A systematic methodology for port-Hamiltonian modeling of multidimensional flexible linear mechanical systems

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Abstract

This article introduces a novel systematic methodology for modeling a class of multidimensional linear mechanical systems that directly allows to obtain their infinite-dimensional port-Hamiltonian representation. While the approach is tailored to systems governed by specific kinematic assumptions, it encompasses a wide range of models found in current literature, including ℓ-dimensional elasticity models (where $\ell = 1, 2, 3$), vibrating strings, torsion in circular bars, classical beam and plate models, among others. The methodology involves formulating the displacement field using primary generalized coordinates via a linear algebraic relation. The non-zero components of the strain tensor are then calculated and expressed using secondary generalized coordinates, enabling the characterization of the skew-adjoint differential operator associated with the port-Hamiltonian representation. By applying Hamilton's principle and employing a specially developed integration by parts formula for the considered class of differential operators, the port-Hamiltonian model is directly obtained, along with the definition of boundary inputs and outputs. To illustrate the methodology, the plate modeling process based on Reddy's third-order shear deformation theory is presented as an example. To the best of our knowledge, this is the first time that a port-Hamiltonian representation of this system is presented in the literature.

Keywords: Infinite-dimensional systems, port-Hamiltonian systems, modeling, Hamilton's principle.

1. Introduction

The modeling of flexible mechanical systems traditionally employs two fundamental approaches: Newton's method and d'Alembert's principle. Newton's method derives equations of motion from the forces acting on the system, whereas d'Alembert's principle uses the concept of virtual work. Applying d'Alembert's principle yields the Euler-Lagrange (E-L) equations, where the Lagrangian functional that depends on the energy of the system is the key quantity used. This approach offers a more compact description of the system, which allows the use of variational techniques to obtain approximate and analytical solutions [1, 2]. Therefore, the E-L equations can also be derived from fundamental physics principles, such as Hamilton's principle [3]. The connection between the E-L equations and Hamiltonian mechanics is demonstrated through the

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Legendre transform and the Poisson structure [4, 5]. In classical Hamiltonian mechanics, the equations describe the temporal evolution of the system through a skew-symmetric matrix called symplectic matrix, which is nothing more than a representation of the set of transformations that leaves the Poisson brackets invariant [5]. In addition, the Poisson structure defines the algebra of the brackets and allows the study of the dynamic properties of the system [6]. In the present work we focus on port-Hamiltonian systems (PHS), which are a class of geometrically defined open physical systems with ports that came from the network modeling of physical systems through bond-graphs, which are associated with a Dirac structure that generalizes the Poisson structure of the classical Hamiltonian approach [7, 8]. PHS are passive systems in which the interconnection structure is separated from the constitutive laws of its energy-storing elements. In these systems, the energy exchanged with the environment flows through their power conjugate physical ports [9]. These models offer a unified representation of complex physical systems, which simplifies the analysis of their dynamic behavior and facilitates the design of effective control strategies [10]. Thus, these properties make them especially suitable for the modeling, control and analysis of complex multiphysics systems, as has been widely reported in the literature on the matter. For a more complete review refer to [11].

Since the introduction of finite-dimensional PHS in [7] and their subsequent extension to infinite dimensions in [8], the usual way to derive the port-Hamiltonian (PH) representation of existing infinite-dimensional models involves choosing energy variables appropriately. Despite the fact that this procedure may require intuition to select the correct set of state variables ensuring the emergence of the skew-adjoint differential operator, it has proven effective for reformulating classical elasticity-based models within the PH framework. Examples include the vibrating string model [12], Euler-Bernoulli beam [13], Timoshenko beam [14], and classical plate models [15, 16, 17]. In [18], it is demonstrated that the E-L equations of flexible mechanisms can be derived by applying Hamilton's principle to a system with a properly defined Lagrangian functional. Then, through a suitable change of variables it can be rewritten as an infinite-dimensional PHS. A more general treatment of linear elastodynamics is presented in [19], where it is shown that defining energy variables as vector or tensor fields leads to PH models associated with coordinate-free skew-adjoint differential operators. It should be noted that although these works typically begin with another model representation, such as the E-L equations, they are grounded in specific kinematic assumptions and constitutive laws. In this line, [20] shows a one-to-one correspondence between E-L equations and field port-Lagrangian systems on variational complexes of jet bundles. When the Lagrangian functional is known, a systematic procedure for the transformation to a port-Lagrangian model is presented. Furthermore, other works, such as [21, 22, 23, 24], explore the relationships between field equations from variational structures and PH representations, the connection between Poisson and Dirac structures, and propose adapted variational methods such as the Hamilton–Pontryagin principle, which internally includes the Legendre transform and allows for direct derivation of port-Lagrangian representations.

This work proposes a novel systematic modeling methodology for obtaining the PH representation of flexible mechanical systems characterized by the Stokes-Dirac structure. We restrict ourselves to systems derived from linear elasticity whose kinematic assumptions fit with a certain class of displacement field. The main contributions include characterizing the integration by parts formula for a class of higher-order multidimensional differential operators and the modeling methodology itself. Key distinctions from previous works include: explicit definition of energy and co-energy variables, facilitating the a priori construction of the skew-adjoint differential operator; applicability for creating models from scratch based on kinematic assumptions and constitutive laws; and explicit definition of boundary variables. The methodology offers the advantage of systematically deriving the PH representation of linear mechanical systems in any spatial dimension. It bypasses the need for extensive algebraic work typically associated with applying variational principles or Newton's methods. Additionally, it eliminates the reliance on intuition in the modeling process by providing explicit definitions of state variables, co-energy variables, and boundary inputs and outputs.

The paper is structured as follows. Section 2 presents the framework of infinite-dimensional PHS and provides a motivation example for the modeling of flexible systems. Section 3 presents the results that support the proposed methodology. To demonstrate the potential of the results, Section 4 presents a "non-classical" Reddy's plate model, which to the best of our knowledge, has not previously been formulated as an infinite-dimensional PHS. Finally, Section 5 concludes the paper and offers perspectives for future work.

2. Background

This section presents the theory that supports the rest of the article. Subsection 2.1 begins by presenting Hamilton's principle for a continuum elastic body. Then, Subsection 2.2 defines infinite-dimensional PHS and shows how they are associated with Stokes-Dirac structures. Lastly, in subsection 2.3 a motivation example is used to define notations and highlight the main difficulties in deriving an appropriate PH model for a flexible system.

2.1. Hamilton's principle for a continuum elastic body

The branch of physics that studies the relationship between external forces, internal stresses, and deformation of an elastic body is known as elasticity. For the mathematical description of elastic problems, particularly those based on linear elasticity, it is necessary to introduce the concepts of the displacement field, stress and strain tensors. For a more in-depth review of elasticity and all related concepts, the reader can refer to [25, 26]. From Figure 1, let $\Omega_T \subset \mathbb{R}^3$ be the total volume of an elastic body in space, the displacement field $\mathbf{u} = \mathbf{u}(\mathtt{X}, t) \in \mathbb{R}^3$ assigns to each point $\mathtt{X}_p \in \Omega_T$ of the body a displacement vector $\mathrm{u}(\mathtt{X}_p, t)$, that specifies its current position at time $t > t_0$ in the deformed configuration regarding an initial undeformed configuration at time t_0.

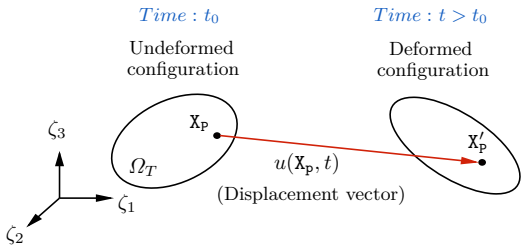


Figure 1: Displacement field and configurations.

In case of small deformation ($\|\nabla \mathbf{u}(\mathtt{X}, t)\| << 1$), the strain tensor $\underline{\varepsilon}(\mathtt{X}, t) \in \mathbb{R}^{3\times3}$, which is a measure of deformation excluding rigid body displacements, is defined as

$$\underline{\varepsilon}(\mathtt{X}, t) = \mathrm{Grad}(\mathbf{u}(\mathtt{X}, t)), \tag{1}$$

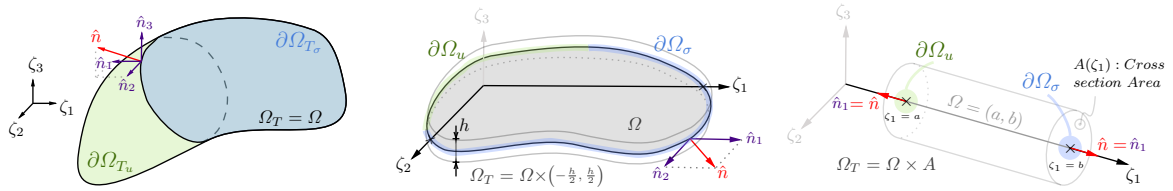
where $\mathrm{Grad}(\cdot) = \frac{1}{2}[(\nabla(\cdot)) + (\nabla(\cdot))^\top]$ is the symmetric part of the tensor gradient operator. The stress tensor is obtained by means of the generalized Hooke's law given by $\underline{\sigma}(\mathtt{X}, t) = \underline{\underline{C}} : \underline{\varepsilon}(\mathtt{X}, t) \in \mathbb{R}^{3\times3}$, with $\underline{\underline{C}} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ a constant symmetric fourth-order constitutive tensor. We consider a linear elastic body with volume $\Omega_T \subset \mathbb{R}^3$ as the one illustrated in Figure 2a. Its boundary is defined by $\partial\Omega_T = \partial\Omega_{T_u} \cup \partial\Omega_{T_\sigma}$ where $\partial\Omega_{T_u}$ and $\partial\Omega_{T_\sigma}$ denote the Dirichlet and Neumann boundary portions, on which displacements and stresses are specified, respectively. The kinetic energy T, the elastic potential energy U, and the total external work W_E are defined as [25]:

$$T = \frac{1}{2}\int_{\Omega_T} \rho(\mathtt{X})\dot{\mathbf{u}}(\mathtt{X}, t) \cdot \dot{\mathbf{u}}(\mathtt{X}, t)\, d\mathtt{X}, \tag{2}$$

$$U = \frac{1}{2}\int_{\Omega_T} \underline{\sigma}(\mathtt{X}, t) : \underline{\varepsilon}(\mathtt{X}, t)\, d\mathtt{X}, \tag{3}$$

$$W_E = \int_{\Omega_T} \mathbf{F}(\mathtt{X}, t) \cdot \mathbf{u}(\mathtt{X}, t)\, d\mathtt{X} + \int_{\partial\Omega_{T_\sigma}} \mathbf{t}(\mathtt{S}_\sigma, t) \cdot \mathbf{u}(\mathtt{S}_\sigma, t)\, d\mathtt{S}_\sigma, \tag{4}$$

where $\mathtt{S}_\sigma \in \partial\Omega_{T_\sigma}$ and $\mathtt{S}_u \in \partial\Omega_{T_u}$ are curvilinear coordinates along the boundary, $\partial_t \mathbf{u}(\mathtt{X}, t) = \dot{\mathbf{u}}(\mathtt{X}, t) \in \mathbb{R}^3$ is the velocity vector, $\rho(\mathtt{X}) \in \mathbb{R}$ is the density of the body, $\mathbf{F}(\mathtt{X}, t) \in \mathbb{R}^3$ is the sum of all external body forces,



(a) 3D domain (b) 2D domain (c) 1D domain

Figure 2: Schemes to illustrate notation.

and $\mathbf{t}(\mathtt{S}_\sigma, t) \in \mathbb{R}^3$ is the sum of all external surface forces (also called tractions). Hamilton's principle states that the true evolution $\mathbf{u} = \mathbf{u}(\mathtt{X}, t)$ of a system, between two specific states $\mathbf{u}_1 = \mathbf{u}(\mathtt{X}, t_1)$ and $\mathbf{u}_2 = \mathbf{u}(\mathtt{X}, t_2)$ at two specific times t_1 and t_2 is a stationary point (a point where the variation is zero) of the action functional, that is [25, Chapter 7.7]

$$\delta \int_{t_1}^{t_2} T - (U + V)\, dt = 0, \tag{5}$$

where $V = -W_E$ is the potential associated to the external forces. Defining as generalized coordinates the variables that define the configuration of the system and defining as configuration space the space generated by these coordinates, Hamilton's principle states that, as the system evolves, a path is traced through the configuration space, where the real path $\mathbf{u}(\mathtt{X}, t)$ taken by the system has a stationary action under admissible small variations ($\delta\mathbf{u}$) in the configuration of the system. Thus, an admissible variation $\delta\mathbf{u}$, also called a virtual displacement, must be consistent with the essential boundary conditions (in $\partial\Omega_{T_u}$) and satisfies

$$\delta\mathbf{u}(\mathtt{S}_u, t) = 0 \text{ on } \partial\Omega_{T_u} \text{ for all } t, \tag{6}$$

$$\delta\mathbf{u}(\mathtt{X}, t_1) = \delta\mathbf{u}(\mathtt{X}, t_2) = 0 \text{ for all } \mathtt{X}. \tag{7}$$

So with the definitions of kinetic energy T, elastic potential energy U and external work W_E, in (2), (3) and (4), respectively, equations (5) to (7) can be used to find the dynamic equations of the system.

2.2. Port-Hamiltonian systems

The theory of PHS provides a framework for the geometric description of interconnected systems. The framework's key features are the precise characterization of power flow between subsystems, the separation of the interconnecting structure from the constitutive relationships of its components, and the exploitation of this structure for analysis and control. Hence, the PH framework is particularly well adapted for the modeling, control, and simulation of complex nonlinear multiphysics systems [10]. Using the so-called input and output ports, the modeling of complex systems can be approached constructively as the interconnection of elementary PH subsystems that may belong to one or more different physical domains (mechanical, electrical, hydraulic, thermal, etc.), where the interconnection is such that the total energy is preserved and the resulting system is also a PHS [9]. In simple terms, the conservative infinite-dimensional system can be written as a PHS from [19] as

$$\begin{aligned} \partial_t x &= \mathcal{J}\,\delta_x H + \mathcal{G}\,u_d \\ y_d &= \mathcal{G}^*\delta_x H \\ u_\partial &= \mathcal{B}_\partial \delta_x H \\ y_\partial &= \mathcal{C}_\partial \delta_x H \end{aligned} \tag{8}$$

where $\partial_t = \partial/\partial t$, x is the state, u_d, y_d are the distributed input and output ports, respectively. $\mathcal{J} = -\mathcal{J}^*$ is a formally skew-adjoint differential operator, $\mathcal{G}$, $\mathcal{G}^*$ are the input map operator and its formal adjoint, respectively. $\delta_x H$ is the variational derivative of the Hamiltonian functional $H(x)$ with respect to x and defines the co-energy variables. $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ are boundary operators that provide the boundary input u_∂ and

boundary output y_∂ [27, 28]. In order to define a PHS in the Stokes-Dirac structure, the operators $\mathcal{J}$, $\mathcal{B}_\partial$ and $\mathcal{C}_\partial$ must satisfy an integration by parts formula (for more details see [19, Assumption 1]).

Definition 1. Let $\mathscr{B}$, $\mathscr{F}$, and $\mathscr{E}$ be Hilbert spaces, where $\mathscr{F}$ is the flow space, its dual $\mathscr{E}$ is the effort space, and $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ is called the bond space of power variables. A Stokes-Dirac structure on $\mathscr{B} = \mathscr{F} \times \mathscr{E}$ is a subspace $\mathscr{D}_s \subset \mathscr{B}$ such that $\mathscr{D}_s = \mathscr{D}_s^\perp$ with respect to a bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ given by [27]

$$\langle\langle (\mathtt{f}_1, \mathtt{f}_{\partial_1}, \mathtt{e}_1, \mathtt{e}_{\partial_1}), (\mathtt{f}_2, \mathtt{f}_{\partial_2}, \mathtt{e}_2, \mathtt{e}_{\partial_2}) \rangle\rangle = \langle \mathtt{e}_1 | \mathtt{f}_2 \rangle_{in}^{\Omega} + \langle \mathtt{e}_2 | \mathtt{f}_1 \rangle_{in}^{\Omega} - \langle \mathtt{e}_{\partial_1} | \mathtt{f}_{\partial_2} \rangle_{in}^{\partial\Omega} - \langle \mathtt{e}_{\partial_2} | \mathtt{f}_{\partial_1} \rangle_{in}^{\partial\Omega},$$

where $\langle \cdot | \cdot \rangle_{in}^{\Omega}$ and $\langle \cdot | \cdot \rangle_{in}^{\partial\Omega}$ are inner products defined over the spatial domain Ω, and its boundary $\partial\Omega$, respectively. From the previous definition we have that for any $(\mathtt{f}, \mathtt{f}_\partial, \mathtt{e}, \mathtt{e}_\partial) \in \mathscr{D}_s$ it is satisfied that $\langle\langle (\mathtt{f}, \mathtt{f}_\partial, \mathtt{e}, \mathtt{e}_\partial), (\mathtt{f}, \mathtt{f}_\partial, \mathtt{e}, \mathtt{e}_\partial) \rangle\rangle = 0$, which is verified in a general way using the Stokes' theorem [27].

Now, consider the infinite-dimensional PHS in (8), if we choose $\mathtt{f} = [\mathtt{f}_s, \mathtt{f}_e, \mathtt{f}_\partial]^\top$ and $\mathtt{e} = [\mathtt{e}_s, \mathtt{e}_e, \mathtt{e}_\partial]^\top$, where $\mathtt{f}_s = \partial_t x$, $\mathtt{f}_e = u_d$, $\mathtt{f}_\partial = u_\partial$, $\mathtt{e}_s = \delta_x H$, $\mathtt{e}_e = -y_d$, $\mathtt{e}_\partial = -y_\partial$, then the set

$$\mathscr{D}_s = \{(\mathtt{f}, \mathtt{e}) \in \mathscr{B} \mid \mathtt{f}_s = \mathcal{J}\mathtt{e}_s + \mathcal{G}\mathtt{f}_e \,,\, \mathtt{e}_e = -\mathcal{G}^*\mathtt{e}_s \,,\, \mathtt{f}_\partial = \mathcal{B}_\partial \mathtt{e}_s \,,\, \mathtt{e}_\partial = -\mathcal{C}_\partial \mathtt{e}_s\}$$

is a Stokes-Dirac structure [19]. With the above definitions for $\mathtt{f}$ and $\mathtt{e}$, it is easy to show that the system (8) is conservative, that is, $\langle\langle (\mathtt{f}, \mathtt{e}), (\mathtt{f}, \mathtt{e}) \rangle\rangle = 0$, and that the energy exchange with the environment is determined by the distributed and boundary ports by the expression

$$\partial_t H = \langle y_d |\, u_d \rangle_{in}^{\Omega} + \langle y_\partial |\, u_\partial \rangle_{in}^{\partial\Omega}.$$

2.3. Motivation example: Timoshenko beam

In this part, we use the Timoshenko beam as an example to highlight the main difficulties in obtaining the PH representation for such system and to introduce several notations and key assumptions for the proposed modeling methodology. First consider the beam scheme in Figure 2c, with $\Omega = (a, b) \subset \mathbb{R}$ the spatial domain, $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\sigma$ the boundary domain, where $\partial\Omega_u = \{a\}$ is the portion where displacements are imposed, and $\partial\Omega_\sigma = \{b\}$ the portion where tractions are imposed. In this example assume no body forces applied to the beam ($\mathbf{F}(\mathtt{X}, t) = 0$). The Timoshenko beam model is based on the kinematic assumption of first-order shear deformation theory [29, Chapter 10.1], which states that plane sections perpendicular to the neutral axis before deformation remain plane but not necessarily perpendicular to the neutral axis after deformation. In Figure 3 are shown the undeformed and deformed configurations. Let $\mathtt{X}$ be an arbitrary point with coordinates $\mathtt{X} = [\zeta_1, \zeta_2, \zeta_3]^\top$, as the plane sections remain plane, the position of $\mathtt{X}$ in the deformed configuration ($\mathtt{X}'$) has coordinates $\mathtt{X}' = \mathtt{X}'(\mathtt{X}, t)$ and are given by

$$\mathtt{X}'(\mathtt{X}, t) = \begin{bmatrix} \zeta_1 - \zeta_3 \sin(\psi(\zeta_1, t)) \\ \zeta_2 \\ w(\zeta_1, t) + \zeta_3 \cos(\psi(\zeta_1, t)) \end{bmatrix} \approx \begin{bmatrix} \zeta_1 - \zeta_3\, \psi(\zeta_1, t) \\ \zeta_2 \\ w(\zeta_1, t) + \zeta_3 \end{bmatrix},$$

where $\sin(\psi(\zeta_1, t)) \approx \psi(\zeta_1, t)$ and $\cos(\psi(\zeta_1, t)) \approx 1$ come from the assumption of small deformation.

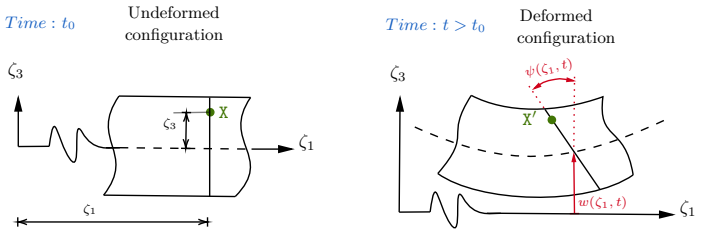


Figure 3: Kinematic assumption of Timoshenko beam.

Then, the kinematic assumption of the Timoshenko beam is equivalent to the displacement field $\mathbf{u}(\mathtt{X},t)\in\mathbb{R}^3$ given by

$$\mathbf{u}(\mathtt{X},t)=\mathtt{X}'(\mathtt{X},t)-\mathtt{X}=\begin{bmatrix}-\zeta_3\,\psi(\zeta_1,t)\\0\\w(\zeta_1,t)\end{bmatrix}, \tag{9}$$

where $w(\zeta_1,t)$ is the vertical displacement of a point in the neutral axis, and $\psi(\zeta_1,t)$ is the total angle rotated by the cross section respecting to the vertical axis. The equations of motion obtained after applying Hamilton's principle are given by (see the details in [29, Chapter 2.2.3])

$$\begin{aligned}
&\text{for all } \zeta_1\in\Omega: && \rho I(\zeta_1)\frac{\partial^2\psi}{\partial t^2}+\kappa GA(\zeta_1)\left(\psi-\frac{\partial w}{\partial\zeta_1}\right)-\frac{\partial}{\partial\zeta_1}\left(EI(\zeta_1)\frac{\partial\psi}{\partial\zeta_1}\right)=0,\\
&&& \rho A(\zeta_1)\frac{\partial^2 w}{\partial t^2}+\frac{\partial}{\partial\zeta_1}\left[\kappa GA(\zeta_1)\left(\psi-\frac{\partial w}{\partial\zeta_1}\right)\right]=0,\\
&\text{for all } \mathtt{s}_\sigma\in\partial\Omega_\sigma: && \hat{M}(\mathtt{s}_\sigma,t)-EI(\mathtt{s}_\sigma)\frac{\partial\psi}{\partial\zeta_1}=0,\\
&&& \hat{V}(\mathtt{s}_\sigma,t)-\kappa GA(\mathtt{s}_\sigma)\left(\frac{\partial w}{\partial\zeta_1}-\psi\right)=0,
\end{aligned} \tag{10}$$

where $\mathtt{s}_\sigma\in\partial\Omega_\sigma$ is a coordinate along the boundary $\partial\Omega_\sigma$, $A(\zeta_1)$ is the cross section area, $I(\zeta_1)$ the second moment of inertia of the cross section, G,E the properties of the material, κ is a correction factor, and $\hat{M}$,$\hat{V}$ are the imposed generalized boundary tractions which represent internal bending moment and internal shearing force, respectively. For the PH representation of the Timoshenko beam model, the energy variables of the system can be chosen as [14]:

$$p_1=\rho I(\zeta_1)\frac{\partial\psi}{\partial t}\quad,\quad p_2=\rho A(\zeta_1)\frac{\partial w}{\partial t}\quad,\quad \epsilon_1=\frac{\partial\psi}{\partial\zeta_1}\quad,\quad \epsilon_2=\left(\frac{\partial w}{\partial\zeta_1}-\psi\right), \tag{11}$$

where p_1 is the angular momentum, p_2 the linear momentum, ϵ_1 is the deformation due to bending (also called curvature) and ϵ_2 is the shear deformation. The Hamiltonian functional that represents the total stored energy in the system is given by

$$H(p,\epsilon)=\frac{1}{2}\int_a^b\left(\frac{p_1^2}{\rho I(\zeta_1)}+\frac{p_2^2}{\rho A(\zeta_1)}+EI(\zeta_1)\epsilon_1^2+\kappa GA(\zeta_1)\epsilon_2^2\right)d\zeta_1, \tag{12}$$

where the first and second terms are the rotational and translational kinetic energies, respectively, and the third and fourth terms are the elastic potential energy due to the bending and shearing, respectively. The power variables (or also called co-energy variables) are given by

$$\begin{aligned}
f_{p_1}&=\frac{\partial p_1}{\partial t}=\rho I(\zeta_1)\frac{\partial^2\psi}{\partial t^2} &,\quad e_{p_1}&=\frac{\delta H}{\delta p_1}=\frac{p_1}{\rho I(\zeta_1)}=\frac{\partial\psi}{\partial t},\\
f_{p_2}&=\frac{\partial p_2}{\partial t}=\rho A(\zeta_1)\frac{\partial^2 w}{\partial t^2} &,\quad e_{p_2}&=\frac{\delta H}{\delta p_2}=\frac{p_2}{\rho A(\zeta_1)}=\frac{\partial w}{\partial t},\\
f_{\epsilon_1}&=\frac{\partial\epsilon_1}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial\zeta_1}\right) &,\quad e_{\epsilon_1}&=\frac{\delta H}{\delta\epsilon_1}=EI(\zeta_1)\epsilon_1=EI(\zeta_1)\frac{\partial\psi}{\partial\zeta_1},\\
f_{\epsilon_2}&=\frac{\partial\epsilon_2}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial\zeta_1}-\psi\right) &,\quad e_{\epsilon_2}&=\frac{\delta H}{\delta\epsilon_2}=\kappa GA(\zeta_1)\epsilon_2=\kappa GA(\zeta_1)\left(\frac{\partial w}{\partial\zeta_1}-\psi\right),
\end{aligned} \tag{13}$$

where f_{p_1} is the inertial moment, f_{p_2} the inertial force, f_{ϵ_1} the bending strain velocity, f_{ϵ_2} shearing strain velocity, e_{p_1} the angular velocity, e_{p_2} the linear velocity, e_{ϵ_1} the internal bending moment and e_{ϵ_2} the internal shearing force. With all the above defined power variables and using the notation $\partial_1=\partial/\partial\zeta_1$, the

PH representation of the Timoshenko beam model is given by

$$\begin{pmatrix} f_{p_1} \\ f_{p_2} \\ f_{\epsilon_1} \\ f_{\epsilon_2} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \partial_1 & 1 \\ 0 & 0 & 0 & \partial_1 \\ \partial_1 & 0 & 0 & 0 \\ -1 & \partial_1 & 0 & 0 \end{bmatrix}}_{\mathcal{J}=-\mathcal{J}^*} \begin{pmatrix} e_{p_1} \\ e_{p_2} \\ e_{\epsilon_1} \\ e_{\epsilon_2} \end{pmatrix} \tag{14}$$
$$u_\partial = \begin{bmatrix} e_{p_1}(a) & e_{p_2}(a) & e_{\epsilon_1}(b) & e_{\epsilon_2}(b) \end{bmatrix}$$
$$y_\partial = \begin{bmatrix} e_{p_1}(b) & e_{p_2}(b) & -e_{\epsilon_1}(a) & -e_{\epsilon_2}(a) \end{bmatrix}.$$

Note that this model has four state variables and therefore they are four equations, where the two first are equivalent to the model presented in (10), and the two last are compatibility equations. From the Timoshenko beam example presented above, some of the difficulties in finding the PH representations are:

- Starting from the displacement field in (9) and knowing the constitutive relationships, the equations of motion (10) are obtained after applying Hamilton's principle, which involves a lot of algebraic work and where integration by parts is required (generally over multidimensional domains).
- The equations of motion in (10) do not always give clues about the structure of the associated PH model or the energy variables. A wrong choice of variables can lead to a model not associated with a skew-adjoint differential operator, making this an iterative process, especially in non-trivial cases.
- While the boundary conditions of (10) arise naturally from Hamilton's principle, the boundary conditions of (14) arise from the energy balance of the system, so they are defined from the Hamiltonian, the structure of the differential operator and the co-energy variables.

These difficulties motivate us to propose a systematic methodology that allows us to overcome them.

2.4. Overview of the proposed modeling methodology

In order to introduce the notations and the general approach developed in Section 3, we first apply it to the proposed example. Notation: Let $\mathtt{X} = \{\zeta_1, \zeta_2, \zeta_3\} \in \Omega_T$ be a set of Cartesian coordinates for a three-dimensional body, such as those illustrated in Figure 2. In general, consider that $\Omega_T = \Omega \times \Omega^c \subset \mathbb{R}^3$ is the total volume of the elastic body, where $\Omega \subset \mathbb{R}^\ell$ is the ℓ-dimensional spatial domain where the parameters of the model are distributed, and Ω^c is a complementary domain such that $\Omega \times \Omega^c = \Omega_T$. Let $\mathtt{x} \subset \mathtt{X}$ with $\mathtt{x} \in \Omega$ be the subset of coordinates where the parameters are distributed, and $\mathtt{x}^c \subset \mathtt{X}$ with $\mathtt{x}^c \in \Omega^c$ the complement of $\mathtt{x}$ such that $\mathtt{X} = \mathtt{x} \cup \mathtt{x}^c$ and $\mathtt{x} \cap \mathtt{x}^c = \{\emptyset\}$, where $\{\emptyset\}$ denotes the empty set. Then, a differential of volume $d\Omega_T$ and the integral of an arbitrary separable function $g(\mathtt{X}) = g_1(\mathtt{x})g_2(\mathtt{x}^c)$ over the volume Ω_T can be expressed as

$$d\Omega_T = d\zeta_3 d\zeta_2 d\zeta_1 = d\mathtt{x}^c\, d\mathtt{x} = d\mathtt{X} \quad , \quad \int_{\Omega_T} g(\mathtt{X})\, d\mathtt{X} = \int_{\Omega} g_1(\mathtt{x}) \int_{\Omega^c} g_2(\mathtt{x}^c)\, d\mathtt{x}^c\, d\mathtt{x}. \tag{15}$$

Note that in case of three-dimensional elasticity (see Figure 2a), we have $\mathtt{x} = \mathtt{X}$ and $\mathtt{x}^c = \{\emptyset\}$, then $\Omega_T = \Omega$ and $d\mathtt{X} = d\mathtt{x}$, which implies $\int_{\Omega_T} g(\mathtt{X})\, d\mathtt{X} = \int_{\Omega} g(\mathtt{X})\, d\mathtt{x}$, then $\int_{\Omega^c} g(\mathtt{X})\, d\mathtt{x}^c = g(\mathtt{X})$.

Considering the Timoshenko beam, the displacement field $\mathbf{u}(\mathtt{X}, t)$ in (9) is first rewritten as

$$\mathbf{u}(\mathtt{X}, t) = \begin{bmatrix} -\zeta_3\, \psi(\zeta_1, t) \\ 0 \\ w(\zeta_1, t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\zeta_3 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\lambda_1(\mathtt{x}^c)} \underbrace{\begin{bmatrix} \psi(\mathtt{x}, t) \\ w(\mathtt{x}, t) \end{bmatrix}}_{\mathbf{r}(\mathtt{x}, t)},$$

where $\mathbf{r}(\mathtt{x}, t)$ is the vector of primary unknowns, and $\lambda_1(\mathtt{x}^c)$ is the mapping of $\mathbf{r}(\mathtt{x}, t)$ that allows us to know the deformed configuration of the three-dimensional beam at any point inside the volume Ω_T. From Figure

2c, we see that the volume of the body can be written as $\Omega_T = \Omega \times A \subset \mathbb{R}^3$, with $\Omega = (a, b) \subset \mathbb{R}$ an open set that defines the spatial domain where the parameters are distributed, and $A \subset \mathbb{R}^2$ is the cross section area.

From (2) and $\mathbf{u}(\mathtt{X}, t)$, $\lambda_1(\mathtt{x}^c)$ allows us to find the momentum variables $p(\mathtt{x}, t)$ defined in (11) as

$$p(\mathtt{x},t) = \rho(\mathtt{x}) \int_{\Omega^c} \lambda_1(\mathtt{x}^c)^\top \lambda_1(\mathtt{x}^c)\, d\mathtt{x}^c\, \dot{\mathbf{r}}(\mathtt{x},t) = \rho(\mathtt{x}) \int_A \begin{bmatrix} \zeta_3^2 & 0 \\ 0 & 1 \end{bmatrix} dA\, \dot{\mathbf{r}}(\mathtt{x},t) = \begin{bmatrix} \rho(\mathtt{x}) I(\mathtt{x}) & 0 \\ 0 & \rho(\mathtt{x}) A(\mathtt{x}) \end{bmatrix} \begin{bmatrix} \dot{\psi}(\mathtt{x},t) \\ \dot{w}(\mathtt{x},t) \end{bmatrix}.$$

From (1), the only non-zero components of the strain tensor are

$$\begin{bmatrix} \varepsilon_{11} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} -\zeta_3\, \partial_1 \psi(\mathtt{x},t) \\ \partial_1 w(\mathtt{x},t) - \psi(\mathtt{x},t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\zeta_3 & 0 \\ 0 & 1 \end{bmatrix}}_{\lambda_2(\mathtt{x}^c)} \underbrace{\begin{bmatrix} \partial_1 & 0 \\ -1 & \partial_1 \end{bmatrix}}_{\mathcal{F}} \underbrace{\begin{bmatrix} \psi(\mathtt{x},t) \\ w(\mathtt{x},t) \end{bmatrix}}_{\mathbf{r}(\mathtt{x},t)},$$

from which it can already be seen that $\mathcal{F}$ and its formal adjoint $\mathcal{F}^*$ characterize the differential operator presented in (14) as $\mathcal{J} = -\mathcal{J}^* = [0 \quad -\mathcal{F}^* \,;\, \mathcal{F} \quad 0]$, and the generalized strains $\epsilon(\mathtt{x}, t)$ defined in (11) are given by $\epsilon(\mathtt{x}, t) = \mathcal{F}\mathbf{r}(\mathtt{x}, t)$. Writing the constitutive matrix as $C = [E \;\; 0\,; 0 \;\; \kappa G]$, from (3) $\lambda_2(\mathtt{x}^c)$ allows us to find the co-energy variables $e_\epsilon(\mathtt{x}, t)$ defined in (13) as

$$e_\epsilon(\mathtt{x},t) = \int_{\Omega^c} \lambda_2(\mathtt{x}^c)^\top\, C\, \lambda_2(\mathtt{x}^c)\, d\mathtt{x}^c\, \epsilon(\mathtt{x},t) = \int_A \begin{bmatrix} E\zeta_3^2 & 0 \\ 0 & \kappa G \end{bmatrix} dA\, \epsilon(\mathtt{x},t) = \begin{bmatrix} EI(\mathtt{x}) & 0 \\ 0 & \kappa G A(\mathtt{x}) \end{bmatrix} \begin{bmatrix} \epsilon_1(\mathtt{x},t) \\ \epsilon_2(\mathtt{x},t) \end{bmatrix}.$$

It is worth to notice that only by using the displacement field $\mathbf{u}(\mathtt{X}, t)$ without any application of the variational method, we are able to obtain the energy variables, co-energy variables and derive the structure of the skew-adjoint differential operator associated with the PH representation. This general idea will be rigorously generalized to a broader class of multidimensional systems in Section 3.

3. Modeling of linear elastic port-Hamiltonian systems

In this section the modeling methodology is presented, where the definitions of the energy and co-energy variables are explicitly presented. Moreover, we introduce all the elements related to the infinite-dimensional PH representation of linear elastic mechanical models.

3.1. Considered class of systems

The model assumptions, the definition of the class of differential operators considered and the Lemma of integration by parts for them are presented below.

Assumption 1. We consider systems defined by linear elasticity in a body of density $\rho = \rho(\mathtt{x}) \in \mathbb{R}$ and whose kinematic assumptions are represented by a displacement field $\mathbf{u}(\mathtt{X}, t) \in \mathbb{R}^3$ with the following structure

$$\mathbf{u}(\mathtt{X}, t) = \lambda_1(\mathtt{x}^c)\, \mathbf{r}(\mathtt{x}, t), \tag{16}$$

where $\lambda_1(\mathtt{x}^c) \in \mathbb{R}^{3\times n}$ is a full rank matrix, and $\mathbf{r}(\mathtt{x}, t) = [r_1(\mathtt{x}, t) \;\cdots\; r_n(\mathtt{x}, t)]^\top \in \mathbb{R}^n$ is defined as the vector of generalized displacements or primary unknowns.

Note that depending on the specific definition of the displacement field $\mathbf{u}(\mathtt{X}, t)$ some components of the strain tensor could be zero. On the other hand, to avoid working directly with higher order tensors, the Voigt-Kelvin vector notation will be used. Then, let $\varepsilon(\mathtt{X}, t) \in \mathbb{R}^d$ be the non-zero components of the Voigt-strain vector $\vec{\varepsilon}(\mathtt{X}, t)$ and $\sigma(\mathtt{X}, t) \in \mathbb{R}^d$ the correlative components, so we can write the constitutive relation between stress and strain as

$$\sigma(\mathtt{X}, t) = C\, \varepsilon(\mathtt{X}, t), \tag{17}$$

where $C = C^\top > 0 \in \mathbb{R}^{d\times d}$ is a suitable constitutive matrix according to the problem (see Appendix B for more details). In the case of a mixed boundary value problem, as represented in Figure 2, we observe that $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\sigma$ represents the boundary of the spatial domain $\Omega \subset \mathbb{R}^\ell$, consisting of complementary sub-boundaries $\partial\Omega_u$ and $\partial\Omega_\sigma$, where essential and natural boundary conditions are applied, respectively. Then, we can deduce from (4) that the total external work W_E can be equivalently expressed as:

$$W_E = \int_\Omega F_\Omega(\mathbf{x},t)\cdot\mathbf{r}(\mathbf{x},t)\,d\mathbf{x} + \int_{\partial\Omega_\sigma} \tau_\partial(\mathbf{s}_\sigma,t)\cdot\mathbf{r}(\mathbf{s}_\sigma,t)\,d\mathbf{s}_\sigma, \tag{18}$$

where $F_\Omega(\mathbf{x},t)\in\mathbb{R}^n$ is defined as the imposed generalized load, and $\tau_\partial(\mathbf{s}_\sigma,t)\in\mathbb{R}^n$ with $\mathbf{s}_\sigma\in\partial\Omega_\sigma$ is defined as the imposed generalized boundary traction.

Assumption 2. Assume that the imposed generalized load can be expressed equivalently as $F_\Omega(\mathbf{x},t) = B_d(u_d(\mathbf{x},t))$, where $u_d(\mathbf{x},t)\in\Omega$ is the distributed external input and $B_d(\cdot)$ is the input map that may be a matrix or a differential operator of the same class as in Definition 2.

Definition 2. Let $\mathtt{X} = \{\zeta_1,\dots,\zeta_\ell\}$ be a set of orthogonal coordinate axes and also the coordinates of an arbitrary point of an open set $\Omega\subset\mathbb{R}^\ell$, $v(\mathtt{X})\in\mathbb{R}^m$ and $w(\mathtt{X})\in\mathbb{R}^n$ two vector functions. Consider a linear differential operator $\mathcal{F}$ and its formal adjoint $\mathcal{F}^*$ such that, $\mathcal{F}\,w(\mathtt{X}) = P_0\,w(\mathtt{X}) + \sum_{k=1}^{\ell}\mathcal{F}_k\,w(\mathtt{X})$ and $\mathcal{F}^*\,v(\mathtt{X}) = P_0^\top\,v(\mathtt{X}) + \sum_{k=1}^{\ell}\mathcal{F}_k^*\,v(\mathtt{X})$, where $\mathcal{F}_k$ and its formal adjoint $\mathcal{F}_k^*$ are given by [27]

$$\mathcal{F}_k\,w(\mathtt{X}) = \sum_{i=1}^{N} P_k(i)\,\partial_k^i\,w(\mathtt{X}), \tag{19}$$

$$\mathcal{F}_k^*\,v(\mathtt{X}) = \sum_{i=1}^{N}(-1)^i P_k(i)^\top\partial_k^i\,v(\mathtt{X}), \tag{20}$$

with $\partial_k^i = \partial^i/\partial\zeta_k^i$, P_0, $P_k(i)\in\mathbb{R}^{m\times n}$ and N the order of the highest derivative with respect to any ζ_k.

Lemma 1. Consider that Definition 2 holds and define the open set $\Omega\subset\mathbb{R}^\ell$ as an ℓ-dimensional domain, its boundary $\partial\Omega$ and $\bar\Omega = \Omega\cup\partial\Omega$ the closure, such that $\mathbf{x}\in\Omega$ and $\mathbf{s}\in\partial\Omega$, where $\mathbf{s}$ is the coordinates of an arbitrary point on $\partial\Omega$. Then for any pair of functions $v(\mathbf{x})\in\mathbb{R}^m$ and $w(\mathbf{x})\in\mathbb{R}^n$ defined in $\bar\Omega$, we have that

$$\int_\Omega \left(v(\mathbf{x})^\top\mathcal{F}\,w(\mathbf{x}) - w(\mathbf{x})^\top\mathcal{F}^*\,v(\mathbf{x})\right)d\mathbf{x} = \sum_{k=1}^{\ell}\sum_{i=1}^{N}\sum_{j=1}^{i}(-1)^{j-1}\int_{\partial\Omega}\partial_k^{i-j}w(\mathbf{s})^\top P_k(i)^\top\hat n_k(\mathbf{s})\,\partial_k^{j-1}v(\mathbf{s})\,d\mathbf{s}, \tag{21}$$

$$= \int_{\partial\Omega}\mathcal{B}(w(\mathbf{s}))^\top\mathcal{Q}_\partial(\mathbf{s})\,\mathcal{B}(v(\mathbf{s}))\,d\mathbf{s}, \tag{22}$$

where $\hat n_k(\mathbf{s})$ is the component of the outward unit normal vector to the boundary projected on the coordinate axis ζ_k, $\mathcal{B}(\cdot)$ is a linear differential operator defined as

$$\mathcal{B}(\cdot) = \left[(\cdot)\quad \partial_1(\cdot)\ \cdots\ \partial_\ell(\cdot)\quad \partial_1^2(\cdot)\ \cdots\ \partial_\ell^2(\cdot)\quad \cdots\quad \partial_1^{N-1}(\cdot)\ \cdots\ \partial_\ell^{N-1}(\cdot)\right]^\top, \tag{23}$$

$\mathcal{Q}_\partial(\mathbf{s})\in\mathbb{R}^{n+(N-1)n\ell\times m+(N-1)m\ell}$ is a matrix given by

$$\mathcal{Q}_\partial(\mathbf{s}) = \begin{bmatrix} P_\partial(\mathbf{s}) & -W_2(\mathbf{s}) & W_3(\mathbf{s}) & -W_4(\mathbf{s}) & \cdots & (-1)^{N-1}W_N(\mathbf{s}) \\ V_2(\mathbf{s}) & -\Lambda_3(\mathbf{s}) & \Lambda_4(\mathbf{s}) & \ddots & \ddots & 0 \\ V_3(\mathbf{s}) & -\Lambda_4(\mathbf{s}) & \Lambda_5(\mathbf{s}) & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ V_{N-1}(\mathbf{s}) & -\Lambda_N(\mathbf{s}) & 0 & & & \vdots \\ V_N(\mathbf{s}) & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{24}$$

with $P_\partial(\mathtt{s}) \in \mathbb{R}^{n\times m}$, $W_i(\mathtt{s}) \in \mathbb{R}^{n\times m\ell}$, $V_i(\mathtt{s}) \in \mathbb{R}^{n\ell\times m}$, and $\Lambda_i(\mathtt{s}) \in \mathbb{R}^{n\ell\times m\ell}$ defined as

$$\begin{aligned} P_\partial(\mathtt{s}) &= \sum_{k=1}^{\ell} P_k(1)^\top \hat{n}_k(\mathtt{s}) \quad , \quad W_i(\mathtt{s}) = \begin{bmatrix} P_1(i)^\top \hat{n}_1(\mathtt{s}) & \cdots & P_\ell(i)^\top \hat{n}_\ell(\mathtt{s}) \end{bmatrix}, \\ V_i(\mathtt{s}) &= \begin{bmatrix} P_1(i)^\top \hat{n}_1(\mathtt{s}) \\ \vdots \\ P_\ell(i)^\top \hat{n}_\ell(\mathtt{s}) \end{bmatrix} \quad , \quad \Lambda_i(\mathtt{s}) = \begin{bmatrix} P_1(i)^\top \hat{n}_1(\mathtt{s}) & & 0 \\ & \ddots & \\ 0 & & P_\ell(i)^\top \hat{n}_\ell(\mathtt{s}) \end{bmatrix}. \end{aligned} \tag{25}$$

Proof. By iteratively applying integration by parts to the left side of (21) and employing the identity provided in [30, Section 3], we derive the expression on the right side of (21). Expanding the terms and grouping them into a quadratic form, following the approach outlined in [27, Theorem 3.1], leads to the expression in (22) with $\mathcal{Q}_\partial(\mathtt{s})$ given in (24) along with their associated matrices given in (25). □

3.2. Modeling methodology

In this subsection, the key steps of the methodology for the systematic modeling of systems based on linear elasticity are presented. The section begins with two propositions that allows to define the energy variables from the kinematic assumptions and the differential operator, and then, using Hamilton's principle, it is shown that this selected results lead to an infinite-dimensional port-Hamiltonian system. Finally, based on the kinematic assumptions and constitutive laws of the system, the methodology is outlined as a stet-by-step series.

Proposition 1. Consider linear elasticity under the kinematic assumption (16), the generalized momentum $p = p(\mathtt{x}, t) \in \mathbb{R}^n$, the mass matrix $\mathcal{M}(\mathtt{x}) = \mathcal{M}(\mathtt{x})^\top > 0 \in \mathbb{R}^{n\times n}$, the total kinetic energy $T(p) \in \mathbb{R}$ and the co-energy variable $e_p(\mathtt{x}, t) \in \mathbb{R}^n$ are defined as

$$p(\mathtt{x}, t) = \mathcal{M}(\mathtt{x})\, \dot{\mathbf{r}}(\mathtt{x}, t), \tag{26}$$

$$\mathcal{M}(\mathtt{x}) = \rho(\mathtt{x}) \int_{\Omega^c} \lambda_1(\mathtt{x}^c)^\top \lambda_1(\mathtt{x}^c)\, d\mathtt{x}^c, \tag{27}$$

$$T(p) = \frac{1}{2} \int_\Omega p(\mathtt{x}, t)^\top \mathcal{M}(\mathtt{x})^{-1} p(\mathtt{x}, t)\, d\mathtt{x}, \tag{28}$$

$$e_p(\mathtt{x}, t) = \mathcal{M}(\mathtt{x})^{-1} p(\mathtt{x}, t). \tag{29}$$

Proof. The kinematic assumption $\mathbf{u}(\mathtt{X}, t)$ according to (16) implies $\dot{\mathbf{u}}(\mathtt{X}, t) = \lambda_1(\mathtt{x}^c)\, \dot{\mathbf{r}}(\mathtt{x}, t)$. Then, by the definition of kinetic energy in (2) we have

$$T = \frac{1}{2} \int_{\Omega_T} \rho(\mathtt{x})\, \dot{\mathbf{u}}(\mathtt{X}, t)^\top \dot{\mathbf{u}}(\mathtt{X}, t)\, d\mathtt{X} = \frac{1}{2} \int_\Omega \dot{\mathbf{r}}(\mathtt{x}, t)^\top \underbrace{\rho(\mathtt{x}) \int_{\Omega^c} \lambda_1(\mathtt{x}^c)^\top \lambda_1(\mathtt{x}^c)\, d\mathtt{x}^c}_{\mathcal{M}(\mathtt{x})} \dot{\mathbf{r}}(\mathtt{x}, t)\, d\mathtt{x},$$

but from (26) we know that $\dot{\mathbf{r}}(\mathtt{x}, t) = \mathcal{M}(\mathtt{x})^{-1} p(\mathtt{x}, t)$, then we can write $T(p)$ as in (28). By definition $e_p(\mathtt{x}, t)$ is the variational derivative of $T(p)$ respect to $p(\mathtt{x}, t)$. □

Proposition 2. Consider linear elasticity under the kinematic assumption (16). Assume that the strain vector $\varepsilon(\mathtt{X}, t) \in \mathbb{R}^d$ can be written as

$$\varepsilon(\mathtt{X}, t) = \lambda_2(\mathtt{x}^c) \mathcal{F}\, \mathbf{r}(\mathtt{x}, t), \tag{30}$$

where $\lambda_2(\mathtt{x}^c) \in \mathbb{R}^{d\times m}$ is a matrix, and $\mathcal{F}$ is a $(m \times n)$ linear differential operator as in Definition 2, both without any zero rows or columns. Then, the generalized strains $\epsilon = \epsilon(\mathtt{x}, t) \in \mathbb{R}^m$, the stiffness matrix $\mathcal{K}(\mathtt{x}) \in \mathbb{R}^{m\times m}$, the elastic potential energy $U(\epsilon) \in \mathbb{R}$, and the co-energy variable $e_\epsilon(\mathtt{x}, t) \in \mathbb{R}^m$ are defined as

$$\epsilon(\mathtt{x},t) = \mathcal{F}\,\mathbf{r}(\mathtt{x},t), \tag{31}$$

$$\mathcal{K}(\mathtt{x}) = \int_{\Omega^c} \lambda_2(\mathtt{x}^c)^\top\, C\, \lambda_2(\mathtt{x}^c) d\mathtt{x}^c, \tag{32}$$

$$U(\epsilon) = \frac{1}{2}\int_{\Omega} \epsilon(\mathtt{x},t)^\top \mathcal{K}(\mathtt{x})\, \epsilon(\mathtt{x},t)\, d\mathtt{x}, \tag{33}$$

$$e_\epsilon(\mathtt{x},t) = \mathcal{K}(\mathtt{x})\epsilon(\mathtt{x},t), \tag{34}$$

with $C = C^\top > 0 \in \mathbb{R}^{d\times d}$ is the constitutive matrix defined as (17). The dimension $m \in \mathbb{N}$ must be chosen such that $\mathcal{K}(\mathtt{x}) = \mathcal{K}(\mathtt{x})^\top > 0$.

Proof. From (30) we have $\varepsilon(\mathtt{X},t) = \lambda_2(\mathtt{x}^c)\epsilon(\mathtt{x},t)$, and from (17) we have $\sigma(\mathtt{X},t) = C\,\lambda_2(\mathtt{x}^c)\,\epsilon(\mathtt{x},t)$. Then, by the definition of the elastic potential energy in (3) we have

$$U = \frac{1}{2}\int_{\Omega_T} \sigma(\mathtt{X},t)^\top \varepsilon(\mathtt{X},t)\, d\mathtt{X} = \frac{1}{2}\int_{\Omega} \epsilon(\mathtt{x},t)^\top \underbrace{\int_{\Omega^c} \lambda_2(\mathtt{x}^c)^\top\, C\, \lambda_2(\mathtt{x}^c)\, d\mathtt{x}^c}_{\mathcal{K}(\mathtt{x})}\, \epsilon(\mathtt{x},t)\, d\mathtt{x}.$$

By definition $e_\epsilon(\mathtt{x},t)$ is the variational derivative of $U(\epsilon)$ respect to $\epsilon(\mathtt{x},t)$. □

Remark 1. Note that the structure of $\mathcal{F}$ is mainly determined by $\mathrm{Grad}(\mathbf{u}(\mathtt{X},t))$, where if there are no differential dependencies between the components of $\mathbf{r}(\mathtt{x},t)$ the operator $\mathcal{F}$ is constant and first order. Also note that for $N > 1$ the matrix $\mathcal{Q}_\partial(\mathtt{s})$ is not full row rank.

Proposition 3. Consider a differential operator $\mathcal{F}$ of order N, $\mathcal{B}(\cdot)$ and $\mathcal{Q}_\partial(\mathtt{s})$ defined in (23) and (24), respectively. Define $T_R \in \mathbb{R}^{n_z\times n+(N-1)n\ell}$ with n_z the number of nonzero rows of $\mathcal{Q}_\partial(\mathtt{s})$, as a constant matrix such that $T_R\mathcal{Q}_\partial(\mathtt{s})$ retain only the nonzero rows of $\mathcal{Q}_\partial(\mathtt{s})$. Assume that $T_R\mathcal{B}(e_p(\mathtt{s},t)) = T_L e_p(\mathtt{s},t)$ with $T_L \in \mathbb{R}^{n_z\times n}$ a constant matrix. Then the following equality holds:

$$\int_{\partial\Omega} \mathcal{B}(e_p(\mathtt{s},t))^\top \mathcal{Q}_\partial(\mathtt{s})\, \mathcal{B}(e_\epsilon(\mathtt{s},t))\, d\mathtt{s} = \int_{\partial\Omega} e_p(\mathtt{s},t)^\top \bar{\mathcal{Q}}_\partial(\mathtt{s})\, \mathcal{B}(e_\epsilon(\mathtt{s},t))\, d\mathtt{s}, \tag{35}$$

with $\bar{\mathcal{Q}}_\partial(\mathtt{s}) = T_L^\top T_R\, \mathcal{Q}_\partial(\mathtt{s}) \in \mathbb{R}^{n\times m+(N-1)m\ell}$.

Proof. Noting that $T_R\mathcal{Q}_\partial(\mathtt{s})$ only retains non-zero rows, and using $T_R\mathcal{B}(e_p(\mathtt{s},t)) = T_L e_p(\mathtt{s},t)$, then we have $\mathcal{B}(e_p(\mathtt{s},t))^\top\mathcal{Q}_\partial(\mathtt{s})\mathcal{B}(e_\epsilon(\mathtt{s},t)) = (T_R\mathcal{B}(e_p(\mathtt{s},t)))^\top T_R\mathcal{Q}_\partial(\mathtt{s})\mathcal{B}(e_\epsilon(\mathtt{s},t)) = e_p(\mathtt{s},t)^\top T_L^\top T_R\,\mathcal{Q}_\partial(\mathtt{s})\mathcal{B}(e_\epsilon(\mathtt{s},t))$. □

Corollary 1. Note that in case of $\mathcal{F}$ of order $N = 1$, we have $\bar{\mathcal{Q}}_\partial(\mathtt{s}) = \mathcal{P}_\partial(\mathtt{s})$ and $\mathcal{B}(e_\epsilon(\mathtt{s},t)) = e_\epsilon(\mathtt{s},t)$.

Theorem 1. Let consider $x(\mathtt{x},t) = [p(\mathtt{x},t)^\top\ \epsilon(\mathtt{x},t)^\top]^\top \in \mathbb{R}^{(n+m)}$ as the state variable with $\mathbf{u}(\mathtt{X},t)$ defined in (16) and total external work W_E in (18). From Propositions 1, 2 and 3 the dynamics of the system defines an infinite-dimensional port-Hamiltonian system of the form

$$\begin{aligned} \underbrace{\begin{bmatrix} f_p \\ f_\epsilon \end{bmatrix}}_{\dot{x}} &= \underbrace{\begin{bmatrix} 0 & -\mathcal{F}^* \\ \mathcal{F} & 0 \end{bmatrix}}_{\mathcal{J}=-\mathcal{J}^*} \underbrace{\begin{bmatrix} e_p \\ e_\epsilon \end{bmatrix}}_{\delta_x H} + \underbrace{\begin{bmatrix} B_d \\ 0 \end{bmatrix}}_{\mathcal{G}} u_d \\ y_d &= \mathcal{G}^*\delta_x H = B_d^*(e_p), \end{aligned} \tag{36}$$

$$H(p,\epsilon) = \frac{1}{2}\int_{\Omega} p(\mathtt{x},t)^\top \mathcal{M}(\mathtt{x})^{-1} p(\mathtt{x},t) + \epsilon(\mathtt{x},t)^\top \mathcal{K}(\mathtt{x})\epsilon(\mathtt{x},t)\, d\mathtt{x}, \tag{37}$$

where $H(p,\epsilon) = T(p) + U(\epsilon)$ is the Hamiltonian of the system, $f_p = \dot{p} = \mathcal{M}\,\ddot{\mathbf{r}}$, $f_\epsilon = \dot{\epsilon} = \mathcal{F}\,\dot{\mathbf{r}}$, $e_p = \delta_p H = \mathcal{M}^{-1}\,p = \dot{\mathbf{r}}$, $e_\epsilon = \delta_\epsilon H = \mathcal{K}\,\epsilon = \mathcal{K}\mathcal{F}\,\mathbf{r}$, and the power exchange with the environment is given by

$$\partial_t H = \int_{\Omega} y_d(\mathtt{x},t)^\top u_d(\mathtt{x},t)\, d\mathtt{x} + \int_{\partial\Omega} e_p(\mathtt{s},t)^\top \bar{\mathcal{Q}}_\partial(\mathtt{s})\, \mathcal{B}(e_\epsilon(\mathtt{s},t))\, d\mathtt{s}. \tag{38}$$

Proof. First of all we have $\mathbf{u}(\mathtt{X},t) = \lambda_1(\mathtt{x}^c)\,\mathbf{r}(\mathtt{x},t)$, $\delta\mathbf{u}(\mathtt{X},t) = \lambda_1(\mathtt{x}^c)\,\delta\mathbf{r}(\mathtt{x},t)$, $\dot{\mathbf{u}}(\mathtt{X},t) = \lambda_1(\mathtt{x}^c)\,\dot{\mathbf{r}}(\mathtt{x},t)$, $\delta\dot{\mathbf{u}}(\mathtt{X},t) = \lambda_1(\mathtt{x}^c)\,\delta\dot{\mathbf{r}}(\mathtt{x},t)$, $\varepsilon(\mathtt{X},t) = \lambda_2(\mathtt{x}^c)\,\mathcal{F}\,\mathbf{r}(\mathtt{x},t)$, $\delta\varepsilon(\mathtt{X},t) = \lambda_2(\mathtt{x}^c)\,\mathcal{F}\,\delta\mathbf{r}(\mathtt{x},t)$, and

$$\begin{aligned}
\delta T = &\int_{\Omega_T} \rho(\mathtt{x})\dot{\mathbf{u}}(\mathtt{X},t)\cdot\delta\dot{\mathbf{u}}(\mathtt{X},t)d\mathtt{X} = \int_{\Omega} \delta\dot{\mathbf{r}}(\mathtt{x},t)^\top \mathcal{M}(\mathtt{x})\dot{\mathbf{r}}(\mathtt{x},t)d\mathtt{x},\\
\delta W_E = &\int_{\Omega} \delta\mathbf{r}(\mathtt{x},t)^\top B_d(u_d(\mathtt{x},t))\,d\mathtt{x} + \int_{\partial\Omega_\sigma} \delta\mathbf{r}(\mathtt{s}_\sigma,t)^\top \tau_\partial(\mathtt{s}_\sigma,t)\,d\mathtt{s}_\sigma,\\
\delta U = &\int_{\Omega_T} \sigma(\mathtt{X},t)\cdot\delta\varepsilon(\mathtt{X},t)\,d\mathtt{X} = \int_{\Omega} e_\epsilon(\mathtt{x},t)^\top\mathcal{F}\,\delta\mathbf{r}(\mathtt{x},t)d\mathtt{x}.
\end{aligned}$$

Applying Lemma 1 to the right side of δU, and considering that due to (6) the integral in the portion of the boundary $\partial\Omega_u$ where the essential boundary conditions are specified is zero, since $\delta\mathbf{r}(\mathtt{s}_u,t)=0$, then we have

$$\delta U = \int_{\Omega} \delta\mathbf{r}(\mathtt{x},t)^\top\mathcal{F}^*\,e_\epsilon(\mathtt{x},t)d\mathtt{x} + \int_{\partial\Omega_\sigma} \mathcal{B}(\delta\mathbf{r}(\mathtt{s}_\sigma,t))^\top\mathcal{Q}_\partial(\mathtt{s}_\sigma)\,\mathcal{B}(e_\epsilon(\mathtt{s}_\sigma,t))\,d\mathtt{s}_\sigma.$$

Since $e_p(\mathtt{x},t) = \dot{\mathbf{r}}(\mathtt{x},t)$, we can use Proposition 3 to write the above expression as

$$\delta U = \int_{\Omega} \delta\mathbf{r}(\mathtt{x},t)^\top\mathcal{F}^*\,e_\epsilon(\mathtt{x},t)d\mathtt{x} + \int_{\partial\Omega_\sigma} \delta\mathbf{r}(\mathtt{s}_\sigma,t)^\top\bar{\mathcal{Q}}_\partial(\mathtt{s}_\sigma)\,\mathcal{B}(e_\epsilon(\mathtt{s}_\sigma,t))\,d\mathtt{s}_\sigma.$$

As a previous step to apply Hamilton's principle we integrate by parts δT respect to time, then we obtain

$$\int_{t_1}^{t_2}\delta T\,dt = -\int_{t_1}^{t_2}\int_{\Omega}\delta\mathbf{r}(\mathtt{x},t)^\top\mathcal{M}(\mathtt{x})\,\ddot{\mathbf{r}}(\mathtt{x},t)\,d\mathtt{x}\,dt \;+\; \int_{\Omega}(\delta\mathbf{r}(\mathtt{x},t)^\top\mathcal{M}(\mathtt{x})\,\dot{\mathbf{r}}(\mathtt{x},t))\Big|_{t_1}^{t_2}d\mathtt{x},$$

where the last term is equal to zero due to (7), that is $\delta\mathbf{r}(\mathtt{x},t_1) = \delta\mathbf{r}(\mathtt{x},t_2) = 0$. Then, with all the above we apply Hamilton's principle and we obtain

$$\int_{t_1}^{t_2}\Bigg[\int_{\Omega}\delta\mathbf{r}(\mathtt{x},t)^\top\left[\mathcal{M}(\mathtt{x})\,\ddot{\mathbf{r}}(\mathtt{x},t) + \mathcal{F}^*\,e_\epsilon(\mathtt{x},t) - B_d(u_d(\mathtt{x},t))\right]d\mathtt{x} \;+\; \ldots$$
$$\ldots\int_{\partial\Omega_\sigma}\delta\mathbf{r}(\mathtt{s}_\sigma,t)^\top\left[\bar{\mathcal{Q}}_\partial(\mathtt{s}_\sigma)\,\mathcal{B}(e_\epsilon(\mathtt{s}_\sigma,t)) - \tau_\partial(\mathtt{s}_\sigma,t)\right]d\mathtt{s}_\sigma\Bigg]dt = 0.$$

So applying Lemma 2 and Lemma 3 respectively to each term in the above expression (see Appendix A), we obtain the following Lagrangian model and Neumann boundary condition, respectively.

$$\forall\mathtt{x}\in\Omega: \quad \mathcal{M}(\mathtt{x})\ddot{\mathbf{r}}(\mathtt{x},t) + \mathcal{F}^*\,e_\epsilon(\mathtt{x},t) - B_d(u_d(\mathtt{x},t)) = 0, \tag{39}$$

$$\forall\mathtt{s}_\sigma\in\partial\Omega_\sigma: \quad \tau_\partial(\mathtt{s}_\sigma,t) = \bar{\mathcal{Q}}_\partial(\mathtt{s}_\sigma)\mathcal{B}(e_\epsilon(\mathtt{s}_\sigma,t)). \tag{40}$$

Note that the equation (39) together with $\dot{\epsilon}(\mathtt{x},t) = \mathcal{F}\,\dot{\mathbf{r}}(\mathtt{x},t) = \mathcal{F}\,e_p(\mathtt{x},t)$ can be written equivalently as in (36) with $H(p,\epsilon)$ the Hamiltonian defined in (37). The power exchanged with the environment is given by

$$\partial_t H = \int_{\Omega}\delta_x H^\top\dot{x}\,d\mathtt{x} = \int_{\Omega} e_p^\top B_d(u_d)\,d\mathtt{x} - \int_{\Omega} e_p^\top\mathcal{F}^*\,e_\epsilon\,d\mathtt{x} + \int_{\Omega} e_\epsilon^\top\mathcal{F}\,e_p\,d\mathtt{x},$$

so applying Lemma 1 to the first term, and also to the last two terms in the equation above we obtain

$$\partial_t H = \int_{\Omega} u_d^\top B_d^*(e_p)\,d\mathtt{x} + \int_{\partial\Omega}\mathcal{B}(u_d)^\top\mathcal{Q}_\partial\,\mathcal{B}(e_p)\,d\mathtt{s} + \int_{\partial\Omega}\mathcal{B}(e_p)^\top\mathcal{Q}_\partial\,\mathcal{B}(e_\epsilon)\,d\mathtt{s}.$$

From the first term above we obtain the power conjugated distributed output $y_d(\mathtt{x},t)$ defined in (36), and the second term is equal to zero because $u_d(\mathtt{x},t)$ is not defined on the boundary ($u_d = 0$ in $\partial\Omega$). So considering the above together with Proposition 3, the last expression is equal to (38). □

Proposition 4. Consider the PHS in Theorem 1. For a mixed boundary value problem the second boundary term in (38) can be written equivalently as

$$\int_{\partial\Omega} e_p(\mathtt{s},t)^\top \bar{\mathcal{Q}}_\partial(\mathtt{s})\,\mathcal{B}(e_\epsilon(\mathtt{s},t))d\mathtt{s} = \int_{\partial\Omega_\sigma} y_\tau(\mathtt{s}_\sigma,t)^\top \tau_\partial(\mathtt{s}_\sigma,t)\,d\mathtt{s}_\sigma + \int_{\partial\Omega_u} v_\partial(\mathtt{s}_u,t)^\top y_v(\mathtt{s}_u,t)\,d\mathtt{s}_u, \tag{41}$$

with

$$\tau_\partial(\mathtt{s}_\sigma,t) = \bar{\mathcal{Q}}_\partial(\mathtt{s}_\sigma)\,\mathcal{B}(e_\epsilon(\mathtt{s}_\sigma,t)) \quad \text{on } \partial\Omega_\sigma, \tag{42}$$

$$v_\partial(\mathtt{s}_u,t) = e_p(\mathtt{s}_u,t) \quad \text{on } \partial\Omega_u, \tag{43}$$

$$y_\tau(\mathtt{s}_\sigma,t) = e_p(\mathtt{s}_\sigma,t) \quad \text{on } \partial\Omega_\sigma, \tag{44}$$

$$y_v(\mathtt{s}_u,t) = \bar{\mathcal{Q}}_\partial(\mathtt{s}_u)\,\mathcal{B}(e_\epsilon(\mathtt{s}_u,t)) \quad \text{on } \partial\Omega_u, \tag{45}$$

where $\tau_\partial(\mathtt{s}_\sigma,t)\in\mathbb{R}^n$ is the imposed generalized boundary traction, $v_\partial(\mathtt{s}_u,t)\in\mathbb{R}^n$ is the imposed generalized boundary velocity. Then, the boundary inputs of (36) are given by $u_\partial(\mathtt{s},t) = [\tau_\partial(\mathtt{s}_\sigma,t)^\top\ v_\partial(\mathtt{s}_u,t)^\top]^\top$, and the boundary outputs by $y_\partial(\mathtt{s},t) = [y_\tau(\mathtt{s}_\sigma,t)^\top\ y_v(\mathtt{s}_u,t)^\top]^\top$.

Proof. The proof is straightforward, given that $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\sigma$ and the definitions from (42) to (45). □

Procedure: The modeling methodology is summarized in the following procedure. To formulate infinite-dimensional port-Hamiltonian models based on kinematic assumptions that lead to a displacement fields of the class $\mathbf{u}(\mathtt{X},t) = \lambda_1(\mathtt{x}^c)\,\mathbf{r}(\mathtt{x},t)$, where the constitutive matrix C and the density of material $\rho(\mathtt{x})$ are known, follows the steps below:

1. Calculate the mass matrix $\mathcal{M}(\mathtt{x})$ from (27) and define the generalized momentum $p(\mathtt{x},t)$ according to (26). That is: $\mathcal{M}(\mathtt{x}) = \rho(\mathtt{x})\int_{\Omega^c}\lambda_1(\mathtt{x}^c)^\top\lambda_1(\mathtt{x}^c)\,d\mathtt{x}^c \quad\longrightarrow\quad p(\mathtt{x},t) = \mathcal{M}(\mathtt{x})\dot{\mathbf{r}}(\mathtt{x},t)$.
2. Compute the nonzero components of the strain tensor $\varepsilon(\mathtt{X},t)$ using (B.4) and factorize them according to (30). Define generalized strains $\epsilon(\mathtt{x},t)$ according to (31) and compute the stiffness matrix $\mathcal{K}(\mathtt{x})$ from (32). That is: $\varepsilon(\mathtt{X},t) = \lambda_2(\mathtt{x}^c)\mathcal{F}\,\mathbf{r}(\mathtt{x},t) \quad\longrightarrow\quad \epsilon(\mathtt{x},t) = \mathcal{F}\,\mathbf{r}(\mathtt{x},t), \quad \mathcal{K}(\mathtt{x}) = \int_{\Omega^c}\lambda_2(\mathtt{x}^c)^\top C\,\lambda_2(\mathtt{x}^c)\,d\mathtt{x}^c$.
3. Apply Theorem 1 to obtain the infinite-dimensional PHS and define the boundary ports u_∂, y_∂ from Proposition 4.

Remark 2. The port-Hamiltonian model obtained in Theorem 1 is different from the one obtained by the Legendre transformation of the Euler-Lagrange system (39) [31]. Furthermore, the latter is a field port-Lagrangian system [20] defined using the state variable $z(\mathtt{x},t) = [p(\mathtt{x},t)^\top\ \ \mathbf{r}(\mathtt{x},t)^\top]^\top \in \mathbb{R}^{2n}$, which leads to an infinite-dimensional system associated with an algebraic skew-symmetric matrix $J = -J^\top$. This system is given by

$$\underbrace{\begin{bmatrix}\dot{p}\\ \dot{\mathbf{r}}\end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{J=-J^\top}\underbrace{\begin{bmatrix}e_p\\ e_r\end{bmatrix}}_{\delta_z H} + \underbrace{\begin{bmatrix}B_d\\ 0\end{bmatrix}}_{\mathcal{G}} u_d \tag{46}$$

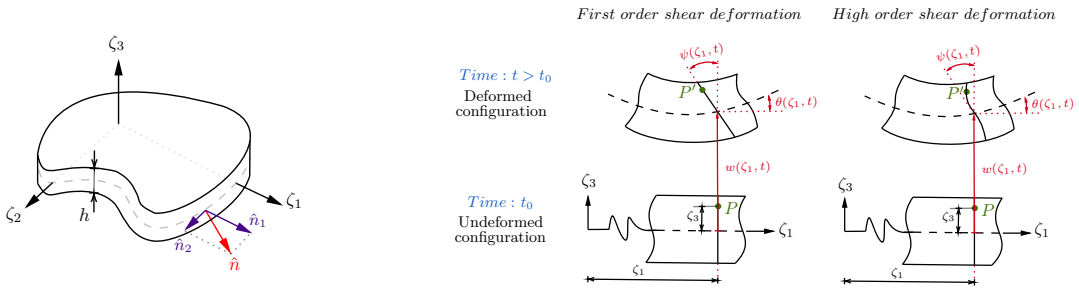
$$y_d = \mathcal{G}^*\delta_z H = B_d^*(e_p),$$

$$H(p,\mathbf{r}) = T(p) + U(\mathbf{r}) = \frac{1}{2}\int_\Omega p(\mathtt{x},t)^\top\mathcal{M}(\mathtt{x})^{-1}p(\mathtt{x},t) + (\mathcal{F}\,\mathbf{r}(\mathtt{x},t))^\top\mathcal{K}(\mathtt{x})\,(\mathcal{F}\,\mathbf{r}(\mathtt{x},t))\,d\mathtt{x}, \tag{47}$$

$$e_r(\mathtt{x},t) = \mathcal{F}^*(\mathcal{K}(\mathtt{x})\,\mathcal{F}\,\mathbf{r}(\mathtt{x},t)), \tag{48}$$

where $e_r(\mathtt{x},t)$ is the variational derivative of the Hamiltonian $H(p,\mathbf{r})$ respect to $\mathbf{r}(\mathtt{x},t)$.

Remark 3. The methodology presented could be further generalized by using the concept of Stokes-Lagrange structure [32]. In a Lagrange structure, constitutive equations are expressed in a more general way using a Lagrange submanifold. Then, differential operators can be included in the mapping $\lambda_1(\mathtt{x}^c)$ or the mass matrix (mass operator). For an extensive treatment, particularly concerning to finite-dimensional linear PHS, we refer the reader to [33].



(a) Plate scheme.

(b) Kinematic assumptions

Figure 4: Plate scheme and kinematic assumptions

4. Reddy's plate model

In this section, the plate model based on Reddy's third-order shear deformation theory [34] is presented. This model generalizes the classic Mindlin's plate model, with the latter being a specific case of Reddy's formulation. In addition, because this theory considers higher order terms, the kinematic assumption is closer to reality, which brings advantages such as these models describe more accurately the shear stress contributions to the elastic potential energy, thus avoiding the use of correction factors and the shear locking problem in finite element approximations (as in the case of Mindlin's plate or the Timoshenko beam). Finally, since there is a better description of shear stresses, these models are particularly useful for describing the dynamics of very thick or layered beams and plates. For more details, see [34] and [35, Chapter 11].

4.1. Model assumptions

The kinematic assumptions in the first-order shear deformation models (Timoshenko beam and Mindlin's plate) are that the plane sections perpendicular to the neutral axis before deformation remain plane but not necessarily perpendicular to the neutral axis after deformation. In the high-order shear deformation models, plane sections perpendicular to the neutral axis before the deformation transform into curved sections after deformation (see Figure 4). The so-called Reddy models are third-order shear deformation models due to fact that the curve is described by a third-order polynomial that always satisfies the condition of zero tangential traction boundary conditions on the surfaces of the plate.

The problem starts with the assumption on the displacement field $\mathbf{u}(\mathtt{X}, t)$ given by

$$\mathbf{u}(\mathtt{X}, t) = \begin{bmatrix} \mathbf{u}_1(\mathtt{X}, t) \\ \mathbf{u}_2(\mathtt{X}, t) \\ \mathbf{u}_3(\mathtt{X}, t) \end{bmatrix} = \begin{bmatrix} -(\zeta_3\psi_1(\zeta_1, \zeta_2, t) + \zeta_3^2\beta_1(\zeta_1, \zeta_2, t) + \zeta_3^3\phi_1(\zeta_1, \zeta_2, t)) \\ -(\zeta_3\psi_2(\zeta_1, \zeta_2, t) + \zeta_3^2\beta_2(\zeta_1, \zeta_2, t) + \zeta_3^3\phi_2(\zeta_1, \zeta_2, t)) \\ w(\zeta_1, \zeta_2, t) \end{bmatrix},$$

where ψ_1, ψ_2 are the angles of rotation of the cross section along each coordinate axis, w is the vertical displacement to the mid-plane, and the functions β_1, β_2, ϕ_1, ϕ_2 will be determined using the condition that transverse shear stresses vanish on the top and bottom surfaces of the plate, that is $\sigma_{13}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$ and $\sigma_{23}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$. For isotropic plates these conditions are equivalent to $\varepsilon_{13}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$ and $\varepsilon_{23}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$ [34]. Then, from (1) we obtain

$$2\varepsilon_{13} = \partial_1 w - \left(\psi_1 + 2\zeta_3\beta_1 + 3\zeta_3^2\phi_1\right) \quad , \quad 2\varepsilon_{23} = \partial_2 w - \left(\psi_2 + 2\zeta_3\beta_2 + 3\zeta_3^2\phi_2\right),$$

and from $\varepsilon_{13}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$ and $\varepsilon_{23}(\zeta_1, \zeta_2, \zeta_3 = \pm h/2, t) = 0$ we obtain $\beta_1, \beta_2 = 0$ and $\phi_1 = \frac{4}{3h^2}(\partial_1 w - \psi_1)$, $\phi_2 = \frac{4}{3h^2}(\partial_2 w - \psi_2)$, where the constant $\alpha = 4/(3h^2)$ is introduced. With all the above

variables, the displacement field $\mathbf{u}(\mathtt{X},t)$ of the Reddy's plate is given by

$$\mathbf{u}(\mathtt{X},t) = \underbrace{\begin{bmatrix} -(\zeta_3-\alpha\zeta_3^3) & 0 & 0 & -\alpha\zeta_3^3 & 0 \\ 0 & -(\zeta_3-\alpha\zeta_3^3) & 0 & 0 & -\alpha\zeta_3^3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\lambda_1(\mathbf{x}^c)} \underbrace{\begin{bmatrix} \psi_1(\mathbf{x},t) \\ \psi_2(\mathbf{x},t) \\ w(\mathbf{x},t) \\ \partial_1 w(\mathbf{x},t) \\ \partial_2 w(\mathbf{x},t) \end{bmatrix}}_{\mathbf{r}(\mathbf{x},t)} . \tag{49}$$

Note that with $\alpha = 0$ the displacement field becomes the same as Mindlin's plate [16]. On the other hand related to the constitutive equations, considering an isotropic material, the stress $\sigma(\mathbf{x},t)$ is obtained from Hooke's law $\sigma(\mathbf{x},t) = C\,\varepsilon(\mathbf{x},t)$, where the constitutive matrix $C = C^\top > 0$ is given by

$$C = \begin{bmatrix} C_b & 0 \\ 0 & C_s \end{bmatrix} \quad , \quad \text{with} \quad C_b = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \quad , \quad C_s = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}, \tag{50}$$

where C_b is the constitutive matrix for plane stress, and E, G, ν are properties of the material.

4.2. Infinite-dimensional port-Hamiltonian model of the Reddy plate

Now that the displacement field is known and written according to (16), the proposed methodology allows us to systematically find the infinite-dimensional port-Hamiltonian model of the plate that is energetically consistent with the kinematic assumptions. Following the proposed methodology we have:

1. From (27) we calculate the mass matrix as:

$$\begin{aligned} \mathcal{M}(\mathbf{x}) &= \rho(\mathbf{x}) \int_{-h/2}^{h/2} \begin{bmatrix} (\zeta_3^2 - 2\alpha\zeta_3^4 + \alpha^2\zeta_3^6) & 0 & 0 & \alpha(\zeta_3^4 - \alpha\zeta_3^6) & 0 \\ 0 & (\zeta_3^2 - 2\alpha\zeta_3^4 + \alpha^2\zeta_3^6) & 0 & 0 & \alpha(\zeta_3^4 - \alpha\zeta_3^6) \\ 0 & 0 & 1 & 0 & 0 \\ \alpha(\zeta_3^4 - \alpha\zeta_3^6) & 0 & 0 & \alpha^2\zeta_3^6 & 0 \\ 0 & \alpha(\zeta_3^4 - \alpha\zeta_3^6) & 0 & 0 & \alpha^2\zeta_3^6 \end{bmatrix} d\zeta_3 \\ &= \rho(\mathbf{x}) \begin{bmatrix} (\bar{I}_2 - 2\alpha\bar{I}_4 + \alpha^2\bar{I}_6) & 0 & 0 & \alpha(\bar{I}_4 - \alpha\bar{I}_6) & 0 \\ 0 & (\bar{I}_2 - 2\alpha\bar{I}_4 + \alpha^2\bar{I}_6) & 0 & 0 & \alpha(\bar{I}_4 - \alpha\bar{I}_6) \\ 0 & 0 & \bar{I}_0 & 0 & 0 \\ \alpha(\bar{I}_4 - \alpha\bar{I}_6) & 0 & 0 & \alpha^2\bar{I}_6 & 0 \\ 0 & \alpha(\bar{I}_4 - \alpha\bar{I}_6) & 0 & 0 & \alpha^2\bar{I}_6 \end{bmatrix}, \end{aligned} \tag{51}$$

where $\bar{I}_i \in \mathbb{R}$ with $i = 0, 1, 2, \ldots$ is defined as

$$\bar{I}_i = \int_{-h/2}^{h/2} \zeta_3^i \, d\zeta_3 = \begin{cases} 0 & ; \text{if } i \text{ is odd} \\ \dfrac{h^{i+1}}{2^i(i+1)} & ; \text{if } i \text{ is even} \end{cases} . \tag{52}$$

Then, from (26) we have

$$p(\mathbf{x},t) = \mathcal{M}(\mathbf{x})\dot{\mathbf{r}}(\mathbf{x},t) = \begin{bmatrix} \rho(\bar{I}_2 - 2\alpha\bar{I}_4 + \alpha^2\bar{I}_6)\dot{\psi}_1(\mathbf{x},t) + \rho\alpha(\bar{I}_4 - \alpha\bar{I}_6)\partial_1\dot{w}(\mathbf{x},t) \\ \rho(\bar{I}_2 - 2\alpha\bar{I}_4 + \alpha^2\bar{I}_6)\dot{\psi}_2(\mathbf{x},t) + \rho\alpha(\bar{I}_4 - \alpha\bar{I}_6)\partial_2\dot{w}(\mathbf{x},t) \\ \rho\bar{I}_0\dot{w}(\mathbf{x},t) \\ \rho\alpha(\bar{I}_4 - \alpha\bar{I}_6)\dot{\psi}_1(\mathbf{x},t) + \rho\alpha^2\bar{I}_6\partial_1\dot{w}(\mathbf{x},t) \\ \rho\alpha(\bar{I}_4 - \alpha\bar{I}_6)\dot{\psi}_2(\mathbf{x},t) + \rho\alpha^2\bar{I}_6\partial_2\dot{w}(\mathbf{x},t) \end{bmatrix} = \begin{bmatrix} p_1(\mathbf{x},t) \\ p_2(\mathbf{x},t) \\ p_3(\mathbf{x},t) \\ p_4(\mathbf{x},t) \\ p_5(\mathbf{x},t) \end{bmatrix} . \tag{53}$$

2. On the other hand, from (B.4) we obtain the non-zero strain components $\varepsilon \subset \vec{\varepsilon}$, that is

$$\varepsilon(\mathtt{X},t)=\begin{bmatrix}\varepsilon_b\\ \hline \varepsilon_s\end{bmatrix}=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_4\\ \hline \varepsilon_5\\ \varepsilon_6\end{bmatrix}=\left[\begin{array}{c} -(\zeta_3-\alpha\zeta_3^3)\,[\partial_1\psi_1(\mathtt{x},t)]-\alpha\zeta_3^3[\partial_1^2 w(\mathtt{x},t)]\\ -(\zeta_3-\alpha\zeta_3^3)\,[\partial_2\psi_2(\mathtt{x},t)]-\alpha\zeta_3^3[\partial_2^2 w(\mathtt{x},t)]\\ -(\zeta_3-\alpha\zeta_3^3)[\partial_2\psi_1(\mathtt{x},t)+\partial_1\psi_2(\mathtt{x},t)]-\alpha\zeta_3^3[\partial_2\,\partial_1 w(\mathtt{x},t)+\partial_1\,\partial_2 w(\mathtt{x},t)]\\ \hline (1-3\alpha\zeta_3^2)[\partial_1 w(\mathtt{x},t)-\psi_1(\mathtt{x},t)]\\ (1-3\alpha\zeta_3^2)[\partial_2 w(\mathtt{x},t)-\psi_2(\mathtt{x},t)]\end{array}\right]. \tag{54}$$

From Proposition 2 we choose $m = 8$ since there are eight functions that are independent of $\mathtt{x}^c$ in the strain vector $\varepsilon(\mathtt{X},t)$ (highlighted in square brackets in (54)). Then we seek to write $\varepsilon(\mathtt{X},t)$ according to (30) considering that $\lambda_2(\mathtt{x}^c)\in\mathbb{R}^{d\times m=5\times 8}$ and $\mathcal{F}$ of dimension $(m\times n)=(8\times 5)$. That is

$$\lambda_2(\mathtt{x}^c)=\begin{bmatrix} -(\zeta_3-\alpha\zeta_3^3) & 0 & 0 & 0 & 0 & -\alpha\zeta_3^3 & 0 & 0\\ 0 & -(\zeta_3-\alpha\zeta_3^3) & 0 & 0 & 0 & 0 & -\alpha\zeta_3^3 & 0\\ 0 & 0 & -(\zeta_3-\alpha\zeta_3^3) & 0 & 0 & 0 & 0 & -\alpha\zeta_3^3\\ 0 & 0 & 0 & (1-3\alpha\zeta_3^2) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (1-3\alpha\zeta_3^2) & 0 & 0 & 0\end{bmatrix}, \tag{55}$$

$$\mathcal{F}=\left[\begin{array}{cc|c|cc} \partial_1 & 0 & 0 & 0 & 0\\ 0 & \partial_2 & 0 & 0 & 0\\ \partial_2 & \partial_1 & 0 & 0 & 0\\ \hline -1 & 0 & \partial_1 & 0 & 0\\ 0 & -1 & \partial_2 & 0 & 0\\ \hline 0 & 0 & 0 & \partial_1 & 0\\ 0 & 0 & 0 & 0 & \partial_2\\ 0 & 0 & 0 & \partial_2 & \partial_1\end{array}\right] \rightarrow \quad \epsilon(\mathtt{x},t)=\mathcal{F}\,\mathbf{r}(\mathtt{x},t)=\begin{bmatrix}\partial_1\psi_1(\mathtt{x},t)\\ \partial_2\psi_2(\mathtt{x},t)\\ \partial_2\psi_1(\mathtt{x},t)+\partial_1\psi_2(\mathtt{x},t)\\ \partial_1 w(\mathtt{x},t)-\psi_1(\mathtt{x},t)\\ \partial_2 w(\mathtt{x},t)-\psi_2(\mathtt{x},t)\\ \partial_1^2 w(\mathtt{x},t)\\ \partial_2^2 w(\mathtt{x},t)\\ \partial_2\partial_1 w(\mathtt{x},t)+\partial_1\partial_2 w(\mathtt{x},t)\end{bmatrix}=\begin{bmatrix}\epsilon_1(\mathtt{x},t)\\ \epsilon_2(\mathtt{x},t)\\ \epsilon_3(\mathtt{x},t)\\ \epsilon_4(\mathtt{x},t)\\ \epsilon_5(\mathtt{x},t)\\ \epsilon_6(\mathtt{x},t)\\ \epsilon_7(\mathtt{x},t)\\ \epsilon_8(\mathtt{x},t)\end{bmatrix}, \tag{56}$$

with $\mathcal{F}$ a differential operator of order $N=1$. From (32) we have

$$\begin{aligned}\mathcal{K}(\mathtt{x})&=\int_{-h/2}^{h/2}\begin{bmatrix}(\zeta_3^2-2\alpha\zeta_3^4+\alpha^2\zeta_3^6)\,C_b & 0 & \alpha(\zeta_3^4-\alpha\zeta_3^6)\,C_b\\ 0 & (1-6\alpha\zeta_3^2+9\alpha^2\zeta_3^4)\,C_s & 0\\ \alpha(\zeta_3^4-\alpha\zeta_3^6)\,C_b & 0 & \alpha^2\zeta_3^6\,C_b\end{bmatrix}d\zeta_3\\ &=\begin{bmatrix}(\bar{I}_2-2\alpha\bar{I}_4+\alpha^2\bar{I}_6)\,C_b & 0 & \alpha(\bar{I}_4-\alpha\bar{I}_6)\,C_b\\ 0 & (\bar{I}_0-6\alpha\bar{I}_2+9\alpha^2\bar{I}_4)\,C_s & 0\\ \alpha(\bar{I}_4-\alpha\bar{I}_6)\,C_b & 0 & \alpha^2\bar{I}_6\,C_b\end{bmatrix}.\end{aligned} \tag{57}$$

3. Considering that there are no distributed inputs, from Theorem 1 we have

$$\begin{bmatrix}\dot{p}(\mathtt{x},t)\\ \dot{\epsilon}(\mathtt{x},t)\end{bmatrix}=\begin{bmatrix}0 & -\mathcal{F}^*\\ \mathcal{F} & 0\end{bmatrix}\begin{bmatrix}\mathcal{M}^{-1}(\mathtt{x}) & 0\\ 0 & \mathcal{K}(\mathtt{x})\end{bmatrix}\begin{bmatrix}p(\mathtt{x},t)\\ \epsilon(\mathtt{x},t)\end{bmatrix}, \tag{58}$$

with Hamiltonian $H(p,\epsilon)$ defined as in (37), and boundary variables defined from Proposition 4. Then

$$\partial_t H=\int_{\partial\Omega}\begin{bmatrix}e_{p_1}\\ e_{p_2}\\ e_{p_3}\\ e_{p_4}\\ e_{p_5}\end{bmatrix}^{\top}\underbrace{\begin{bmatrix}\hat{n}_1 & 0 & \hat{n}_2 & 0 & 0 & 0 & 0 & 0\\ 0 & \hat{n}_2 & \hat{n}_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \hat{n}_1 & \hat{n}_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \hat{n}_1 & 0 & \hat{n}_2\\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{n}_2 & \hat{n}_1\end{bmatrix}}_{\bar{\mathcal{Q}}_\partial(\mathtt{s})=P_\partial(\mathtt{s})}\begin{bmatrix}e_{\epsilon_1}\\ e_{\epsilon_2}\\ e_{\epsilon_3}\\ e_{\epsilon_4}\\ e_{\epsilon_5}\\ e_{\epsilon_6}\\ e_{\epsilon_7}\\ e_{\epsilon_8}\end{bmatrix}d\mathtt{s}=\int_{\partial\Omega}y_\partial^\top u_\partial\,d\mathtt{s}. \tag{59}$$

Remark 4. Despite the fact that the methodology uses the Voigt-Kelvin notation (see Appendix B) and a Cartesian reference system to obtain the models, using the similarities $\mathbb{L} \sim \text{Grad}$, and $-\mathbb{L}^* \sim \text{Div}$, then $\text{Grad} = -\text{Div}^*$ (see [16, Theorem 4] for the proof), these models can be written with tensor notation which has the advantage of being independent of the coordinate system.

4.3. Tensor representation of the port-Hamiltonian Reddy plate model

In order to write the Reddy plate model using tensor notation, first note that the differential operator $\mathcal{F}$ defined in (56) and its formal adjoint $\mathcal{F}^*$ can be written using intrinsic tensor operators, and second, we can rewrite the generalized displacements $\mathbf{r}(\mathtt{x}, t)$ in (49) as

$$\mathcal{F} = \begin{bmatrix} \text{Grad} & 0 & 0 \\ -1_2 & \text{grad} & 0 \\ 0 & 0 & \text{Grad} \end{bmatrix} \quad , \quad \mathcal{F}^* = - \begin{bmatrix} \text{Div} & 1_2 & 0 \\ 0 & \text{div} & 0 \\ 0 & 0 & \text{Div} \end{bmatrix} \quad , \quad \mathbf{r}(\mathtt{x}, t) = \begin{bmatrix} \psi(\mathtt{x}, t) \\ w(\mathtt{x}, t) \\ \theta(\mathtt{x}, t) \end{bmatrix}, \tag{60}$$

where $\psi(\mathtt{x}, t) = [\psi_1(\mathtt{x}, t) \quad \psi_2(\mathtt{x}, t)]^\top \in \mathbb{R}^2$ groups the angles of rotation of the cross section along each coordinate axis, $w(\mathtt{x}, t) \in \mathbb{R}$ remains representing the vertical displacement of a point in the mid-plane of the plate, and $\theta(\mathtt{x}, t) = [\partial_1 w(\mathtt{x}, t) \quad \partial_2 w(\mathtt{x}, t)]^\top \in \mathbb{R}^2$ groups the first spatial derivatives of $w(\mathtt{x}, t)$ with respect to each coordinate axis. The mass matrix $\mathcal{M}(\mathtt{x})$ in (51) and stiffness matrix $\mathcal{K}(\mathtt{x})$ in (57) can be rewritten as

$$\mathcal{M}(\mathtt{x}) = \rho(\mathtt{x}) \begin{bmatrix} c_1 1_2 & 0 & c_2 1_2 \\ 0 & \bar{I}_0 & 0 \\ c_2 1_2 & 0 & c_3 1_2 \end{bmatrix} \quad , \quad \underline{\underline{\mathcal{K}}}(\mathtt{x}) = \begin{bmatrix} c_1 \underline{\underline{C_b}}(\cdot) & 0 & c_2 \underline{\underline{C_b}}(\cdot) \\ 0 & c_4 C_s & 0 \\ c_2 \underline{\underline{C_b}}(\cdot) & 0 & c_3 \underline{\underline{C_b}}(\cdot) \end{bmatrix}, \tag{61}$$

with $c_1 = (\bar{I}_2 - 2\alpha\bar{I}_4 + \alpha^2\bar{I}_6)$, $c_2 = \alpha(\bar{I}_4 - \alpha\bar{I}_6)$, $c_3 = \alpha^2\bar{I}_6$, $c_4 = (\bar{I}_0 - 6\alpha\bar{I}_2 + 9\alpha^2\bar{I}_4)$, C_s the constitutive matrix for shear stress as defined in (50), and $\underline{\underline{C_b}}(\cdot) = \frac{E}{1-\nu^2}\left[(1-\nu)(\cdot) + \nu \text{tr}(\cdot) 1_2\right]$ the constitutive tensor for plane stress. With the above we redefine the energy variables as

$$p(\mathtt{x}, t) = \begin{bmatrix} p_\psi(\mathtt{x}, t) \\ p_w(\mathtt{x}, t) \\ p_\theta(\mathtt{x}, t) \end{bmatrix} = \begin{bmatrix} \rho(\mathtt{x}) c_1 \dot{\psi}(\mathtt{x}, t) + \rho(\mathtt{x}) c_2 \dot{\theta}(\mathtt{x}, t) \\ \rho(\mathtt{x}) \hat{I}_0 \dot{w}(\mathtt{x}, t) \\ \rho(\mathtt{x}) c_2 \dot{\psi}(\mathtt{x}, t) + \rho(\mathtt{x}) c_3 \dot{\theta}(\mathtt{x}, t) \end{bmatrix}, \quad \underline{\epsilon}(\mathtt{x}, t) = \begin{bmatrix} \epsilon_\psi(\mathtt{x}, t) \\ \epsilon_w(\mathtt{x}, t) \\ \epsilon_\theta(\mathtt{x}, t) \end{bmatrix} = \begin{bmatrix} \text{Grad}(\psi(\mathtt{x}, t)) \\ \text{grad}(w(\mathtt{x}, t)) - \psi(\mathtt{x}, t) \\ \text{Grad}(\theta(\mathtt{x}, t)) \end{bmatrix}, \tag{62}$$

and co-energy variables as

$$e_p(\mathtt{x}, t) = \begin{bmatrix} e_{p_\psi}(\mathtt{x}, t) \\ e_{p_w}(\mathtt{x}, t) \\ e_{p_\theta}(\mathtt{x}, t) \end{bmatrix} = \begin{bmatrix} \dot{\psi}(\mathtt{x}, t) \\ \dot{w}(\mathtt{x}, t) \\ \dot{\theta}(\mathtt{x}, t) \end{bmatrix} \quad , \quad \underline{e_\epsilon}(\mathtt{x}, t) = \begin{bmatrix} e_{\epsilon_\psi}(\mathtt{x}, t) \\ e_{\epsilon_w}(\mathtt{x}, t) \\ e_{\epsilon_\theta}(\mathtt{x}, t) \end{bmatrix} = \begin{bmatrix} c_1 \underline{\underline{C_b}}(\epsilon_\psi(\mathtt{x}, t)) + c_2 \underline{\underline{C_b}}(\epsilon_\theta(\mathtt{x}, t)) \\ c_4 C_s \, \epsilon_w(\mathtt{x}, t) \\ c_2 \underline{\underline{C_b}}(\epsilon_\psi(\mathtt{x}, t)) + c_3 \underline{\underline{C_b}}(\epsilon_\theta(\mathtt{x}, t)) \end{bmatrix}, \tag{63}$$

where $\epsilon_\psi(\mathtt{x}, t)$, $\epsilon_\theta(\mathtt{x}, t)$, $e_{\epsilon_\psi}(\mathtt{x}, t)$, $e_{\epsilon_\theta}(\mathtt{x}, t) \in \mathbb{R}^{2\times 2}$ are second-order tensor fields, $p_\psi(\mathtt{x}, t)$, $p_\theta(\mathtt{x}, t)$, $\epsilon_w(\mathtt{x}, t)$, $e_{p_\psi}(\mathtt{x}, t)$, $e_{p_\theta}(\mathtt{x}, t)$, $e_{\epsilon_w}(\mathtt{x}, t) \in \mathbb{R}^2$ are vector fields, and $p_w(\mathtt{x}, t)$, $e_{p_w}(\mathtt{x}, t) \in \mathbb{R}$ are scalar fields. Then, the Reddy's plate model in (58) written using tensor notation is given by

$$\underbrace{\left[\begin{array}{c} \dot{p}_\psi \\ \dot{p}_w \\ \dot{p}_\theta \\ \hline \dot{\epsilon}_\psi \\ \dot{\epsilon}_w \\ \dot{\epsilon}_\theta \end{array}\right]}_{\dot{x}} = \underbrace{\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & \text{Div} & 1_2 & 0 \\ 0 & 0 & 0 & 0 & \text{div} & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{Div} \\ \hline \text{Grad} & 0 & 0 & 0 & 0 & 0 \\ -1_2 & \text{grad} & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{Grad} & 0 & 0 & 0 \end{array}\right]}_{\mathcal{J} = -\mathcal{J}^*} \underbrace{\left[\begin{array}{c} e_{p_\psi} \\ e_{p_w} \\ e_{p_\theta} \\ \hline e_{\epsilon_\psi} \\ e_{\epsilon_w} \\ e_{\epsilon_\theta} \end{array}\right]}_{\delta_x H}, \tag{64}$$

with Hamiltonian functional $H(p,\underline{\epsilon}) > 0 \in \mathbb{R}$ given by

$$H(p,\underline{\epsilon}) = \frac{1}{2}\int_{\Omega}\Big([\mathcal{M}^{-1}p]\cdot p + [c_1\underline{\underline{C_b}}(\epsilon_\psi) + c_2\underline{\underline{C_b}}(\epsilon_\theta)]:\epsilon_\psi + c_4 C_s \epsilon_w\cdot\epsilon_w + [c_2\underline{\underline{C_b}}(\epsilon_\psi) + c_3\underline{\underline{C_b}}(\epsilon_\theta)]:\epsilon_\theta\Big)\,d\mathtt{x}. \tag{65}$$

The boundary variables are obtained from the energy balance which is given by

$$\begin{aligned}\partial_t H = \langle \delta_x H,\, \dot{x}\rangle_{in}^{\Omega} = \int_{\Omega}\big\{&[e_{p_\psi}\cdot \mathrm{Div}(e_{\epsilon_\psi}) + e_{\epsilon_\psi} : \mathrm{Grad}(e_{p_\psi})] + [e_{p_\theta}\cdot \mathrm{Div}(e_{\epsilon_\theta}) + e_{\epsilon_\theta} : \mathrm{Grad}(e_{p_\theta})] \;+\\ &[e_{p_w}\,\mathrm{div}(e_{\epsilon_w}) + e_{\epsilon_w}\cdot \mathrm{grad}(e_{p_w})]\big\}\,d\mathtt{x},\end{aligned} \tag{66}$$

so applying the integration by parts theorem for symmetric tensors to the first two terms of the integral above (see [19, Theorem 8]), and the divergence theorem to the third term, then we obtain

$$\partial_t H = \int_{\partial\Omega}\left[(e_{\epsilon_\psi}\,\hat{n})\cdot e_{p_\psi} + (e_{\epsilon_\theta}\,\hat{n})\cdot e_{p_\theta} + e_{p_w}\,(e_{\epsilon_w}\cdot\hat{n})\right]d\mathtt{s} = \int_{\partial\Omega} y_\partial^\top u_\partial\,d\mathtt{s}, \tag{67}$$

which is completely analogous to the expression in (59).

Remark 5. Note that the energy and co-energy variables related to the generalized strains in the model (64) can be written in terms of the variables of the model (58) as

$$\epsilon_\psi(\mathtt{x},t) = \begin{bmatrix}\epsilon_1(\mathtt{x},t) & \frac{1}{2}\epsilon_3(\mathtt{x},t)\\ \frac{1}{2}\epsilon_3(\mathtt{x},t) & \epsilon_2(\mathtt{x},t)\end{bmatrix},\quad \epsilon_w(\mathtt{x},t) = \begin{bmatrix}\epsilon_4(\mathtt{x},t)\\ \epsilon_5(\mathtt{x},t)\end{bmatrix},\quad \epsilon_\theta(\mathtt{x},t) = \begin{bmatrix}\epsilon_6(\mathtt{x},t) & \frac{1}{2}\epsilon_8(\mathtt{x},t)\\ \frac{1}{2}\epsilon_8(\mathtt{x},t) & \epsilon_7(\mathtt{x},t)\end{bmatrix},$$

$$e_{\epsilon_\psi}(\mathtt{x},t) = \begin{bmatrix}e_{\epsilon_1}(\mathtt{x},t) & e_{\epsilon_3}(\mathtt{x},t)\\ e_{\epsilon_3}(\mathtt{x},t) & e_{\epsilon_2}(\mathtt{x},t)\end{bmatrix},\quad e_{\epsilon_w}(\mathtt{x},t) = \begin{bmatrix}e_{\epsilon_4}(\mathtt{x},t)\\ e_{\epsilon_5}(\mathtt{x},t)\end{bmatrix},\quad e_{\epsilon_\theta}(\mathtt{x},t) = \begin{bmatrix}e_{\epsilon_6}(\mathtt{x},t) & e_{\epsilon_8}(\mathtt{x},t)\\ e_{\epsilon_8}(\mathtt{x},t) & e_{\epsilon_7}(\mathtt{x},t)\end{bmatrix},$$

where $\epsilon_i(\mathtt{x},t)$, $e_{\epsilon_i}(\mathtt{x},t)$ with $i = 1,...,8$ are the energy and co-energy variables related to the generalized strains of the model (58), respectively.

Remark 6. The third-order shear deformation theory presented in this section is also applicable for beams, and note that dynamic models based on the first-order shear deformation theory are obtained from the Reddy's theory by setting $\alpha = 0$ [35]. In this case, $\alpha = 0$ leads to the well known Mindlin plate model, and the port-Hamiltonian representation obtained by this methodology is equivalent to the one obtained first in [15], and then generalized using tensor notation in [16].

5. Conclusion and future work

In this paper a novel three-steps methodology is proposed to systematically derive the infinite-dimensional port-Hamiltonian representation of multidimensional flexible linear mechanical systems, subject to a given class of kinematic assumptions and constitutive relationships. The methodology involves assuming the factorization of the displacement field to define the mass matrix and generalized momentum variables. Following this, the non-zero components of the strain tensor are factorized in terms of generalized strain variables, leading to the computation of the stiffness matrix and characterization of the differential operator. Finally, an energetically consistent port-Hamiltonian representation of the model is proposed, primarily demonstrated using Lemma 1, which is defined for the considered class of differential operators and Hamilton's principle.

It is shown that the proposed methodology allows to derive port-Hamiltonian representations in multidimensional spatial domains. It is effective in finding classical models such as the Timoshenko beam, Mindlin's plate or the general three-dimensional elasticity problem (among others), and other less classical models such as the ones based on more specific kinematic assumptions like the Reddy's third-order shear deformation theory. The main advantages of this procedure regarding the usual existing methods in the literature are: first, it considerably reduces the amount of algebraic work to derive these models since no variational

principle has to be applied and integration by parts has been done once for all, in general, over multidimensional domains. Second, no intuition is required to choose the set of state variables that guarantees the existence of an associated skew-adjoint differential operator, because they are explicitly defined. In addition, the structure of the differential operator is determined as soon as the state variables are chosen. Third, an expression of the boundary variables is proposed which allows to define the boundary inputs and boundary outputs ports such that the proposed model satisfy the Stoke-Dirac structure. Lastly, since the method starts from kinematic assumptions and constitutive laws, it is not only suitable for rewriting pre-existing models within the port-Hamiltonian framework, but also potentially allows to directly derive new models in PH form.

As future work we will consider the extension of this methodology to the case of nonlinear elasticity, regarding both geometric and material nonlinearities. Also, as discussed in [36], Hamilton's principle facilitates the modeling of constrained and multiphysics systems. Since our methodology is primarily grounded in Hamilton's principle, further research in this topic promises to extend our findings to encompass complex multiphysics systems. Examples include [37] which deal with magneto-thermo-viscoelastic interactions, [38] focusing on bidimensional deformation of initially stressed thermoelastic mediums, and [39] concerning photo-thermal interactions in a semi conductor material, among others. This extension would enable us to systematically derive their PH representations and motivate interesting control scenarios. Furthermore, in the same way that variational methods such as Hamilton's principle unify Lagrangian modeling and finite element discretization, it remains to be studied under what conditions or choices, this methodology unifies both port-Hamiltonian modeling and structure-preserving finite element discretization.

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Appendix A. Lemmas from variational calculus

The two following lemmas from variational calculus serve as justification to obtain the equations of motion and the boundary conditions of models based on Hamilton's principle.

Lemma 2. ([40], p.224.) Let $\mathcal{W}$ be an inner product space, and consider a C^0 field $h : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ with $\bar{\Omega}$ the closure $\bar{\Omega} = \Omega \cup \partial\Omega$. If the equation

$$\int_{t_1}^{t_2} \int_{\Omega} h(\mathbf{x}, t) \cdot \eta(\mathbf{x}, t)\, d\mathbf{x}\, dt = 0 \tag{A.1}$$

holds for every C^∞ field $\eta : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on $\partial\Omega$, then $h(\mathbf{x}, t) = 0$ on $\bar{\Omega} \times [t_1, t_2]$.

Lemma 3. ([40], p.224.) Suppose that $\partial\Omega$ consists of complementary regular sub-surfaces $\partial\Omega_u$ and $\partial\Omega_\sigma$. Let $\mathcal{W}$ be an inner product space, and consider a function $g : \partial\Omega_\sigma \times [t_1, t_2] \to \mathcal{W}$ that is piecewise regular and continuous in time. If the equation

$$\int_{t_1}^{t_2} \int_{\partial\Omega_\sigma} g(\mathbf{s}, t) \cdot \eta(\mathbf{s}, t)\, d\mathbf{s}\, dt = 0 \tag{A.2}$$

holds for every C^∞ field $\eta : \bar{\Omega} \times [t_1, t_2] \to \mathcal{W}$ that vanishes at time t_1, at time t_2, and on $\partial\Omega_u$, then $g(\mathbf{s}, t) = 0$ on $\partial\Omega_\sigma \times [t_1, t_2]$.

Appendix B. Voigt-Kelvin notation

Since the stress and strain tensors are symmetric, they only have six independent components. The Voigt-Kelvin notation defines

$$\begin{array}{llllll} \sigma_1 = \sigma_{11}, & \sigma_2 = \sigma_{22}, & \sigma_3 = \sigma_{33}, & \sigma_4 = \sigma_{12}, & \sigma_5 = \sigma_{13}, & \sigma_6 = \sigma_{23}, \\ \varepsilon_1 = \varepsilon_{11}, & \varepsilon_2 = \varepsilon_{22}, & \varepsilon_3 = \varepsilon_{33}, & \varepsilon_4 = 2\varepsilon_{12}, & \varepsilon_5 = 2\varepsilon_{13}, & \varepsilon_6 = 2\varepsilon_{23}, \end{array}$$

where the independent components of both tensors are grouped into the so-called Voigt-stress vector $\vec{\sigma}(\mathtt{X}, t) \in \mathbb{R}^6$ and Voigt-strain vector $\vec{\varepsilon}(\mathtt{X}, t) \in \mathbb{R}^6$, which are respectively given by

$$\vec{\sigma}(\mathtt{X}, t) = \begin{bmatrix} \sigma_1(\mathtt{X}, t) & \cdots & \sigma_6(\mathtt{X}, t) \end{bmatrix}^\top, \tag{B.1}$$

$$\vec{\varepsilon}(\mathtt{X}, t) = \begin{bmatrix} \varepsilon_1(\mathtt{X}, t) & \cdots & \varepsilon_6(\mathtt{X}, t) \end{bmatrix}^\top. \tag{B.2}$$

In addition, using the Voigt-Kelvin notation it is possible to express the constitutive relation as

$$\vec{\sigma}(\mathtt{X}, t) = C\,\vec{\varepsilon}(\mathtt{X}, t) \quad \sim \quad \underline{\sigma}(\mathtt{X}, t) = \underline{\underline{C}} : \underline{\varepsilon}(\mathtt{X}, t),$$

where $C = C^\top > 0 \in \mathbb{R}^{6\times 6}$ is a constitutive matrix. For example, for isotropic materials the constitutive fourth-order tensor for 3D elasticity ($\underline{\underline{C_{3D}}}(\cdot) = 2\mu(\cdot) + \lambda \mathrm{tr}(\cdot) 1_3$), and for plane stress in 2D elasticity ($\underline{\underline{C_{2D}}}(\cdot) = \frac{E}{1-\nu^2}\left[(1-\nu)(\cdot) + \nu \mathrm{tr}(\cdot) 1_2\right]$) reduce to

$$\underline{\sigma_{3D}} = \underline{\underline{C_{3D}}} : \underline{\varepsilon_{3D}} \quad \Leftrightarrow \quad \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix},$$

$$\underline{\sigma_{2D}} = \underline{\underline{C_{2D}}} : \underline{\varepsilon_{2D}} \quad \Leftrightarrow \quad \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix},$$

where μ and λ are the Lamé constants defined as $\mu = \frac{E}{2(1+\nu)} = G$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, where E is Young' modulus, ν is Poisson's ratio (ratio between transverse elongation and axial shortening) and the constant $\mu = G$ is also known as the shear modulus. Note also that the tensor contraction now reduces to

$$\underline{\sigma}(\mathtt{X}, t) : \underline{\varepsilon}(\mathtt{X}, t) \;=\; \vec{\sigma}(\mathtt{X}, t) \cdot \vec{\varepsilon}(\mathtt{X}, t). \tag{B.3}$$

Assuming a Cartesian reference system $\mathtt{X} = \{\zeta_1, \zeta_2, \zeta_3\}$, the Voigt-strain vector can be obtained by

$$\vec{\varepsilon}(\mathtt{X}, t) = \mathbb{L}\,\mathbf{u}(\mathtt{X}, t) \quad \sim \quad \underline{\varepsilon}(\mathtt{X}, t) = \mathrm{Grad}(\mathbf{u}(\mathtt{X}, t)), \tag{B.4}$$

with $\mathbb{L}$ a linear differential operator of dimension (6×3) given by

$$\mathbb{L} = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ \partial_2 & \partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \\ 0 & \partial_3 & \partial_2 \end{bmatrix} \sim \mathrm{Grad}, \qquad \text{with } \partial_k = \frac{\partial}{\partial \zeta_k},\ k = \{1, 2, 3\}. \tag{B.5}$$

This is also true for two-dimensional tensor fields, where in that case $\mathbb{L}_{2D}$ is a linear differential operator of

dimension (3×2) given by

$$\mathbb{L}_{2D} = \begin{bmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{bmatrix} \sim \text{Grad}, \qquad \text{with } \partial_k = \frac{\partial}{\partial \zeta_k},\ k = \{1, 2\}. \tag{B.6}$$

Note that $\text{Div}(\underline{\sigma}) = \nabla \cdot \underline{\sigma} = \mathbb{L}^\top \vec{\sigma} = -\mathbb{L}^* \vec{\sigma} \sim -\text{Grad}^*(\underline{\sigma})$, where $\text{Div}(\cdot)$ is the tensor divergence operator. Then by similarity it can be seen that $\text{Div} = -\text{Grad}^*$ (see [16, Theorem 4] for the proof).

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