta, t) \mathrm{d}\zeta = \mathcal{H}(t), \quad x_c(0) = \mathcal{H}(0)$$

i.e., $x_c$ is a measure of the total energy of (13). Setting $\alpha = -1$, $\Gamma = 1$ and $\bar{k} = -\mathcal{H}(0)$,

$$x_c(t) = \int_0^L (u(\zeta, t) - u_0)\, \mathrm{d}\zeta = \Delta \mathcal{H}, \quad x_c(0) = 0$$

with $\Delta \mathcal{H} = \mathcal{H}(t) - \mathcal{H}(0)$. In this case, $x_c$ measures the energy supplied by the controller to the process. Setting $\alpha = -1$, $\Gamma = \mathcal{H}(0)$ and $\bar{k} = -\mathcal{H}(0)$ the controller state characterizes the normalized total energy of (13), $x_c(t) = \mathcal{H}/\mathcal{H}(0)$ if $x_c(0) = 1$, and the normalized energy supplied by the controller, $x_c(t) = \Delta \mathcal{H}/\mathcal{H}_0$ if $x_c(0) = 0$.

Notice that the conditions in Proposition 1 impose no constraint on the matrix $G_c(x_c, \mathbf{u}_c)$. Hence, $G_c$ is a degree of freedom used to guarantee the convergence of the closed-loop system to the desired equilibrium profile.

**Proposition 3.** *The boundary controller (16) with*

$$G_c^\top = \frac{\alpha}{\Gamma T|_L} \begin{bmatrix} g(x_c, \mathbf{y}) \\ 0 \end{bmatrix} \tag{27}$$

*where*

$$g(x_c, \mathbf{y}) = \phi_L(x_c)(T - T^*)|_L + \phi_0(x_c)(T - T^*)|_0$$

*with $\phi_0(x_c) \geq 0$ and $\phi_L(x_c) > \phi_0(x_c) \left( \frac{L^2 \phi_0(x_c)}{2k} - 1 \right)$, exponentially stabilizes (13) at the desired equilibrium profile $T^*$.*

*Proof.* Consider the Lyapunov functional candidate

$$\mathcal{V}(T, T^*) = \int_0^L \frac{1}{2} (T - T^*)^2 \mathrm{d}\zeta. \tag{28}$$

Its variational derivative is $\delta_s \mathcal{V} = \partial_s T (T - T^*) = T/c_v (T - T^*)$, where the fundamental property $\partial_s T = T/c_v \geq 0$ is derived from Gibbs' and Maxwell relations [28]. Hence,

$$\dot{\mathcal{V}} = \int_0^L \delta_s \mathcal{V} \partial_t s \mathrm{d}\zeta = \int_0^L \delta_s \mathcal{V} (r_s \partial_\zeta e_s + \partial_\zeta (r_s e_s))\, \mathrm{d}\zeta.$$

Defining $\bar{r}_s = \frac{k}{T^2}\partial_\zeta(T - T^*)$, using integration by parts and the fact that $\int_0^L \partial_\zeta\left(\partial_s T \bar{r}_s e_s\right)(T - T^*)\,\mathrm{d}\zeta - \int_0^L \partial_s T(T-T^*)\left(r_s \partial_\zeta e_s + \partial_\zeta\left(r_s e_s\right)\right)\mathrm{d}\zeta = 0$, $\dot{\mathcal{V}}$ is rewritten as

$$
\begin{aligned}
\dot{\mathcal{V}} = &\int_0^L \partial_\zeta\left(\partial_s T\left(T - T^*\right)\bar{r}_s e_s\right)\mathrm{d}\zeta \\
&- \int_0^L \partial_s T \bar{r}_s e_s \partial_\zeta\left(T - T^*\right)\mathrm{d}\zeta \\
&- \int_0^L \partial_\zeta\left(\partial_s T \bar{r}_s e_s\right)\left(T - T^*\right)\mathrm{d}\zeta \\
&+ \int_0^L \partial_s T(T - T^*)\left(r_s \partial_\zeta e_s + \partial_\zeta\left(r_s e_s\right)\right)\mathrm{d}\zeta \\
= &\frac{T}{c_v}(T - T^*)\bar{r}_s e_s\Big|_0^L - \int_0^L \frac{k}{c_v}\left(\partial_\zeta(T - T^*)\right)^2\mathrm{d}\zeta
\end{aligned}
$$

Since $\partial_\zeta T|_0 = \partial_\zeta T^*|_0 = \partial_\zeta T^*|_L = 0$, we get that $\bar{r}_s e_s|_0 = 0$ and $\mathbf{u} = -q_s|_L = \bar{r}_s e_s|_L$, then

$$
\dot{\mathcal{V}} = \frac{T}{c_v}(T - T^*)\Big|_L \mathbf{u} - \sigma \tag{29}
$$

with $\sigma = \int_0^L \frac{k}{c_v}\left(\partial_\zeta(T - T^*)\right)^2\mathrm{d}\zeta \geq 0$. Setting $\mathbf{u} = 0$ assures that $\dot{\mathcal{V}}$ is non-positive, however since $\sigma$ only depends on the temperature's spatial gradient, it will vanish on any profile which has the same spatial slope as the desired equilibrium profile. To guarantee the convergence to the desired equilibrium additional entropy is assigned to the system through the controller. From the power preserving interconnection (17) with $\mathbf{u}' = 0$, the controller term $G_c$ can be designed as

$$
G_c^\top = -\frac{\alpha}{\Gamma}\begin{bmatrix}\mathbf{u} \\ 0\end{bmatrix} \tag{30}
$$

Choosing the control signal as the function

$$
\mathbf{u}(x_c, \mathbf{y}) = -\frac{1}{T|_L}\left(\phi_L(x_c)(T - T^*)|_L + \phi_0(x_c)(T - T^*)|_0\right) \tag{31}
$$

with $\phi_L(x_c)$ and $\phi_0(x_c)$ non-negative functions of the controller state variable, we obtain

$$
\begin{aligned}
\dot{\mathcal{V}} = &- \int_0^L \frac{k}{c_v}(\partial_\zeta(T - T^*))^2\mathrm{d}\zeta - \frac{\phi_L}{c_v}\left(T|_L - T^*|_L\right)^2 \\
&- \frac{\phi_0}{c_v}\left(T|_L - T^*|_L\right)\left(T|_0 - T^*|_0\right).
\end{aligned} \tag{32}
$$

Using the inequality $\left(\int_0^L w(\zeta)\mathrm{d}\zeta\right)^2 \leq L^2 \int_0^L w^2(\zeta)\mathrm{d}\zeta$ and completing squares we get

$$
\begin{aligned}
\dot{\mathcal{V}} \leq &- \int_0^L \frac{k}{2c_v}(\partial_\zeta(T - T^*))^2\mathrm{d}\zeta - \frac{\phi_L}{c_v}\left(T|_L - T^*|_L\right)^2 \\
&- \frac{\phi_0}{c_v}\left(T|_L - T^*|_L\right)\left(T|_0 - T^*|_0\right) \\
&- \frac{k}{2c_v L^2}\left(\int_0^L (\partial_\zeta(T - T^*))\mathrm{d}\zeta\right)^2 \\
\leq &- \int_0^L \frac{k}{2c_v}(\partial_\zeta(T - T^*))^2\mathrm{d}\zeta - \frac{\phi_L}{c_v}\left(T|_L - T^*|_L\right)^2 \\
&- \frac{\phi_0}{c_v}\left(T|_L - T^*|_L\right)\left(T|_0 - T^*|_0\right) \\
&- \frac{k}{2c_v L^2}\left((T|_L - T^*|_L) - (T|_0 - T^*|_0)\right)^2
\end{aligned}
$$

Expanding the last term, regrouping, and completing squares, we obtain that:

$$
\begin{aligned}
\dot{\mathcal{V}} \leq &- \int_0^L \frac{k}{2c_v}(\partial_\zeta(T - T^*))^2\mathrm{d}\zeta \\
&- \left(\frac{\phi_L}{c_v} - \frac{\phi_0}{c_v}\left(\frac{L^2\phi_0}{2k} - 1\right)\right)\left(T|_L - T^*|_L\right)^2 \\
&- \frac{1}{c_v}\left(\sqrt{\frac{L^2}{2k}}\left(\phi_0 - \frac{k}{L^2}\right)\left(T|_L - T^*|_L\right)\right. \\
&\left. + \sqrt{\frac{k}{2L^2}}\left(T|_0 - T^*|_0\right)\right)^2 \\
\leq &- \int_0^L \frac{k}{2c_v}(\partial_\zeta(T - T^*))^2\mathrm{d}\zeta \\
&- \left(\frac{\phi_L}{c_v} - \frac{\phi_0}{c_v}\left(\frac{L^2\phi_0}{2k} - 1\right)\right)\left(T|_L - T^*|_L\right)^2
\end{aligned}
$$

Hence, if $\phi_0 \geq 0$ and $\phi_L > \phi_0\left(\frac{L^2\phi_0}{2k} - 1\right)$ we obtain that $\dot{\mathcal{V}} < 0$ for all $T \neq T^*$ and $\dot{\mathcal{V}} = 0$ when $T = T^*$. The exponential stability follows applying Poincaré's inequality as in [26]. □

The boundary controller defined by (16) and (27) can be interpreted as an energy/entropy shaping controller. Indeed, using the thermodynamic relation $\mathrm{d}T = \frac{T}{c_v}\mathrm{d}s$ [28] and a Taylor expansion, we obtain that $\frac{1}{2}(T - T^*)^2 = \frac{1}{2}\left(\frac{T^*}{c_v}\right)^2(s - s^*)^2 + \mathcal{O}\left((s - s^*)^3\right)$, where $s^*$ is the equilibrium entropy profile at $T^*$, and from (32) we observe that the effect of the BC is to change i) the dynamic equilibrium and ii) the shape of the closed-loop entropy function.

It is worth noting that the boundary condition (BC) does not rely on any assumptions regarding the relationship between entropy and temperature. Therefore, the BC is independent of the function that maps entropy into temperature. Additionally, it is essential to emphasize that the proposed BC enables the system to reach the desired equilibrium profile without any restrictions on the initial and boundary conditions.

**Remark 2.** *The design of $G_c$ is still valid when the measurements of $T|_0$ are not accessible by the controller. In this case, it is sufficient to set $\phi_0(x_c) = 0$, and then from* (32) *it is obtained that the closed-loop system is exponentially stable.*

The following Corollary considers the case where the reflective boundary condition at $\zeta = 0$ is substituted by a control signal, i.e., actuation at both spatial boundary sides.

**Corollary 1.** *Consider* (13) *fully actuated. The boundary controller* (16) *with*

$$G_c^\top = \frac{\alpha}{\Gamma}\left(\begin{bmatrix} -\dfrac{km^*}{T|_L} \\ -\frac{km^*}{T|_0} \end{bmatrix} + \frac{1}{c_v}\Phi(x_c)\begin{bmatrix} T(T-T^*)|_L \\ -T(T-T^*)|_0 \end{bmatrix}\right)$$

*where $\Phi(x_c) = \Phi(x_c)^\top > 0$ and $m^*$ is the desired slope of the target equilibrium profile, exponentially stabilizes* (13) *at the desired equilibrium profile $T^*$.*

Table 1: Copper material and simulation parameters

| Material parameter [29] | | | |
|---|---|---|---|
| Specific heat | | 385 J/kg°K | |
| Thermal conductivity | | 398 W/m°K | |
| Density | | 8960 kg/m$^3$ | |
| **Simulation parameters** | | | |
| $C_0$ | 273.15°K | $c_v$ | 344.96 J/m°K |
| $L$ | 0.1 m | $k$ | 0.0398 Wm/°K |

## 4. Simulation results

The structure-preserving discretization of the irreversible port-Hamiltonian system is still an open problem. As a consequence, for simulation purposes we use the discretization proposed in [7] for the Temperature formulation of the heat equation, see (9), with the heat flux at $x = L$ as the boundary input. Then, the input of this discretized system is equal to the controller outputs times $T|_L$, allowing us to implement the control laws designed in this work. We consider a copper bar of length $L = 0.1m$ and a cross-sectional area of $10^{-4}$m$^2$. Copper material properties [29] and simulation parameters are summarized in Table 1. We consider the following constitutive relation for the temperature $T(\zeta, t) = C_0 e^{s(\zeta,t)/c_v}$ [30], where $C_0$ is a constant, and a constant initial temperature profile

$$T_0 = T(\zeta, 0) = 200\zeta + 330 \quad (°\text{K}), \quad \forall\zeta \in [0, 0.1]$$

Similarly, the desired temperature equilibrium profile is defined as

$$T^* = 325 \quad (°\text{K}), \quad \zeta \in [0, 0.1]$$

The boundary controller acts on the entropy flux at the boundary $\zeta = L$, i.e., $\mathbf{u} = -q_s|_L$. To move $T_0$ to $T^*$ we use the boundary controller described in Proposition 3. Using the controller (16) with $\alpha = -1$, $\Gamma = \mathcal{H}_0$ and $\bar{k} = 0$, the controller state variable represents a measurement of the normalized total energy if $x_c(0) = 1$, i.e.,

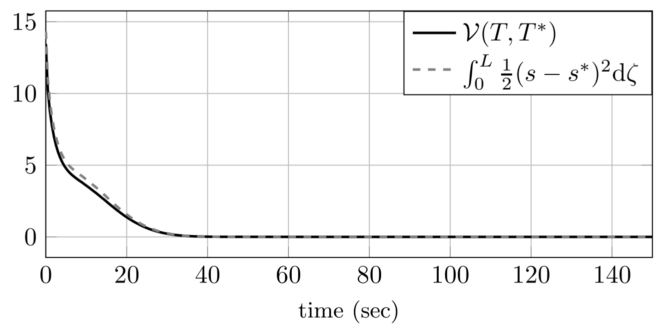
$$x_c = \frac{1}{\mathcal{H}_0}\int_0^L u(\zeta, t)\mathrm{d}\zeta.$$

Assuming temperature measurements at both boundaries we select

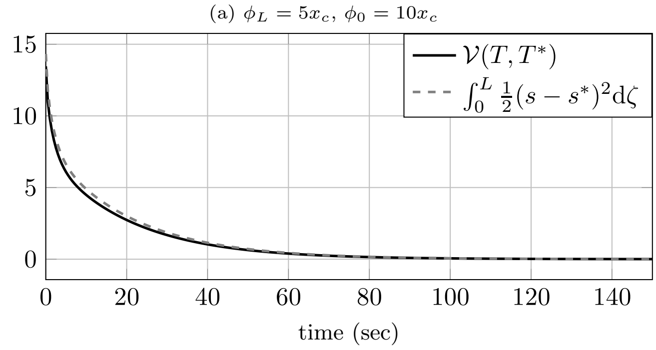
$$\phi_L(x_c) = 5x_c \quad \text{and} \quad \phi_0(x_c) = 10x_c,$$

and get the following control law

$$\begin{aligned}\mathbf{u} &= -\frac{x_c}{T|_L}\left(5(T-T^*)|_L + 10(T-T^*)|_0\right) \\ &= -\frac{5(T-T^*)|_L + 10(T-T^*)|_0}{\mathcal{H}_0 T|_L}\int_0^L u(\zeta,t)\mathrm{d}\zeta \qquad (33)\end{aligned}$$




(a) $\phi_L = 5x_c$, $\phi_0 = 10x_c$




(b) $\phi_L = 5x_c$, $\phi_0 = 0$

Figure 2: Behavior of the Lyapunov function

Figure 2 shows the behavior of the Lyapunov function (28) (solid lines) and the entropy shaping functional $\displaystyle\int_0^L \frac{1}{2}(s - s^*)\mathrm{d}\zeta$ (dashed lines). Figure 2a is the result of using the control law (33) obtaining an approximated settling time of $t = 40s$. Figure 2b is obtained considering temperature measurements only at the actuated boundary, i.e., with $\phi_0 = 0$, resulting in an approximated settling time of $t = 80s$. In both graphics, it is observed that the entropy functional and $\mathcal{V}$ have almost the same shape. It is also observed that considering temperature measurements only at the actuated boundary is sufficient to guarantee exponential convergence to the target equilibrium profile. On the other hand, when having measurements from both boundaries, the convergence rate is notably increased. In this example, the controller with all the boundary measurements converges 2 times faster than with measurements from only one boundary.

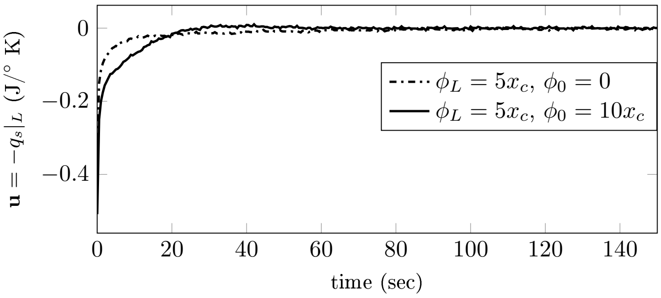
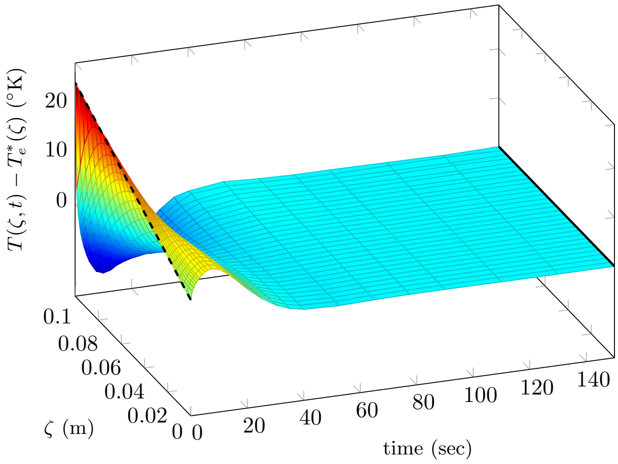
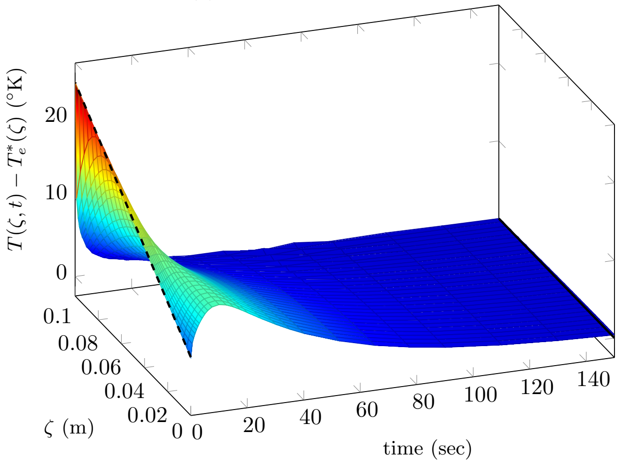



Figure 3: Boundary entropy fluxes

Figure 3 shows the boundary entropy fluxes for the controller with and without $T|_0$. Note that when $\phi_0 = 0$, the control action initially approaches 0 rapidly and then slowly extracts entropy (heat) for an extended period of time. On the other hand, when $\phi_0 = 10x_c$, The entropy extraction/injection is greater during the first 20s, and then stabilizes at 0 quickly. This greater extraction of entropy induces an undershoot in the trajectories of the temperature error $T(\zeta, t) - T^*$, as shown in Figure 4a, helping to decrease the temperature error in the opposite boundary. Figure 4b shows how with $\phi_0 = 0$ the temperature error at the controlled boundary converges to 0 quickly. However, the convergence at the opposite boundary is slower.




(a) $\phi_L = 5x_c$, $\phi_0 = 10x_c$




(b) $\phi_L = 5x_c$, $\phi_0 = 0$

Figure 4: Temperature error, $T(\zeta, t) - T^*(\zeta)$, through the trajectories.

## 5. Conclusion

An exponentially stabilizing energy-shaping and entropy-assignment boundary controller for the heat equation defined on a 1D spatial domain with actuation at one boundary and a reflective boundary condition at the other boundary has been proposed. Using the IPHS formulation and motivated by passivity-based control techniques developed for PHS defined on 1D spatial domains, a boundary controller that exponentially stabilizes the plant at the desired equilibrium profile has been developed. The existence of structural invariant functions has been characterized in order to shape the closed-loop energy and assign the required closed-loop entropy. The proposed boundary controller encompasses the cases in which measurements are available at only one or at both boundaries. Numerical simulations have been used to illustrate the design approach. Future work aims to extend these control design techniques to a larger class of IPHS.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## References


[1] H. Ramirez, B. Maschke, D. Sbarbaro, Irreversible port-Hamiltonian systems: A general formulation of irreversible processes with application to the CSTR, Chemical Engineering Science 89 (2013) 223–234.

[2] H. Ramirez, Y. Le Gorrec, An irreversible port-Hamiltonian formulation of distributed diffusion processes, IFAC-PapersOnLine 49 (24) (2016) 46–51.

[3] H. Ramirez, Y. Le Gorrec, B. Maschke, F. Couenne, On the passivity based control of irreversible processes: A port-Hamiltonian approach, Automatica 64 (2016) 105–111.

[4] H. Ramirez, Y. L. Gorrec, B. Maschke, Boundary controlled irreversible port-Hamiltonian systems, Chemical Engineering Science 248 (2022) 117107.

[5] V. Duindam, A. Macchelli, S. Stramigioli, H. Bruyninckx, Modeling and Control of Complex Physical Systems, Springer Berlin Heidelberg, 2009.

[6] A. van der Schaft, D. Jeltsema, Port-Hamiltonian Systems Theory: An Introductory Overview, now Publishers Inc, Boston, MA, USA, 2014.

[7] A. Serhani, G. Haine, D. Matignon, Anisotropic heterogeneous n-D heat equation with boundary control and observation: II. Structure-preserving discretization, IFAC-PapersOnLine 52 (7) (2019) 57–62.

[8] S. Micu, E. Zuazua, Regularity issues for the null-controllability of the linear 1-d heat equation, Systems and Control Letters 60 (6) (2011) 406–413.

[9] M. Krstic, Lyapunov adaptive stabilization of parabolic PDEs - Part I: A benchmark for boundary control, Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC '05 2005 (2005) 3164–3169.

[10] P. Martin, L. Rosier, P. Rouchon, Null controllability of the heat equation using flatness, Automatica 50 (12) (2014) 3067–3076.

[11] Q.-Q. Hu, F.-F. Jin, B.-Q. Yan, Boundary Stabilization of Heat Equation with Multi-Point Heat Source, Mathematics 9 (8) (2021) 834.

[12] J. Lohéac, E. Trélat, E. Zuazua, Nonnegative control of finite-dimensional linear systems, Annales de l'Institut Henri Poincare (C) Analyse Non Lineaire 38 (2) (2021) 301–346.

[13] P. Cannarsa, G. Da Prato, J. P. Zolesio, Dynamical shape control of the heat equation, Systems and Control Letters 12 (2) (1989) 103–109.

[14] M. B. Cheng, V. Radisavljevic, W. C. Su, Sliding mode boundary control of a parabolic PDE system with parameter variations and boundary uncertainties, Automatica 47 (2) (2011) 381–387.

[15] L. A. Mora, Y. Le Gorrec, H. Ramirez, Available energy-based interconnection and entropy assignment (ABI-EA) boundary control of the heat equation: an Irreversible Port Hamiltonian approach, in: The 2022 American Control Conference (ACC), Atlanta, Georgia, USA, 2022, pp. 2397–2402.

[16] A. A. Alonso, B. E. Ydstie, Process systems, passivity and the second law of thermodynamics, Computers & Chemical Engineering 20 (1996) S1119–S1124.

[17] B. Ydstie, Passivity based control via the second law, Computers & Chemical Engineering 26 (7-8) (2002) 1037–1048.

[18] Y. Le Gorrec, A. Macchelli, H. Ramirez, H. Zwart, Energy shaping of boundary controlled linear port Hamiltonian systems, IFAC Proceedings Volumes, 19th IFAC World Congress 47 (3) (2014) 1580–1585.

[19] A. Macchelli, Y. L. Gorrec, H. Ramírez, H. Zwart, F. Califano, Control Design for Linear Port-Hamiltonian Boundary Control Systems: An Overview, in: G. Sklyar, A. Zuyev (Eds.), Stabilization of Distributed Parameter Systems: Design Methods and Applications, Vol. 2 of SEMA SIMAI Springer Series, Springer, Cham, 2021, pp. 57–72.

[20] A. Macchelli, Y. Le Gorrec, H. Ramirez, H. Zwart, On the synthesis of boundary control laws for distributed port-Hamiltonian systems, IEEE Transactions on Automatic Control 62 (4) (2017) 1700–1713.

[21] Y. Le Gorrec, H. Zwart, B. Maschke, Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators, SIAM Journal on Control and Optimization 44 (5) (2005) 1864–1892.

[22] J. A. Villegas, Y. Le Gorrec, H. Zwart, B. Maschke, Boundary control for a class of dissipative differential operators including diffusion systems, Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (2006) 297–304.

[23] H. Zwart, Y. Le Gorrec, B. Maschke, Building systems from simple hyperbolic ones., Systems and control letters 23 (1) (2017) 1864–1892.

[24] R. Rashad, F. Califano, A. J. van der Schaft, S. Stramigioli, Twenty years of distributed port-Hamiltonian systems: a literature review, IMA Journal of Mathematical Control and Information 37 (4) (2020) 1400–1422.

[25] A. Serhani, G. Haine, D. Matignon, Anisotropic heterogeneous n-D heat equation with boundary control and observation: I. Modeling as port-Hamiltonian system, IFAC-PapersOnLine 52 (7) (2019) 51–56.

[26] M. Krstic, A. Smyshlyaev, Boundary Control of PDEs: A Course on Backstepping Designs, Advances in Design and Control SIAM's, Society for Industrial and Applied Mathematics, USA, 2008.

[27] A. J. van der Schaft, $L_2$-Gain and Passivity Techniques in Nonlinear Control, 2nd Edition, Springer-Verlag, New York, USA, 2000.

[28] D. Kondepudi, I. Prigogine, Modern Thermodynamics: From Heat Engines to Dissipative Structures, John Wiley & Sons, Chichester, England, 1998.

[29] W. Gale, T. Totemeier (Eds.), Smithells Metals Reference Book, 8th Edition, Butterworth-Heinemann, Burlington, USA, 2004.

[30] R. J. LeVeque, Finite Volume Methods for Hyperbolic Problems, Cambrige Texts in Applied Mathematics, Cambridge University Press, Cambridge, U.K., 2002.