Observer-based H_∞ fault-tolerant control for stochastic parabolic systems

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A B S T R A C T

This paper proposes an observer-based H_∞ fault-tolerant controller for a class of stochastic parabolic systems (SPSs) subject to actuator failures. An augmented SPS that includes both the state vector and the error vector is given. The coupling difficulty between faults and system states is solved by using matrix decompositions and inequality techniques. The designed observer-based controller ensures that the SPSs achieve the mean square finite horizon H_∞ performance. Finally, a simulation of CPU chip thermal fault is provided to illustrate the effectiveness of the proposed scheme.

1. Introduction

Due to aging, oxidation, and other factors, failures or imperfect behaviors of the actuator or system components are inevitable. To address this issue, different researchers have investigated fault-tolerant control (FTC), which can maintain the overall performance of systems even when the system components fail. Over the past decade, FTC has become one of the key control objectives reliability control design. The study of FTC has made significant contributions to nonlinear systems [1], multi-agent systems [2], cooperative heterogeneous systems [3], stochastic ordinary differential systems [4,5], etc. To mention a few examples, in [4], a novel sliding mode observer approach was proposed to address the problem of FTC for stochastic Markovian systems with sensor and actuator faults, as well as disturbances. In [5], an FTC design was developed using a reduced-order dynamic estimator for quantum systems with faults. Another research frontier, the fault-tolerant boundary control of deterministic flexible manipulator systems [6] has also made significant progress. However, due to inevitable random interference, many manufacturing processes with thermal failures are modeled by stochastic partial differential equations (SPDEs), and the related research has become an interesting topic. Significant achievements have been made in sampled-data control [7], intermittent control [8], boundary control [9], robust control [10], H_∞ control [11], sliding mode control [12], non-fragile control [13]. These results have greatly enriched the control theory of SPDEs. For example, [13] considered the robust non-fragile boundary control with gain variation for SPDEs with input quantization. Earlier, [14] considered an adaptive boundary control for a class of stochastic parabolic systems. By the stochastic approximation technique and the regularization method, the unknown coefficient was estimated. Furthermore, few articles have addressed the FTC for SPDEs. Stochastic parabolic system (SPS), as a classical model in SPDEs, has been limited research, although it has been applied in many fields, such as magnetohydrodynamic (MHD) [15], water pollution [16], heat conduction [17] and so on. Therefore, it is worthwhile to establish a realistic FTC strategy for SPS with the desired robustness performance.

Motivated by the above reasons, this paper derives an FTC scheme for SPSs, which serves as a starting point for FTC of SPDEs. The adaptive FTC, which has been developed for deterministic systems and few stochastic nonlinear systems, will also be progressively considered. The coupling of spatial diffusion, the stochastic term and actuator faults in SPSs presents significant challenges to performance analysis. With these difficulties and the unmeasurable system states, we propose a feasible scheme for SPSs to mitigate the effects of faults and disturbances and ensuring that the controlled system achieves mean square finite-horizon H_∞ performance. Furthermore, explicit expressions are provided for the controller and observer gain matrices. The main contribution of this paper is twofold. First, inspired by [18], this study extends the FTC scheme to the stochastic partial differential systems (i.e., SPSs). An observer-based H_∞ FTC is designed to compensate for the combined effects of actuator faults and external disturbances. Second, without simplifying or transforming the dynamics of the infinite-dimensional system, an LMI scheme is proposed that computes both the control gain and observation gain.

Throughout this paper, $(\Omega, \mathcal{F}_t, \mathbb{P})$ denotes a complete probability space adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $\mathbb{L}^2(\mathbb{R}^n)$ is the set of square-integrable function spaces. $\mathcal{H}^1(a,b)$ denotes the Sobolev space defined on (a,b) consisting of absolutely continuous functions whose derivatives

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are square-integrable. $B > 0$ ($B \geq 0$) indicates B is a positive definite (semi-definite) matrix. $\|y(x,t)\|^2 = \int_0^1 y^{\mathrm{T}} y\mathrm{d}x$. The symbol $*$ denotes the symmetric element of the symmetric matrix. He(A) represents $A^{\mathrm{T}} + A$. col$\{a_1, a_2, \ldots, a_n\}$ is an n-dimensional column vector with elements $a_1, a_2, \ldots, a_n$. diag$\{b_1, b_2, \ldots, b_n\}$ is the n dimensional diagonal matrix with elements $b_1, b_2, \ldots, b_n$. $\|Z(x,t)\|^2$ denotes $\int_0^1 Z^{\mathrm{T}} \tilde{P} Z\mathrm{d}x$, where $\tilde{P} = \mathrm{diag}\{P, P\}$ and P is a positive definite matrix.

2. Problem statements and preliminaries

Consider the following disturbed stochastic parabolic system (SPS) with actuator faults and disturbances

$$\begin{cases} \mathrm{d}y(x,t) = \Bigg[Ay(x,t) + B\dfrac{\partial^2 y(x,t)}{\partial x^2} + u^F(x,t) \\ \qquad + v(x,t) \Bigg] \mathrm{d}t + Cy(x,t)\mathrm{d}W(t), \\ \qquad x \in (0,1), \; t \in [0,\infty), \\ y(x,0) = \phi(x), \\ y_x(0,t) = \mathbf{0}, \quad y_x(1,t) = \mathbf{0}, \end{cases} \tag{1}$$

where $y(x,t) \in \mathbb{L}^2(\mathbb{R}^n)$ is the system state. x and t are the spatial and time variables, respectively. Matrices $A, B, C \in \mathbb{R}^{n\times n}$ and $B > 0$. $u^F(x,t) \in \mathbb{R}^n$ is the actuator with an unknown fault. $v(x,t)$ is the unknown external disturbance. $W(t)$ is defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$, a one-dimensional standard Brownian motion. $\mathbb{E}(\cdot)$ is the mathematical expectation associated with $\mathbb{P}$. $\phi(x) \in \mathbb{R}^n$ is an initial continuous function.

The actuator with a fault is given as follows

$$u^F(x,t) = F_a u(x,t), \tag{2}$$

where $F_a \in \mathbb{R}^{n\times n}$ is the unknown effectiveness factor, $u(x,t)$ is the control input.

Remark 1. In light of [19], the fault F_a can be detected and isolated using a fault diagnosis scheme. Therefore, we do not consider the fault diagnosis scheme and directly assume that the fault F_a has been diagnosed in advance.

Based on [20], assume that the actuator effectiveness factor F_a can be written in the following form

$$F_a = \mathrm{diag}\{f_1, f_2, \ldots, f_n\},$$

where $0 \leq f_i \leq 1$ is the ith unknown constant for $i = 1, 2, \ldots, n$. Then there are three types for f_i, $i = 1, 2, \ldots, n$.

$$\begin{cases} f_i = 0, & \text{Actuator is completely ineffective;} \\ f_i = 1, & \text{No fault happens;} \\ Others, & \text{Actuator is partially effective.} \end{cases}$$

In this paper, we focus on the last case.

We make the following assumption, so that the known fault information can be effectively used, which will also help us obtain less conservative results.

Assumption 2. Assume that there exist known constants $\underline{f}_i$ and $\overline{f}_i$, such that $0 < \underline{f}_i \leq f_i \leq \overline{f}_i < 1$ for $i = 1, 2, \ldots, n$. If we define matrices

$$\hat{F}_a = \mathrm{diag}\{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\}, \quad \check{F}_a = \mathrm{diag}\{\check{f}_1, \check{f}_2, \ldots, \check{f}_n\},$$

$$D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}, \; |D| = \mathrm{diag}\{|d_1|, |d_2|, \ldots, |d_n|\},$$

where

$$\hat{f}_i = \frac{\overline{f}_i + \underline{f}_i}{2}, \quad \check{f}_i = \frac{\overline{f}_i - \underline{f}_i}{\underline{f}_i + \overline{f}_i}, \quad d_i = \frac{f_i - \hat{f}_i}{\hat{f}_i},$$

for $i = 1, 2, \ldots, n$. Then $|D| \leq \check{F}_a \leq I_n$ by element and F_a can be rewritten as

$$F_a = \hat{F}_a(I_n + D).$$

Hence, the actuator with fault (2) can be integrated into the following form

$$u^F(x,t) = \hat{F}_a(I_n + D)u(x,t). \tag{3}$$

Hereafter, we omit (x,t) and denote $\frac{\partial y(x,t)}{\partial t}$ as y_t and $\frac{\partial^2 y(x,t)}{\partial x^2}$ as y_{xx}.

Note that the control input u always depends on the full-domain states of system (1). However, system states in the domain cannot be completely acquired in most of engineering applications and under this condition, control input u becomes invalid. Therefore, we construct an observer as follows to estimate system states

$$\begin{cases} \mathrm{d}\hat{y} = \Big(A\hat{y} + B\hat{y}_{xx} + \mathcal{L}\big(\hat{y}(1,t) - y(1,t)\big) + u\Big)\mathrm{d}t, \\ \hat{y}(x,0) = \hat{\phi}(x), \\ \hat{y}_x(0,t) = \mathbf{0}, \quad \hat{y}_x(1,t) = \mathbf{0}, \end{cases} \tag{4}$$

where $\hat{y} \in \mathbb{R}^n$ is the estimation of y and the state at boundary $x = 1$ is measurable. $\mathcal{L} \in \mathbb{R}^{n\times n}$ is the observer gain. $\hat{\phi}(x)$ is the estimation of $\phi(x)$.

The observer-based controller is designed as follows

$$u = -K\hat{y}, \tag{5}$$

where matrix $K \in \mathbb{R}^{n\times n}$ is the control gain, which needs to be determined.

Set $e = y - \hat{y}$ as the error. Let $\tilde{\phi}(x) = \phi(x) - \hat{\phi}(x)$, $Z = \mathrm{col}\{y, e\}$, $\Psi(x) = \mathrm{col}\{\phi(x), \tilde{\phi}(x)\}$. Then the following augmented closed-loop system can be established

$$\begin{cases} \mathrm{d}Z = \Big[\tilde{A}Z + \tilde{B}Z_{xx} + \bar{I}\mathcal{L}\bar{I}^{\mathrm{T}} Z(1,t) + \tilde{u}^F + \tilde{I}v \Big] \mathrm{d}t \\ \qquad + \tilde{C}Z\mathrm{d}W(t), \\ Z(x,0) = \Psi(x), \\ Z_x(0,t) = \mathbf{0}, \quad Z_x(1,t) = \mathbf{0}, \end{cases} \tag{6}$$

where $\tilde{A} = \mathrm{diag}\{A, A\}$, $\tilde{B} = \mathrm{diag}\{B, B\}$, $\bar{I} = \mathrm{col}\{0_n, I_n\}$, $\tilde{C} = \begin{pmatrix} C & 0_n \\ C & 0_n \end{pmatrix}$, $\tilde{u}^F = \begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D) \end{pmatrix} K\hat{y}$, $\tilde{I} = \mathrm{col}\{I_n, I_n\}$.

According the results in [21], Example 3.4, the well-posedness can be guaranteed under the designed FTC (5) for linear system (6).

The following lemmas will be instrumental in the subsequent analysis.

Lemma 3 ([22]). *If there exists vector $z \in \mathcal{H}^1(a,b)$ with $z(a) = 0$ or $z(b) = 0$. Then for matrix $\mathcal{R} > 0$, the following inequality holds*

$$\int_a^b z^{\mathrm{T}}(s)\mathcal{R}z(s)\mathrm{d}s \leq \frac{4(b-a)^2}{\pi^2} \int_a^b (\frac{\mathrm{d}z}{\mathrm{d}s})^{\mathrm{T}} \mathcal{R} (\frac{\mathrm{d}z}{\mathrm{d}s})\mathrm{d}s.$$

Lemma 4 ([23]). *Given matrices $\mathcal{E}, \mathcal{F}, \Lambda \in \mathbb{R}^{n\times n}$. If $\Lambda^{\mathrm{T}}\Lambda \leq I_n$, then for any constant $\varsigma > 0$ and vectors $\eta, \zeta \in \mathbb{R}^n$, the following inequality holds*

$$\eta^{\mathrm{T}}\mathcal{E}\Lambda\mathcal{F}\zeta + \zeta^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\Lambda^{\mathrm{T}}\mathcal{E}^{\mathrm{T}}\eta \leq \varsigma\eta^{\mathrm{T}}\mathcal{E}\mathcal{E}^{\mathrm{T}}\eta + \frac{1}{\varsigma}\zeta^{\mathrm{T}}\mathcal{F}^{\mathrm{T}}\mathcal{F}\zeta.$$

Our objective is to achieve the mean square finite horizon H_∞ performance for the augmented system (6) by solving for the control gain K and observer gain $\mathcal{L}$. In the subsequent section, based on inequality techniques and stochastic analysis theory, sufficient conditions in terms of LMIs are provided for system (6) to achieve H_∞ fault-tolerant performance.

3. Main results

Firstly, an augmented H_∞ performance definition is provided as follows.

Definition 5. If for given constants $0 < T_f < \infty$, $\gamma > 0$, the following inequality holds

$$\begin{aligned}&\int_0^{T_f}\int_0^1 \mathbb{E}(Z^{\mathrm{T}}Z)\mathrm{d}x\mathrm{d}t \\ &\le \gamma^2\int_0^{T_f}\int_0^1 \mathbb{E}(v^{\mathrm{T}}v)\mathrm{d}x\mathrm{d}t + \mathbb{E}\|Z(x,0)\|^2,\end{aligned} \tag{7}$$

then system (6) achieves mean square finite-horizon H_∞ performance.

Remark 6. The initial value $Z(x,0)$ can be regarded as a disturbance. Hence, we can also provide an alternative definition as follows

$$\begin{aligned}&\int_0^{T_f}\int_0^1 \mathbb{E}(Z^{\mathrm{T}}Z)\mathrm{d}x\mathrm{d}t \\ &\le \gamma^2\Big(\int_0^{T_f}\int_0^1 \mathbb{E}(v^{\mathrm{T}}v)\mathrm{d}x\mathrm{d}t + \mathbb{E}\|Z(x,0)\|^2\Big).\end{aligned} \tag{8}$$

The given H_∞ performance definition includes the case of initial value $Z(x,0) = 0$, which has been studied in [24].

The following theorem provides a sufficient condition under which the system (6) with controller (5) achieves the H_∞ fault-tolerant performance.

Theorem 7. *If there exist positive definite matrix P and positive constants T_f, γ, ϵ, such that for given matrices $K, \mathcal{L}$, the following inequality holds*

$$\mathcal{G} = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ * & \Omega_{22}\end{bmatrix} \prec 0, \tag{9}$$

where

$$\begin{aligned}\Omega_{11} &= \mathrm{He}(\tilde{P}\tilde{A} - \frac{\pi^2}{8}\tilde{P}\tilde{B} + \tilde{P}GKE_n + \tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}) \\ &\quad + \tilde{C}^{\mathrm{T}}\tilde{P}\tilde{C} + \frac{1}{\epsilon}E_n^{\mathrm{T}}K^{\mathrm{T}}KE_n + \epsilon\tilde{P}\tilde{I}\hat{F}_a\hat{F}_a\tilde{I}^{\mathrm{T}}\tilde{P} \\ &\quad + \gamma^{-2}\tilde{P}\tilde{I}\tilde{I}^{\mathrm{T}}\tilde{P} + I_{2n},\end{aligned}$$

$$\Omega_{12} = -\tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}},\ \Omega_{22} = -\frac{\pi^2}{8}\mathrm{He}(\tilde{P}\tilde{B}),\ \tilde{P} = \mathrm{diag}\{P, P\},$$

$E_n = \begin{pmatrix} I_n & -I_n\end{pmatrix}$, $G = \begin{pmatrix} -\hat{F}_a \\ I_n - \hat{F}_a\end{pmatrix}$, *then system* (6) *with observer-based fault-tolerant controller* (5) *achieves mean square finite-horizon H_∞ performance.*

Proof. Choose the Lyapunov functional $V(t) = \int_0^1 Z^{\mathrm{T}}\tilde{P}Z\mathrm{d}x$, then the corresponding infinitesimal generator $\mathfrak{L}V(t)$ is given by

$$\begin{aligned}\mathfrak{L}V(t) = \int_0^1 & Z^{\mathrm{T}}\Big(\mathrm{He}(\tilde{P}\tilde{A}) + \tilde{C}^{\mathrm{T}}\tilde{P}\tilde{C}\Big)Z + \mathrm{He}\Big(Z^{\mathrm{T}}\tilde{P}\tilde{B}Z_{xx} \\ &+ Z^{\mathrm{T}}\tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}Z(1,t) + Z^{\mathrm{T}}\tilde{P}\tilde{u}^F + Z^{\mathrm{T}}\tilde{P}\tilde{I}v\Big)\mathrm{d}x.\end{aligned}$$

Let $E_n = \begin{pmatrix} I_n & -I_n\end{pmatrix}$. From $\tilde{u}^F = \begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D)\end{pmatrix}K\hat{y}$, we have

$$\begin{aligned}&\int_0^1 Z^{\mathrm{T}}\tilde{P}\tilde{u}^F\mathrm{d}x \\ &= \int_0^1 Z^{\mathrm{T}}\tilde{P}\begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D)\end{pmatrix}K\hat{y}\mathrm{d}x \\ &= \int_0^1 Z^{\mathrm{T}}\tilde{P}\begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D)\end{pmatrix}KE_nZ\mathrm{d}x.\end{aligned} \tag{10}$$

Set matrices $G = \begin{pmatrix} -\hat{F}_a \\ I_n - \hat{F}_a\end{pmatrix}$, we can derive that

$$\begin{aligned}&\int_0^1 Z^{\mathrm{T}}\tilde{P}\begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D)\end{pmatrix}KE_nZ\mathrm{d}x \\ &= \int_0^1 (y^{\mathrm{T}}P, e^{\mathrm{T}}P)\begin{pmatrix} -\hat{F}_a(I_n + D) \\ I_n - \hat{F}_a(I_n + D)\end{pmatrix}KE_nZ\mathrm{d}x \\ &= \int_0^1 \Big((y^{\mathrm{T}}, e^{\mathrm{T}})\begin{pmatrix} P & 0 \\ 0 & P\end{pmatrix}\begin{pmatrix} -\hat{F}_a \\ I_n - \hat{F}_a\end{pmatrix}KE_nZ \\ &\qquad - (y^{\mathrm{T}}, e^{\mathrm{T}})\begin{pmatrix} P & 0 \\ 0 & P\end{pmatrix}\begin{pmatrix} I_n \\ I_n\end{pmatrix}\hat{F}_aDKE_nZ\Big)\mathrm{d}x \\ &= \int_0^1 Z^{\mathrm{T}}\tilde{P}GKE_nZ - Z^{\mathrm{T}}\tilde{P}\tilde{I}\hat{F}_aDKE_nZ\mathrm{d}x.\end{aligned} \tag{11}$$

For any positive scalar ϵ, Lemma 4 leads to

$$\begin{aligned}&-\int_0^1 Z^{\mathrm{T}}\mathrm{He}(\tilde{P}\tilde{I}\hat{F}_aDKE_n)Z\mathrm{d}x \\ &\le \int_0^1 Z^{\mathrm{T}}(\frac{1}{\epsilon}E_n^{\mathrm{T}}K^{\mathrm{T}}KE_n + \epsilon\tilde{P}\tilde{I}\hat{F}_a\hat{F}_a\tilde{I}^{\mathrm{T}}\tilde{P})Z\mathrm{d}x.\end{aligned} \tag{12}$$

Set $\bar{Z} = Z - Z(1,t)$, integration by parts and Lemma 3 yield

$$\begin{aligned}\int_0^1 Z^{\mathrm{T}}\tilde{P}\tilde{B}Z_{xx}\mathrm{d}x &= -\int_0^1 Z_x^{\mathrm{T}}\tilde{P}\tilde{B}Z_x\mathrm{d}x \\ &\le \int_0^1 -\frac{\pi^2}{8}\bar{Z}^{\mathrm{T}}\tilde{P}\tilde{B}\bar{Z} - \frac{\pi^2}{8}Z^{\mathrm{T}}\tilde{P}\tilde{B}Z\mathrm{d}x.\end{aligned}$$

Above analyses follow that

$$\begin{aligned}&\mathfrak{L}V(t) \\ &\le \int_0^1 Z^{\mathrm{T}}\Big(\mathrm{He}(\tilde{P}\tilde{A} - \frac{\pi^2}{8}\tilde{P}\tilde{B} + \tilde{P}GKE_n + \tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}) \\ &\quad + \tilde{C}^{\mathrm{T}}\tilde{P}\tilde{C} + \frac{1}{\epsilon}E_n^{\mathrm{T}}K^{\mathrm{T}}KE_n + \epsilon\tilde{P}\tilde{I}\hat{F}_a\hat{F}_a\tilde{I}^{\mathrm{T}}\tilde{P}\Big)Z \\ &\quad + \mathrm{He}\big(Z^{\mathrm{T}}\tilde{P}\tilde{I}v - Z^{\mathrm{T}}\tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}\bar{Z} - \frac{\pi^2}{8}\bar{Z}^{\mathrm{T}}\tilde{P}\tilde{B}\bar{Z}\big)\mathrm{d}x.\end{aligned}$$

Note that $V(0) = \|Z(x,0)\|^2$ and $V(T_f) > 0$, then the following inequality holds

$$\begin{aligned}0 &= \mathbb{E}\int_0^{T_f}\mathfrak{L}V(t)\mathrm{d}t + \mathbb{E}V(0) - \mathbb{E}V(T_f) \\ &\le \mathbb{E}\int_0^{T_f}\mathfrak{L}V(t)\mathrm{d}t + \mathbb{E}V(0).\end{aligned}$$

Keeping condition (9) in mind, we can directly obtain the following inequality

$$\begin{aligned}&\mathbb{E}\int_0^{T_f}\int_0^1 (Z^{\mathrm{T}}Z - \gamma^2 v^{\mathrm{T}}v)\mathrm{d}x\mathrm{d}t - \int_0^1 \mathbb{E}V(0)\mathrm{d}x \\ &\le \mathbb{E}\int_0^{T_f}\int_0^1 -\gamma^2(v - \gamma^{-2}\tilde{I}^{\mathrm{T}}\tilde{P}Z)^{\mathrm{T}}(v - \gamma^{-2}\tilde{I}^{\mathrm{T}}\tilde{P}Z) \\ &\quad + Z^{\mathrm{T}}[I_{2n} + \gamma^{-2}\tilde{P}\tilde{I}\tilde{I}^{\mathrm{T}}\tilde{P} + \mathrm{He}(\tilde{P}\tilde{A} - \frac{\pi^2}{8}\tilde{P}\tilde{B} \\ &\quad + \tilde{P}GKE_n + \tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}) + \tilde{C}^{\mathrm{T}}\tilde{P}\tilde{C} + \frac{1}{\epsilon}E_n^{\mathrm{T}}K^{\mathrm{T}}KE_n \\ &\quad + \epsilon\tilde{P}\tilde{I}F_a\hat{F}_a\tilde{I}^{\mathrm{T}}\tilde{P}]Z - \mathrm{He}\big(Z^{\mathrm{T}}\tilde{P}\tilde{I}\mathcal{L}\tilde{I}^{\mathrm{T}}\bar{Z} \\ &\quad + \frac{\pi^2}{8}\bar{Z}^{\mathrm{T}}\tilde{P}\tilde{B}\bar{Z}\big)\mathrm{d}x\mathrm{d}t \\ &\le \mathbb{E}\int_0^{T_f}\int_0^1 \Phi^{\mathrm{T}}\mathcal{G}\Phi\mathrm{d}x\mathrm{d}t < 0,\end{aligned}$$

where $\Phi = \mathrm{col}\{Z, \bar{Z}\}$, $\mathcal{G}$ is defined as (9), which concludes the proof. □

Remark 8. The matrix decompositions and inequality techniques are novelly introduced in equality (11) to address the difficulty of coupling arising from unknown faults and system states.

Theorem 7 gives the sufficient condition such that the observer-based controller can stabilize the system in the H_∞ sense. However, the computation of the gain matrices K and $\mathcal{L}$ remains challenging. Hence, we introduce the following result which gives us an efficient way to compute the two gains via an LMI approach.

Theorem 9. *Given positive constants T_f, γ, ϵ, fault matrices $\hat{F}_a$, if there exist invertible positive definite matrix $M \in \mathbb{R}^{n\times n}$ and matrix $N, R \in \mathbb{R}^{n\times n}$, such that the following LMI holds*

$$\begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & 0_n & 0_n & \tilde{\Omega}_{15} & \tilde{\Omega}_{16} & \tilde{\Omega}_{17} & 0_n \\ * & \tilde{\Omega}_{22} & 0_n & \tilde{\Omega}_{24} & \tilde{\Omega}_{25} & 0_n & 0_n & \tilde{\Omega}_{28} \\ * & * & \tilde{\Omega}_{33} & 0_n & 0_n & 0_n & 0_n & 0_n \\ * & * & * & \tilde{\Omega}_{44} & 0_n & 0_n & 0_n & 0_n \\ * & * & * & * & \tilde{\Omega}_{55} & 0_n & 0_n & 0_n \\ * & * & * & * & * & \tilde{\Omega}_{66} & 0_n & 0_n \\ * & * & * & * & * & * & \tilde{\Omega}_{77} & 0_n \\ * & * & * & * & * & * & * & \tilde{\Omega}_{88} \end{bmatrix} < 0, \tag{13}$$

where

$$\tilde{\Omega}_{11} = \mathrm{He}(AM - \frac{\pi^2}{8}BM - \hat{F}_a N) + \gamma^{-2} I_n + \epsilon \hat{F}_a \hat{F}_a,$$

$$\tilde{\Omega}_{22} = \mathrm{He}(AM - \frac{\pi^2}{8}BM - (I_n - \hat{F}_a)N + R) + \gamma^{-2} I_n + \epsilon \hat{F}_a \hat{F}_a,$$

$$\tilde{\Omega}_{12} = \gamma^{-2} I_n + \epsilon \hat{F}_a \hat{F}_a + \hat{F}_a N + N^{\mathrm{T}}(I_n - \hat{F}_a),\ \tilde{\Omega}_{15} = N^{\mathrm{T}},$$

$$\tilde{\Omega}_{16} = MC^{\mathrm{T}},\ \tilde{\Omega}_{17} = \tilde{\Omega}_{28} = M,\ \tilde{\Omega}_{24} = -R,$$

$$\tilde{\Omega}_{25} = -N^{\mathrm{T}},\ \tilde{\Omega}_{33} = \tilde{\Omega}_{44} = -\frac{\pi^2}{8}\mathrm{He}(BM),\ \tilde{\Omega}_{55} = -\epsilon I_n,$$

$$\tilde{\Omega}_{66} = -\frac{M}{2},\ \tilde{\Omega}_{77} = \tilde{\Omega}_{88} = -I_n,$$

then system (6) with (5) achieves mean square finite-horizon H_∞ performance. Furthermore, the controller gain and observer gain are designed as $K = NM^{-1}$ and $\mathcal{L} = RM^{-1}$.

Proof. Take $M = P^{-1}$. By Pre-and post-multiplying (9) by $\mathrm{diag}\{M, M, M, M\}$ and its transpose, we obtain

$$\bar{\mathcal{G}} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & 0_n & 0_n \\ * & \bar{\Omega}_{22} & 0_n & \tilde{\Omega}_{24} \\ * & * & \tilde{\Omega}_{33} & 0_n \\ * & * & * & \tilde{\Omega}_{44} \end{bmatrix} < 0, \tag{14}$$

where

$$\bar{\Omega}_{11} = \mathrm{He}(AM - \frac{\pi^2}{8}BM - \hat{F}_a N) + \gamma^{-2} I_n + MM + 2MC^{\mathrm{T}}M^{-1}CM + \frac{1}{\epsilon}N^{\mathrm{T}}N + \epsilon \hat{F}_a \hat{F}_a,$$

$$\bar{\Omega}_{12} = \gamma^{-2} I_n - \frac{1}{\epsilon}N^{\mathrm{T}}N + \epsilon \hat{F}_a \hat{F}_a + \hat{F}_a N + N^{\mathrm{T}}(I_n - \hat{F}_a),$$

$$\bar{\Omega}_{22} = \mathrm{He}(AM - \frac{\pi^2}{8}BM + R - (I_n - \hat{F}_a)N) + \gamma^{-2} I_n + MM + \frac{1}{\epsilon}N^{\mathrm{T}}N + \epsilon \hat{F}_a \hat{F}_a,$$

and the other parameters are defined as (13).

By virtue of Schur complement lemma [23], inequality (14) can be rewritten as condition (13). We complete the proof. □

Remark 10. Under the designed controller (5), compared to directly using the unknown effectiveness factor F_a, the application of Assumption 2 yields less conservative results and captures richer fault information. Specifically, inequalities (10)–(12) derived without relying on Assumption 2 demonstrate that $\int_0^1 \mathrm{He}(Z^{\mathrm{T}}\tilde{P}\tilde{u}^F)\mathrm{d}x \le \int_0^1 Z^{\mathrm{T}} (\mathrm{He}(\tilde{P}\begin{pmatrix} 0_n \\ I_n \end{pmatrix} KE_n) + \frac{1}{\epsilon}E_n^{\mathrm{T}}K^{\mathrm{T}}KE_n + \epsilon\tilde{P}\tilde{\mathcal{I}}\tilde{\mathcal{I}}^{\mathrm{T}}\tilde{P})Z\mathrm{d}x$. Since $\begin{pmatrix} -\hat{F}_a \\ I_n - \hat{F}_a \end{pmatrix} \le \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$ and $\hat{F}_a\hat{F}_a \le I_n$, it means that $\mathrm{He}(\tilde{P}GKE_n) + \epsilon\tilde{P}\tilde{\mathcal{I}}\hat{F}_a\hat{F}_a\tilde{\mathcal{I}}^{\mathrm{T}}\tilde{P} \le \mathrm{He}(\tilde{P}\begin{pmatrix} 0_n \\ I_n \end{pmatrix} KE_n) + \epsilon\tilde{P}\tilde{\mathcal{I}}\tilde{\mathcal{I}}^{\mathrm{T}}\tilde{P}$ holds. Hence, the obtained result in the paper is less conservative.

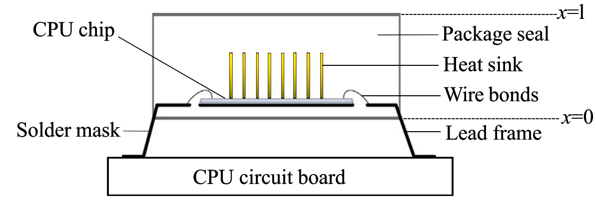

Fig. 1. CPU thermal model with surface package.

4. A simulation of thermal fault-tolerant performance for a CPU chip

In this section, we illustrate the effectiveness of the proposed methods by studying the thermal fault-tolerance performance of a CPU chip.

When a CPU chip is operating, heat can be dissipated through the heat sink in the package seal, see Fig. 1 and [25,26] for more details. We assume that the package seal is insulated. Then, we focus on the following one-dimensional heat conduction model

$$\begin{cases} \rho c_p \mathrm{d}T = \left[k_s T_{xx} + aT + u_T^F + v_T\right]\mathrm{d}t + cT\mathrm{d}W(t), \\ \qquad x \in (0,1),\ t \in [0,\infty), \\ -k_s T_x(0,t) = 0, \quad -k_s T_x(1,t) = 0, \\ T(x,0) = T_0, \end{cases} \tag{15}$$

where $T(x,t)$ is the temperature field distribution on the CPU chip. k_s denotes the thermal conductivity coefficient. ρc_p represents the volumetric heat capacity coefficient. a is the internal temperature effect. c is the coefficient. T_0 is the initial temperature of the electronic component at $t = 0$. $u_T^F(t)$ is the heat sink with unknown fault.

The disturbance v_T and stochastic factor $W(t)$ in the chip thermal model may result from heat loss between the packaging seal and the external environment temperature, or from other unmodeled heat generation.

Remark 11. Potential causes of heat sink failure include:

- Internal short circuits caused by poor heat sink design or manufacturing.
- Local device thermal damage caused by operational overload.
- External physical damage, such as device aging.

Set initial temperature $T_0 = \mathrm{col}\{0.5\cos(2\pi x), 0.01\ \cos(2\pi x)\}$ and stochastic disturbance temperature $v_T = \sin(2x)\cos(\pi t)$. Choose system parameters as $\rho c_p = 1$, $k_s = 2$, $a = 0.1$, $c = 0.1$, $\epsilon = 1$, $\gamma = 1$. Assume that failure factor F is given as $0.4, 0.7, 0.9$, respectively. Then there exist upper and lower bounds of the fault $\overline{F}$ and $\underline{F}$, such that the corresponding fault information $\hat{F}$, $\check{F}$ and D can be derived. By solving LMI (13) in Theorem 9, control gains K, observer gains $\mathcal{L}$ and the calculated performance indexes γ^* are obtained and are shown as Table 1.

In order to demonstrate the effectiveness of the designed controller, we set $u(x,t) = 0$ and further derive the performance index $\gamma^\dagger$, which is greater than $\gamma = 1$. It shows that the proposed controller significantly improves the robustness of the closed-loop system.

Fig. 2 shows that the system state without the controller is not mean square stable while the controlled system with various failure factors is mean square stable. This more clearly illustrates the effectiveness of

Table 1
Performance indexes with and without FTC.

F	$\overline{F}$	$\underline{F}$	$\hat{F}$	$\check{F}$	D	K	$\mathcal{L}$	P	γ^*	$\gamma^\dagger$
0.4	0.41	0.37	0.39	0.05	0.03	0.48	−2.22	1.12	0.72	1.7
0.7	0.72	0.68	0.7	0.03	0	0.93	−2.36	0.95	0.5	1.65
0.9	0.93	0.85	0.89	0.04	0.01	1.2	−2.4	0.82	0.47	1.72

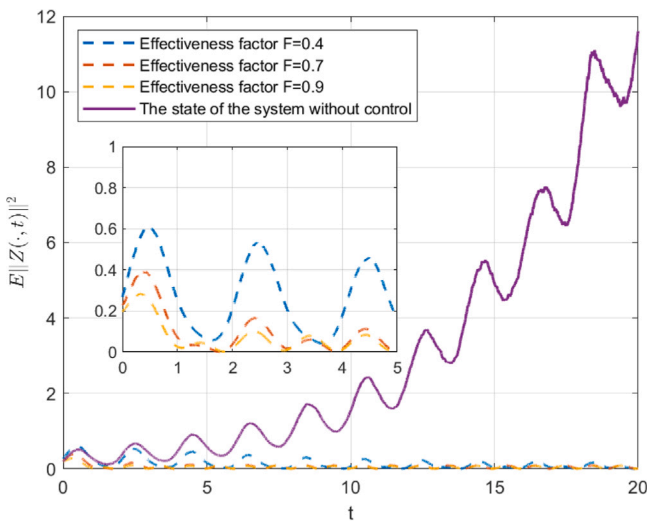

Fig. 2. The norms of system states with and without FTC.

the fault-tolerant controller. But it is worth noting that there are still few methods to deal with multiplicative failure of the actuator. The designed controller is still sensitive to faults, so the robustness results we obtained could be further optimized in the future.

5. Conclusion

This paper studies the observer-based H_∞ fault-tolerant control with actuator faults for a type of stochastic parabolic system. The proposed scheme compensates for the effects of the system under partial actuator failures and external disturbances. The obtained sufficient conditions can be expressed as a set of solutions to LMI. Finally, the control scheme is applied to verify the thermal fault robustness of a CPU chip.

CRediT authorship contribution statement

Yunzhu Wang: Writing – original draft, Formal analysis. **Kai-Ning Wu:** Supervision, Methodology, Conceptualization. **Yongxin Wu:** Writing – review & editing.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] X. Jin, Adaptive fault tolerant tracking control for a class of stochastic nonlinear systems with output constraint and actuator faults, Systems Control Lett. 107 (2017) 100–109, http://dx.doi.org/10.1016/j.sysconle.2017.07.007.

[2] H. Yang, M. Staroswiecki, B. Jiang, J.Y. Liu, Fault tolerant: cooperative control for a class of nonlinear multi-agent systems, Systems Control Lett. 60 (4) (2011) 271–277.

[3] H.J. Ma, G.H. Yang, Adaptive fault tolerant control of cooperative heterogeneous systems with actuator faults and unreliable interconnections, IEEE Trans. Autom. Control 61 (11) (2016) 3240–3255.

[4] H.Y. Li, H.J. Gao, P. Shi, X.D. Zhao, Fault-tolerant control of Markovian jump stochastic systems via the augmented sliding mode observer approach, Automatica 50 (2014) 1825–1834.

[5] S. Wang, D. Dong, Fault-tolerant control of linear quantum stochastic systems, IEEE Trans. Autom. Control 62 (6) (2017) 2929–2935.

[6] Z. Zhao, Z. Liu, W. He, K. Hong, H. Li, Boundary adaptive fault-tolerant control for a flexible Timoshenko arm with backlash-like hysteresis, Automatica 130 (2021) 109690.

[7] K. Wu, Y. Wang, Z. Wang, Spatial sampled–data control for stochastic reaction–diffusion systems, J. Franklin Inst. 357 (17) (2020) 12538–12554.

[8] X. Liu, K. Wu, W. Zhang, Intermittent boundary stabilization of stochastic reaction–diffusion Cohen–Grossberg neural networks, Neural Netw. 131 (2020) 1–13.

[9] X. Han, K. Wu, X. Ding, B. Yang, Boundary control of stochastic reaction-diffusion systems with Markovian switching, Internat. J. Robust Nonlinear Control 30 (2020) 4129–4148.

[10] J. Qiu, S. Ding, H. Gao, S. Yin, Fuzzy-model-based reliable static output feedback H-infinity control of nonlinear hyperbolic PDE systems, IEEE Trans. Fuzzy Syst. 24 (2) (2016) 388–400.

[11] X. Zhang, H. Wu, H infinity boundary control for a class of nonlinear stochastic parabolic distributed parameter systems, Internat. J. Robust Nonlinear Control 29 (2019) 4665–4680.

[12] X. Song, M. Wang, S. Song, Z. Wang, Observer–based sliding mode control for stochastic hyperbolic PDE systems with quantized output signal, Appl. Math. Comput. 393 (125643) (2021) http://dx.doi.org/10.1016/j.amc.2020.125643.

[13] R. Abinandhitha, S. Rathinasamy, F. Kong, A. Parivallal, Robust non-fragile boundary control for non-linear parabolic PDE systems with semi-Markov switching and input quantization, Eur. J. Control 69 (9) (2022) 100713, http://dx.doi.org/10.1016/j.ejcon.2022.100713.

[14] S. Aihara, On adaptive boundary control for stochastic parabolic systems with unknown potential coefficient, IEEE Trans. Autom. Control 42 (3) (1997) 350–363.

[15] Z. Tan, Y.J. Wang, Global existence and large-time behavior of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force, Nonlinear Anal.- Theory Methods Appl. 71 (11) (2009) 5866–5884.

[16] A. Hamdi, Identification of a time-varying point source in a system of two coupled linear diffusion-advection-reaction equations: application to surface water pollution, Inverse Problems 25 (11) (2009) 115009.

[17] M.B. Cheng, V. Radisavljevic, C.C. Chang, C.F. Lin, W.C. Su, Sampled-data singularly perturbed boundary control for a heat conduction system with noncollocated observation, IEEE Trans. Autom. Control 54 (6) (2009) 1305–1310.

[18] K. Wu, X. Liu, P. Shi, C.C. Lim, Boundary control of linear stochastic reaction-diffusion systems, Internat. J. Robust Nonlinear Control 29 (1) (2019) 268–282.

[19] W. Mu, J. Wang, W. Feng, Fault detection and fault-tolerant control of actuators and sensors in distributed parameter systems, J. Franklin Inst. 354 (8) (2017) 3341–3363.

[20] R. Sakthivel, S. Selvi, K. Mathiyalagan, Fault-tolerant sampled-data control of flexible spacecraft with probabilistic time delays, Nonlinear Dynam. 79 (2015) 1835–1846.

[21] W. Liu, Well–posedness of stochastic partial differential equations with Lyapunov condition, J. Differential Equations 255 (2013) 572–592.

[22] W.D. Kammler, A First Course in Fourier Analysis, Cambridge University Press, Cambridge, 2007.

[23] P.P. Khargonekar, I.R. Petersen, K.M. Zhou, Robust stabilization of uncertain linear-systems-quadratic stabilizability and H infinity-control theory, IEEE Trans. Autom. Control 35 (3) (1990) 356–361.

[24] X. Han, K. Wu, X. Ding, B. Yang, Boundary control of stochastic reaction-diffusion systems with Markovian switching, Internat. J. Robust Nonlinear Control 30 (10) (2020) 4129–4148.

[25] Y. Wang, J. Liu, Identification of non–smooth boundary heat dissipation by partial boundary data, Appl. Math. Lett. 69 (2017) 42–48.

[26] H. Jang, C. Cheng, Nonlinear optimal on-line heat-dissipation control methodology in electronic devices, Int. J. Heat Mass Transfer 52 (2009) 2049–2058.