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Boundary stabilization of random reaction-diffusion systems

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Abstract The boundary stabilization of a class of reaction-diffusion systems perturbed by second-order processes is investigated in this work. It extends the results from random ordinary differential equations to random reaction-diffusion systems (RRDSs). First, the stability analysis of RRDSs with boundary input is presented. Using the Lyapunov method and stochastic process estimation, two criteria of asymptotic stability are established in 2-nd moment and in probability, by applying Wirtinger's inequality and the weak law of large numbers. Second, based on the obtained stability criteria, a class of boundary control problems is solved by constructing a Lyapunov functional and designing integral boundary controllers. Additionally, the influence of nonlinear terms and the diffusion coefficient on stability is analyzed. Finally, numerical simulations demonstrate the effectiveness of the boundary controller.

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1 Introduction

In many branches of engineering and science, systems are often disturbed by stochastic noise from the external environment. Such systems are typically represented by the stochastic differential equations (SDEs) $\mathrm{d}x = f(x,t)\mathrm{d}t + g(x,t)\mathrm{d}B(t)$ driven by Brownian motion (BM). The statistical characteristics of BM are extensively employed in mathematical analysis. Based on the well known *Itô* formula, the fundamental theory of various types of stochastic systems has been introduced in [1, 17, 16]. SDEs have seen significant development, with numerous theoretical results have been reported in [28, 27, 9, 32]. However, it is worth noting that BM is not differential. In addition, the variance of BM is unbounded since its mean power is infinite. This leads SDEs seem not to be the most applicable model. In other words, not all environmental disturbances can be well modelled by white noise, deduced from the BM [12]. The disadvantage of SDEs has driven the work of [21] and [2] on random differential equations. In contrast to the stochastic system, the second-order processes $\xi(t)$ is introduced in the random differential system $\dot{x} = f(x,t) + g(x,t)\xi(t)$. Modelling with random systems in practical applications is more reasonable from the energy perspective than SDEs due to the finite mean power of $\xi(t)$. Recently, the research on random systems has attracted much attention [10, 8, 22, 11]. The existence-and-uniqueness of solutions have been investigated for non-linear random systems by Wu et al. in [29], and the fundamental theoretical results for the analysis of the system have been established. Based on this work, Zhang et al. extended the results to switching systems in [8] and applied the results to analyze the stationary process of a spring pendulum suspended from a randomly vibrating ceiling. All the aftermentioned works have been established for systems described by ordinary differential equations (ODEs).

Diffusion is an important phenomenon to present many applications in the real world, and the reaction-diffusion systems (RDSs), as a class of partial differential equations (PDEs), are suitable to describe the diffusion effect. RDSs have a widely application in practical engineering, such as the internal temperature of the lithium battery is modelled by RDSs [31]; RDSs are also used to describe the heat dissipation of the CPU [3]. Different results have been reported for stochastic RDSs in [4,15,13]. However, real applications are often disturbed by energy-limited noise, therefore it is of engineering interest to investigate the random reaction-diffusion systems (RRDSs) with coloured noise.

In control theory, analyzing stability is a crucial and foundational task. In [29], Wu investigated several types of stability for random systems and established the theoretical framework for Lyapunov stability analysis. The stability analysis of RDSs is also well developed with the help of Lyapunov's second method. The impulsive RDSs is considered in [24]. Han in [5] proposed sufficient conditions to ensure that RDNNs are stochastic finite-time bounded. With the development of modern control theory, different control strategies has been investigated. Reaction-diffusion systems typically employ two fundamental control strategies: distributed control and boundary control. However, in some environments of the applications, such as high internal environmental temperatures (inside a boiler) or limited component size (inside a CPU or battery), implementing distributed actuations at every point in the spatial domain is nearly unfeasible. Thus boundary control seems to be a more efficient and less costly control strategy. The boundary control theory of reaction-diffusion systems has been developed in recent years [14,26,7]. Nevertheless, there has been limited focus on RDSs influenced by colored noise. It is interesting to study the RRDSs, while the following challenges should be investigated.

I Existing Lyapunov stability results for ODEs are not directly applicable to RRDSs. The inclusion of spatial diffusion terms and boundary control introduces challenges in extending these theories. Consequently, the stability analysis of RRDSs remains an open and challenging problem.

II The stability of stochastic differential systems driven by BM was explored in [16] using the *Itô* formula and stopping time theory. For RRDSs with colored noise, foundational tools such as the *Itô* formula are no longer applicable. This necessitates the development of alternative analytical methods, making the stability analysis of RRDSs a complex task.

III Applying the results to boundary control presents several challenges. A key difficulty lies in determining the appropriate form of the boundary controller while ensuring it effectively integrates boundary information into the system analysis. Moreover, understanding how system parameters influence stability requires careful consideration, adding complexity to the overall process.

Based on the above discussion, this study develops a Lyapunov stability theoretical framework for RRDSs. The results are utilized in a specific type of boundary control problem. Additionally, the theoretical results are demonstrated through numerical examples. The key contributions of this research are outlined as follows.

I To handle diffusion terms and incorporate boundary information, this research develops new criteria for RRDSs (conditions (4)-(7) and (13)-(16)), differing from existing ODEs stability theories. By constructing an integral Lyapunov functional, it establishes conditions for exponential stability in the second moment and asymptotic stability in probability, offering a new theoretical foundation for RRDSs stability analysis.

II Unlike stochastic RDSs, this research leverages the weak law of large numbers (WLLN) and inequality techniques to develop a probability-based criterion for ensuring the asymptotic stability of RRDSs. This approach offers a novel framework for analyzing the asymptotic stability of random partial differential systems.

III Applying the obtained stability results, a class of boundary stabilization problem is investigated by designing an integral boundary controller. Furthermore, these results facilitate the investigation of how Lipschitz constants and diffusion coefficients impact the stability of RRDSs.

Notations: The symbol $|\cdot|$ denotes the Euclidean norm. $\mathbb{R}^{n\times m}$ is used to denote the set of real matrices with dimensions $n \times m$. The unit matrix of size $n \times n$ is denoted by I_n, while the "$\top$" indicates the transpose of a vector or matrix. For a matrix Z, "Z^{-1}" denotes the inverse of Z. The function $\mathrm{sym}(Z)$ is defined as $Z^{\mathrm{T}} + Z$, and the symbol $\lambda_{\max}(Z)$ stands for the maximum eigenvalue of the symmetric matrix Z. The expression $Z < 0$ (or ≤ 0) indicates that Z is a real symmetric matrix that is negative definite (or negative semi-definite). The space $L^2([0,1];\mathbb{R}^n)$ represents the Hilbert space of square-integrable vector functions $\theta(x)$, where the L^2-norm is given by $\|\theta(\cdot)\|^2 = \int_0^1 \theta(x)^{\mathrm{T}}\theta(x)\mathrm{d}x$. $\mathrm{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}+;\mathbb{R}+)$ denotes the set of non-negative functions $V(y,t)$ on $\mathbb{R}^n \times \mathbb{R}+$, which are continuously differentiable to the 2-nd order with respect to y and differentiable to the 1-st order with respect to t. Lastly, $W^{1,2}([0,1];\mathbb{R}^n)$ refers to the Sobolev space of vector functions $\zeta(x) : [0,1] \to \mathbb{R}^n$ that are absolutely continuous, with derivatives $\frac{\mathrm{d}^l y(x)}{\mathrm{d}x^l}$ that are square integrable up to order l.

2 Preliminaries

In this work, we consider a non-linear RRDS described by

$$\frac{\partial y(x,t)}{\partial t} = f(y(x,t)) + A\frac{\partial^2 y(x,t)}{\partial x^2} + g(y(x,t))\xi(t), \quad (1)$$

where $\xi(t) \in \mathbb{R}^m$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ that meets the usual conditions, represents a stochastic process (SP) rather than white noise. $y(x,t) = [y_1(x,t), \ldots, y_n(x,t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state of the system. A represent positive definite diffusion matrix. $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are non-linear functions.

The initial value function for the RRDS (1) is specified as follows

$$y(x,0) = \varphi(x), \quad (2)$$

where $\varphi \in L^2([0,1];\mathbb{R}^n)$.

The boundary conditions (Neumann) for the RRDS (1) are defined as follows

$$\frac{\partial y(x,t)}{\partial x}\Big|_{x=0} = 0, \qquad \frac{\partial y(x,t)}{\partial x}\Big|_{x=1} = \mu(t), \quad (3)$$

where $\mu(t)$ is the known function.

Assumption 1 *For non-linear functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, there exist positive constants L_1 and L_2 such that the following inequalities are satisfied for any $\rho \in \mathbb{R}^n$*

$$f^{\mathrm{T}}(\rho)f(\rho) \le L_1\rho^{\mathrm{T}}\rho,$$
$$\lambda_{\max}(g^{\mathrm{T}}(\rho)g(\rho)) \le L_2\rho^{\mathrm{T}}\rho.$$

Assumption 2 ([29]) *The process $\xi(t)$ is adapted to the filtration $\mathcal{F}_t$ and is piecewise continuous. Moreover, there is a constant $M > 0$ such that*

$$\sup_{t\ge 0}\mathbb{E}|\xi(t)|^2 < M.$$

Remark 1 Assumption 1 ensures that f and g satisfy Lipschitz condition. The Lemma 3 in [29] follows that $|\xi(t)| < \infty$ almost surely for $\forall t \in [0,T]$ when T is determined. In fact, the term $g(y(x,t))\xi(t)$ in the system (1) can be considered to also satisfy the Lipschitz condition. Combining the results of Chapter 8 in [20], there exists a unique classical solution for the system (1) on $[0,T]$. Moreover, since $T \ge 0$ is arbitrary, this result is valid for the entire time interval.

Remark 2 It should be noted that Assumption 2 is a standard assumption, which can be inferred from the results in [29] that both widely stationary and strictly stationary processes satisfy $\sup_{t\ge 0}\mathbb{E}|\xi(t)|^2 < M$. From physically point of view, energy of random disturbances in nature is generally finite, then physical feasibility requires that the mean power of $\xi(t)$ is bounded, which implies Assumption 2. In more details, Wu et al. in [30] explained the reasonableness of Assumption 2 by mathematical theoretical analysis and physical examples, respectively.

To guarantee $y(x,t) \equiv 0$ being the equilibrium, let the functions f and g vanish at the origin, i.e. $f(0) = 0, g(0) = 0$. The following sets of functions are specified before the definition is presented.
$\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \big|\ \gamma(0) = 0,\ \gamma$ is continuous, strictly increasing$\}$;
$\mathcal{KL} = \{\beta : \mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+ \big|\ \beta(\cdot,t) \in \mathcal{K}, \forall t \in \mathbb{R}_+$ and $\beta(s,t)$ strictly decreases to 0 as $t \to +\infty$ for $\forall s \ge 0\}$;
$\mathcal{K}_\infty = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \big|\ \gamma \in \mathcal{K}$ and γ is unbounded$\}$.

Definitions of stability for SDEs are provided in [1, 18, 16]. Recently, Wu introduced some stability definitions for random systems in [29], which are now adapted for RRDSs in this paper.

Definition 1 If there exists parameters $k_1, k_2 > 0$ such that the following holds for $\forall t \ge 0$

$$\mathbb{E}\|y(\cdot,t)\|^2 \le k_1\|\varphi(\cdot)\|^2 e^{-k_2 t},$$

then, we say that RRDS (1) is exponentially stable in 2-nd moment (ES-2-M).

In the sequel, we use ES-2-M for both exponentially stable in 2-nd moment and exponential stability in 2-nd moment when there is no confusion arises.

Definition 2 For any $\varepsilon > 0$, if there exists a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that the following holds for $\forall t \ge 0$

$$P\left\{\|y(\cdot,t)\|^2 \le \gamma\left(\|\varphi(\cdot)\|^2\right)\right\} \ge 1 - \varepsilon,$$

then, we say that RRDS (1) is stable in probability.

Definition 3 For any $\varepsilon > 0$, if exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and $T > 0$ such that the following holds for $\forall t \geq T$

$$P\left\{\|y(\cdot,t)\|^2 \leq \beta(\|\varphi(\cdot)\|^2, t)\right\} \geq 1-\varepsilon,$$

then, we say that RRDS (1) is attractive in probability.

For any $\varepsilon > 0$, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that the following holds for $\forall t \geq 0$

$$P\left\{\|y(\cdot,t)\|^2 \leq \beta(\|\varphi(\cdot)\|^2, t)\right\} \geq 1-\varepsilon,$$

then, we say that RRDS (1) is asymptotically stable in probability (AS-P).

In the sequel, we use AS-P for both asymptotically stable in probability and asymptotic stability in probability when there is no confusion arises.

Lemma 1 (Wirtinger's inequality [25]) *Consider a vector function $\theta \in W^{1,2}([0,1];\mathbb{R}^n)$ such that $\theta(0) = 0$ or $\theta(1) = 0$. For a matrix $\Gamma > 0$, the following holds*

$$\int_0^1 \theta^{\mathrm{T}}(s)\Gamma\theta(s)\mathrm{d}s \leq \frac{4}{\pi^2}\int_0^1 \left(\frac{\mathrm{d}\,\theta(s)}{\mathrm{d}s}\right)^{\mathrm{T}} \Gamma \left(\frac{\mathrm{d}\,\theta(s)}{\mathrm{d}s}\right)\mathrm{d}s.$$

3 Stability analysis of RRDSs

In the following analysis, we will investigate the stability for RRDSs both in 2-nd moment and in probability. By utilizing the Lyapunov functional method, the WLLN and inequality techniques, a theoretical framework on stability of RRDSs will be established preliminarily.

3.1 Exponential stability in moment

In the first part, we will consider the exponential stability in moment sense of system (1). To derive the key results, we impose the following assumption on the SP $\xi(t)$.

Assumption 3 ([29]) *For a stationary process $\xi(t)$, there exists a function $\varsigma(\cdot)$ such that for any $\sigma > 0$ and $t_1 \geq 0$, the following inequality holds:*

$$\mathbb{E}\exp\left\{\sigma\int_0^{t_1}|\xi(s)|\mathrm{d}s\right\} \leq \exp\left\{\varsigma(\sigma)t_1\right\}.$$

Remark 3 The reasonability of Assumption 3 has been stated in Section VI of [29], and the fact that a stationary Gaussian process satisfies this estimate has been determined.

In the sequel, we suppress the variables (x,t) wherever no confusion arises.

Theorem 4 *For the system (1), under Assumptions 1-3, assume there exist functions $W \in \mathrm{C}^{2,1}(\mathbb{R}^n\times\mathbb{R}+;\mathbb{R}+)$, a non-negative function $\varrho(\cdot) \geq 0$, and positive constants m_1, m_2, and c such that*

$$\mathrm{trace}\left(\frac{1}{2}\left(W_{yy}^{\mathrm{T}}A + A^{\mathrm{T}}W_{yy}\right)\right) \geq \varrho(t), \tag{4}$$

$$|W_y(y,t)g(y)| \leq cW(y,t), \tag{5}$$

$$\int_0^1 \Psi\mathrm{d}x \leq \int_0^1 -\varsigma(c)W(y,t)\mathrm{d}x, \tag{6}$$

$$m_1\|y(\cdot,t)\|^2 \leq \int_0^1 W(y,t)\mathrm{d}x \leq m_2\|y(\cdot,t)\|^2, \tag{7}$$

where

$$\begin{aligned}\Psi =& W_t(y,t) + W_y(y,t)f(y) + W_y(y(1,t),t)A\mu(t)\\ &- \frac{\pi^2\varrho(t)}{4}(y-y(1,t))^{\mathrm{T}}(y-y(1,t)).\end{aligned}$$

Then system (1) achieves ES-2-M.

Proof. Consider the integral Lyapunov functional given by

$$V(t) = \int_0^1 W(y,t)\mathrm{d}x.$$

To find its derivative along RRDS (1)

$$\begin{aligned}\dot{V}(t) =& \int_0^1 W_t(y,t) + W_y(y,t)y_t\mathrm{d}x\\ =& \int_0^1 W_t(y,t) + W_y(y,t)(f(y) + Ay_{xx}\\ &+ g(y)\xi(t))\mathrm{d}x\\ =& \int_0^1 W_t(y,t) + W_y(y,t)f(y) + W_y(y,t)Ay_{xx}\\ &+ W_y(y,t)g(y)\xi(t)\mathrm{d}x.\end{aligned} \tag{8}$$

By applying integration by parts and using condition (4), it becomes evident that

$$\begin{aligned}&\int_0^1 W_y(y,t)Ay_{xx}\mathrm{d}x\\ =& W_y(y,t)Ay_x\Big|_{x=0}^{x=1} - \int_0^1 (W_y(y,t))_x Ay_x\mathrm{d}x\\ =& W_y(y(1,t),t)A\mu(t) - \int_0^1 (W_y(y,t))_x Ay_x\mathrm{d}x\\ =& W_y(y(1,t),t)A\mu(t) - \int_0^1 y_x^{\mathrm{T}}W_{yy}(y,t)Ay_x\mathrm{d}x\\ =& W_y(y(1,t),t)A\mu(t) - \int_0^1 \mathrm{trace}\left(\frac{1}{2}\left(W_{yy}^{\mathrm{T}}A + A^{\mathrm{T}}W_{yy}\right)\right)\\ & y_x^{\mathrm{T}}y_x\mathrm{d}x\\ \leq& W_y(y(1,t),t)A\mu(t) - \varrho(t)\int_0^1 y_x^{\mathrm{T}}y_x\mathrm{d}x.\end{aligned}$$

Define $\bar{y}(x,t) = y(x,t) - y(1,t)$. It is straightforward to show that $\bar{y}(1,t) = 0$ and $y_x^{\mathrm{T}} y_x = \bar{y}_x^{\mathrm{T}} \bar{y}_x$. Furthermore, it follows Lemma 1 that we get

$$
\begin{aligned}
&\int_0^1 W_y(y,t) A y_{xx} \mathrm{d}x \\
=&W_y(y(1,t),t) A\mu(t) - \varrho(t) \int_0^1 \bar{y}_x^{\mathrm{T}} \bar{y}_x \mathrm{d}x. \\
\leq&W_y(y(1,t),t) A\mu(t) - \frac{\pi^2 \varrho(t)}{4} \int_0^1 (y - y(1,t))^{\mathrm{T}} \\
&\times (y - y(1,t)) \mathrm{d}x.
\end{aligned}
\tag{9}
$$

Substituting (9) into (8) and referring to conditions (5) - (6) yield the following

$$
\begin{aligned}
&\dot{V}(t) \\
\leq& \int_0^1 W_t(y,t) + W_y(y,t) f(y) + W_y(y(1,t),t) \\
&\times A\mu(t) - \frac{\pi^2 \varrho(t)}{4} (y - y(1,t))^{\mathrm{T}} (y - y(1,t)) \\
&+ W_y(y,t) g(y) \xi(t) \mathrm{d}x \\
\leq& \int_0^1 -\lambda W(y,t) + c W(y,t) |\xi(t)| \mathrm{d}x \\
=&(-\lambda + c|\xi(t)|) V(t),
\end{aligned}
\tag{10}
$$

where λ is a positive constant such that $\int_0^1 \Psi \mathrm{d}x \leq \int_0^1 -\lambda W(y,t) \mathrm{d}x \leq \int_0^1 -\varsigma(c) W(y,t) \mathrm{d}x$.

From (10) and (7), we arrive at

$$
\begin{aligned}
&m_1 \|y(\cdot,t)\|^2 \\
\leq& V(t) \leq V(0) \exp\left\{ \int_0^t (-\lambda + c|\xi(s)|) \mathrm{d}s \right\} \\
\leq& m_2 \|y(\cdot,0)\|^2 e^{-\lambda t} \exp\left\{ c \int_0^t |\xi(s)| \mathrm{d}s \right\}.
\end{aligned}
\tag{11}
$$

Taking expectations on both sides of (11), we have

$$
\mathbb{E}\|y(\cdot,t)\|^2 \leq \frac{m_2}{m_1} \|\varphi(\cdot)\|^2 e^{-\lambda t} \mathbb{E} \exp\left\{ c \int_0^t |\xi(s)| \mathrm{d}s \right\}.
$$

By adopting Assumption 3, it is followed that

$$
\mathbb{E}\|y(\cdot,t)\|^2 \leq \frac{m_2}{m_1} \|\varphi(\cdot)\|^2 \exp\left[-(\lambda - \varsigma(c)) t \right]. \tag{12}
$$

It is evident that system (1) can achieve ES-2-M.

As of now, the exponential stability in the sense of moment was discussed under the Assumption 3. It is known that stability in 2-nd moment guarantees stability in probability, that is, the latter requires a much weaker condition. Next, we will consider the stability in probability without Assumption 3.

3.2 Asymptotic stability in probability

In this part, by virtue of the WLLN, a sufficient condition to ensure the AS-P for system (1) will be established. At first, we state the WLLN.

Assumption 5 ([29]) *The SP $\xi(t)$ satisfies the WLLN, that is, for any $\varepsilon > 0, \kappa > 0$, there exists a $T > 0$ such that for all $t \geq T$*

$$
P\left\{ \left| \frac{1}{t} \int_0^t |\xi(s)|^2 \mathrm{d}s - \mathbb{E}|\xi(t)|^2 \right| \geq \kappa \right\} \leq \varepsilon.
$$

Remark 4 Based on the results of [19,23], we recognize that the mean-ergodic widely stationary process, the variance-ergodic widely stationary process, and the ergodic strictly stationary process all obey the WLLN. More details on reasonability of Assumption 5 has been provided in Section VI of [29].

Theorem 6 *For system (1), under Assumptions 1-2 and Assumption 5, assume that there exist functions $W \in \mathrm{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\varrho(\cdot) \geq 0$, $\mathcal{K}_\infty$ functions $\underline{\chi}$, $\bar{\chi}$ and positive constant c such that*

$$
\operatorname{trace}\left(\frac{1}{2} \left(W_{yy}^{\mathrm{T}} A + A^{\mathrm{T}} W_{yy} \right) \right) \geq \varrho(t), \tag{13}
$$

$$
|W_y(y,t) g(y)| \leq c W(y,t), \tag{14}
$$

$$
\int_0^1 \Psi \mathrm{d}x \leq \int_0^1 -2c\sqrt{M} W(y,t) \mathrm{d}x, \tag{15}
$$

$$
\underline{\chi}(\|y(\cdot,t)\|^2) \leq \int_0^1 W(y,t) \mathrm{d}x \leq \bar{\chi}(\|y(\cdot,t)\|^2), \tag{16}
$$

where

$$
\begin{aligned}
\Psi =& W_t(y,t) + W_y(y,t) f(y) + W_y(y(1,t),t) A\mu(t) \\
&- \frac{\pi^2 \varrho(t)}{4} (y - y(1,t))^{\mathrm{T}} (y - y(1,t)).
\end{aligned}
$$

Then system (1) achieves AS-P.

Proof. Following Definition 3, we will demonstrate asymptotic stability by establishing both the attraction and stability of the system.

To achieve this, we will employ a Lyapunov functional with the same form as the one in Theorem 4

$$
V(t) = \int_0^1 W(y,t) \mathrm{d}x.
$$

Then, following the line of the proof for Theorem 4 and in light of condition (15), we can obtain that

$$
\begin{aligned}
\dot{V}(t) \leq& \int_0^1 W_t(y,t) + W_y(y,t) f(y) W_y(y(1,t),t) A\mu(t) \\
&- \frac{\pi^2}{4} \varrho(t) (y - y(1,t))^{\mathrm{T}} (y - y(1,t)) \\
&+ W_y(y,t) g(y) \xi(t) \mathrm{d}x \\
=&(-\lambda + c|\xi(t)|) V(t),
\end{aligned}
$$

where λ is positive constant such that $\int_0^1 \Psi \mathrm{d}x \leq \int_0^1 -\lambda W(y,t)\mathrm{d}x \leq \int_0^1 -2c\sqrt{M}W(y,t)\mathrm{d}x$.

It can be deduced that

$$\begin{aligned} V(t) \leq & V(0)\exp\left\{\int_0^t (-\lambda + c|\xi(s)|)\mathrm{d}s\right\} \\ = & V(0)e^{-\lambda t}\exp\left\{c\int_0^t |\xi(s)|\mathrm{d}s\right\}. \end{aligned} \tag{17}$$

Claim 1 (Attraction): By specifying

$$\mathcal{W} = \left\{\left|\frac{1}{t}\int_0^t |\xi(s)|^2\mathrm{d}s - \mathbb{E}|\xi(t)|^2\right| \leq \kappa\right\},$$

for each $\varepsilon > 0$ and $\kappa \in (0, 3M)$. In combination with Assumption 5, we get that there exists a $T > 0$ such that $P\{\mathcal{W}\} \geq 1 - \varepsilon$ for all $t \geq T$.

Together with $\sup_{t\geq 0}\mathbb{E}|\xi(t)|^2 < M$, it implies

$$\int_0^t |\xi(s)|^2\mathrm{d}s \leq (\mathbb{E}|\xi(t)|^2 + \kappa)t \leq 4Mt.$$

The above inequality yields the following

$$\int_0^t |\xi(s)|\mathrm{d}s \leq \sqrt{t\left(\int_0^t |\xi(s)|^2\mathrm{d}s\right)} \leq 2\sqrt{M}t, \tag{18}$$

for all $\omega \in \mathcal{W},\ t \geq T$.

Substituting (18) into (17), we have

$$V(t) \leq V(0)\exp\left\{\left(-\lambda + 2c\sqrt{M}\right)t\right\}.$$

In accordance with conditions (16) and $P\{\mathcal{W}\} \geq 1 - \varepsilon$, the following formula is valid, for $t \geq T$,

$$P\left\{\|y(\cdot,t)\|^2 \leq \frac{\bar{\chi}(\|\varphi(\cdot)\|^2)}{\underline{\chi}}e^{\left(-\lambda+2c\sqrt{M}\right)t}\right\} \geq 1 - \varepsilon. \tag{19}$$

So far, the proof of attraction is complete.

Claim 2 (Stability): By employing Chebyshev's inequality and considering Assumption 2, we have that for every ε, there exists a $\kappa_0 > 0$ such that

$$P\{|\xi(t)| > \kappa_0\} \leq \mathbb{E}|\xi(t)|^2/\kappa_0^2 < M/\kappa_0^2 = \varepsilon, \quad t \geq 0.$$

Together with (17)

$$P\{V(t) \leq V(0)\exp\{(-\lambda + c\kappa_0)T\}\} \geq 1 - \varepsilon,\ t \leq T,$$

which means that, for $t \leq T$,

$$P\left\{\|y(\cdot,t)\|^2 \leq \frac{\bar{\chi}(\|\varphi(\cdot)\|^2)}{\underline{\chi}}e^{(-\lambda+c\kappa_0)T}\right\} \geq 1 - \varepsilon. \tag{20}$$

It's convenient to get the following from (19) and (20)

$$P\{\|y(\cdot,t)\|^2 \leq \gamma(\|\varphi(\cdot)\|^2)\} \geq 1 - \varepsilon, \quad t \geq 0, \tag{21}$$

where

$$\begin{aligned} \gamma(\|\varphi(\cdot)\|^2) = & \underline{\chi}^{-1}\left(\bar{\chi}(\|\varphi(\cdot)\|^2)\right)e^{(-\lambda+c\kappa_0)T} \\ & + \underline{\chi}^{-1}\left(\bar{\chi}(\|\varphi(\cdot)\|^2)\right)e^{\left(-\lambda+2c\sqrt{M}\right)T}. \end{aligned}$$

Definition 2 is obviously satisfied, thus Claim 2 is verified.

Combining (19) with (21) (attraction and stability) gives

$$P\{\|y(\cdot,t)\|^2 \leq \beta(\|\varphi(\cdot)\|^2, t)|\} \geq 1 - \varepsilon, \quad t \geq 0,$$

where $\beta(\cdot,\cdot)$ is a class-$\mathcal{KL}$ function. From Definition 3, it can be concluded that the system achieves AS-P. This completes the proof.

Remark 5 Section V in [29] have discussed global asymptotic stability, where $\underline{\alpha} \leq V(x) \leq \bar{\alpha}$ is equal to radially unbounded and $V_x f(x,t) \leq -c_1 V(x(t))$, $|V_x g(x,t)| \leq c_2 V(x(t))$ imply distinct criteria for nominal term and noise term. However, the object of investigation is ordinary differential random nonlinear systems, and the published results do not work for partial differential systems. To develop a framework for the stability analysis of RRDSs, the new sufficient conditions (4)-(7) and (13)-(16) are presented in Theorem 4 and Theorem 6.

In this section, we have established the Lyapunov stability theory for RRDSs, focusing primarily on the qualitative analysis of system behavior. Building on these stability results, we will now delve deeper into the boundary control problem of the system, which serves as the ultimate objective of this study.

4 Boundary control of RRDSs

In this section, boundary control problem for RRDSs will be considered to support our theoretical results. The boundary inputs $\mu(t)$ in Section 3 will be designed as a boundary controller in integral form and the sufficiency conditions for the system to achieve boundary stabilization will be provided.

Still considering system (1)

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = f(y(x,t)) + A\frac{\partial^2 y(x,t)}{\partial x^2} + g(y(x,t))\xi(t), \\ \left.\dfrac{\partial y(x,t)}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial y(x,t)}{\partial x}\right|_{x=1} = \mu(t), \\ y(x,0) = \varphi(x). \end{cases}$$

The boundary control strategy employed is described by

$$\mu(t) = U\int_0^1 y(x,t)\mathrm{d}x, \tag{22}$$

where $U \in \mathbb{R}^{n\times n}$ denotes the boundary control gain to be determined.

At first, as an application of Theorem 4, we consider the ES-2-M of RRDS (1) with boundary controller (22).

Theorem 7 *If there exists matrix U, positive constants ε and ε_1 such that the following holds*

$$\begin{pmatrix} \Phi_1 & -(AU)^{\mathrm{T}} \\ -AU & \Phi_2 \end{pmatrix} < 0, \tag{23}$$

where

$$\Phi_1 = \left[\varepsilon + \varepsilon^{-1}L_1 + \varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)\right] I_n + \mathrm{sym}(AU),$$

$$\Phi_2 = -\frac{\pi^2}{2}A + \varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right) I_n.$$

Then system (1) achieves ES-2-M under boundary controller (22).

Proof. The integral Lyapunov functional is selected in the form

$$V(t) = \int_0^1 W(y,t)\mathrm{d}x = \int_0^1 y^{\mathrm{T}}y\mathrm{d}x.$$

One seen that $W(y,t)$ satisfies (7). Then we arrive at

$$\mathrm{trace}\left(\frac{1}{2}\left(W_{yy}^{\mathrm{T}}A + A^{\mathrm{T}}W_{yy}\right)\right) = \mathrm{trace}\left(2A\right) > 0. \tag{24}$$

Taking $\varrho(t) = \mathrm{trace}\,(2A)$, then condition (4) holds. By virtue of Assumption 1 and inequality $2X^{\mathrm{T}}Y \le \varepsilon X^{\mathrm{T}}X + \varepsilon^{-1}Y^{\mathrm{T}}Y$, gives

$$\begin{aligned} W_y(y,t)g(y) =& 2y^{\mathrm{T}}g(y) \\ \le& \varepsilon_1 y^{\mathrm{T}}y + \varepsilon_1^{-1}L_2 y^{\mathrm{T}}y \\ =& \left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right) y^{\mathrm{T}}y. \end{aligned} \tag{25}$$

Based on the fact that $\bar{y} = y - y(1,t)$, we have

$$\begin{aligned} &\int_0^1 W_t(y,t) + W_y(y,t)f(y) + W_y(y(1,t),t)A\mu(t) \\ &- \frac{\pi^2\varrho(t)}{4}(y - y(1,t))^{\mathrm{T}}(y - y(1,t))\mathrm{d}x \\ \le& \int_0^1 2y^{\mathrm{T}}f(y) + 2[y(x,t) - \bar{y}(x,t)]^{\mathrm{T}}A\mu(t) - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}}A\bar{y}\mathrm{d}x. \end{aligned}$$

In light of the technology of (25), we obtain

$$2y^{\mathrm{T}}f(y) \le \varepsilon y^{\mathrm{T}}y + \varepsilon^{-1}L_1 y^{\mathrm{T}}y.$$

The boundary controller (22) indicates that

$$\begin{aligned} &\int_0^1 W_t(y,t) + W_y(y,t)f(y) + W_y(y(1,t),t)A\mu(t) \\ &- \frac{\pi^2\varrho(t)}{4}(y - y(1,t))^{\mathrm{T}}(y - y(1,t))\mathrm{d}x \\ \le& \int_0^1 \varepsilon y^{\mathrm{T}}y + \varepsilon^{-1}L_1 y^{\mathrm{T}}y + 2(y-\bar{y})^{\mathrm{T}}AUy - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}}A\bar{y}\mathrm{d}x \\ =& \int_0^1 \begin{pmatrix} y \\ \bar{y} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \hat{\Phi}_1 & -(AU)^{\mathrm{T}} \\ -AU & -\frac{\pi^2}{2}A \end{pmatrix} \begin{pmatrix} y \\ \bar{y} \end{pmatrix} \mathrm{d}x, \end{aligned}$$

where

$$\hat{\Phi}_1 = \left(\varepsilon + \varepsilon^{-1}L_1\right) I_n + \mathrm{sym}(AU).$$

Condition (23) implies that

$$\begin{pmatrix} \hat{\Phi}_1 & -(AU)^{\mathrm{T}} \\ -AU & -\frac{\pi^2}{2}A \end{pmatrix} < -\varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right) I_n. \tag{26}$$

Thus, we derive that

$$\begin{aligned} &\int_0^1 W_t(y,t) + W_y(y,t)f(y) + W_y(y(1,t),t)A\mu(t) \\ &- \frac{\pi^2\varrho(t)}{4}(y - y(1,t))^{\mathrm{T}}(y - y(1,t))\mathrm{d}x \\ \le& -\varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)\int_0^1 (y^{\mathrm{T}}y + \bar{y}^{\mathrm{T}}\bar{y})\mathrm{d}x \\ \le& -\varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)\int_0^1 (y^{\mathrm{T}}y)\mathrm{d}x. \end{aligned} \tag{27}$$

As a result of Theorem 4, it follows by (24) - (27) that system (1) is ES-2-M. The proof is complete.

Remark 6 According to condition (23) in Theorem 7, a larger minimum eigenvalue of A facilitates satisfying inequality (23). In other words, a higher diffusion coefficient is advantageous for the ES-2-M of system (1). We illustrate this fact in the numerical simulation.

Remark 7 From Theorem 4 and Theorem 7, it can be observed that the convergence rate of the mean square exponential stability can be controlled. Specifically, inequality (12) shows that the convergence rate is bounded by the function $e^{(-(\lambda-\varsigma(c))t)}$ Moreover, from (26) in Theorem 7, it is evident that by selecting an appropriate control gain U, the parameter $-\varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)$ can be adjusted, thereby ultimately controlling the convergence rate of the system state.

Secondly, asymptotic stabilization in probability sense will be discussed in the remainder.

Theorem 8 *If there exists matrix U, positive constants ε and ε_1 such that the following holds*

$$\begin{pmatrix} \Phi_3 & -(AU)^{\mathrm{T}} \\ -AU & \Phi_4 \end{pmatrix} < 0, \tag{28}$$

where

$$\Phi_3 = \left[\varepsilon + \varepsilon^{-1}L_1 + 2(\varepsilon_1 + \varepsilon_1^{-1}L_2)\sqrt{M}\right] I_n + \mathrm{sym}(AU),$$

$$\Phi_4 = -\frac{\pi^2}{2}A + \left[2(\varepsilon_1 + \varepsilon_1^{-1}L_2)\sqrt{M}\right] I_n.$$

Then system (1) achieves AS-P under boundary controller (22).

Proof. Still considering the Lyapunov functional

$$V(t)=\int_0^1 y^{\mathrm{T}}y\mathrm{d}x.$$

Following the line of the proof for Theorem 7, it can be shown that

$$\mathrm{trace}\Big(\frac{1}{2}\left(W_{yy}^{\mathrm{T}}A+A^{\mathrm{T}}W_{yy}\right)\Big)=\mathrm{trace}\,(2A)=\varrho(t)>0,$$

and

$$2y^{\mathrm{T}}g(y)\leq\left(\varepsilon_1+\varepsilon_1^{-1}L_2\right)y^{\mathrm{T}}y. \tag{29}$$

Condition (28) implies that

$$\begin{pmatrix}\hat{\Phi}_3 & -(AU)^{\mathrm{T}}\\ -AU & -\frac{\pi^2}{2}A\end{pmatrix}<\left(-2(\varepsilon_1+\varepsilon_1^{-1}L_2)\sqrt{M}\right)I_n,$$

where $\hat{\Phi}_3=\left(\varepsilon+\varepsilon^{-1}L_1\right)I_n+\mathrm{sym}(AU)$. Based on this fact, we derive that

$$\begin{aligned}&\int_0^1 W_t(y,t)+W_y(y,t)f(y)+W_y(y(1,t),t)A\mu(t)\\&-\frac{\pi^2\varrho(t)}{4}(y-y(1,t))^{\mathrm{T}}(y-y(1,t))\mathrm{d}x\\\leq&\int_0^1\begin{pmatrix}y\\ \bar{y}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\hat{\Phi}_3 & -(AU)^{\mathrm{T}}\\ -AU & -\frac{\pi^2}{2}A\end{pmatrix}\begin{pmatrix}y\\ \bar{y}\end{pmatrix}\mathrm{d}x\\\leq&-2(\varepsilon_1+\varepsilon_1^{-1}L_2)\sqrt{M}\int_0^1(y^{\mathrm{T}}y)\mathrm{d}x.\end{aligned}$$

In accordance with Theorem 6, it can be concluded that the system achieves AS-P.

Remark 8 From equation (29), we can deduce that matrix-valued function $g(y)$ is limited by L_2. Then it follows that the smaller L_2 is, the less $\xi(t)$ disturbs the system. In other words, smaller L_2 benefits to the AS-P for system (1). This fact is also evident in condition (28).

5 Numerical simulations

In the subsequent discussion, the validity of the obtained results is illustrated using the example of a temperature management system for cylindrical lithium-ion batteries.

In recent years, the rapid advancement of new energy vehicles has led to the widespread use of lithium-ion batteries. The stability of the temperature in these batteries is crucial for both economic and safety considerations. Research in [31,6] demonstrates that a reaction-diffusion system is effective for modeling the temperature behavior of lithium-ion batteries.

We consider a cylindrical lithium-ion battery cell as shown in Fig. 1. Assuming the same temperature on across circle of equal radius centered on the battery pole, hence it is sufficient to analyze the radial temperature change of the battery pole.

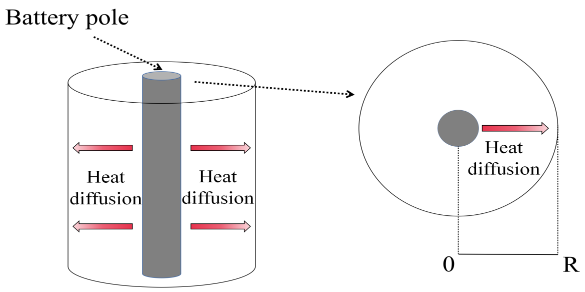


Fig. 1 Temperature diffusion of cylindrical lithium-ion battery

The forward Euler method and the central difference scheme are used to construct the numerical scheme.

Example 1 The temperature system of lithium-ion battery is described as

$$\frac{\partial T(x,t)}{\partial t}=f(T(x,t))+A\frac{\partial^2 T(x,t)}{\partial x^2}+g(T(x,t))\xi(t),\quad x\in(0,R),t>0, \tag{30}$$

where $T(x,t)\in\mathbb{R}^2$ is the radially distributed temperature inside the battery cell. $x=0$ denotes the centre of battery and $x=R$ denotes the surface of battery, where R is the radius of the cylindrical lithium battery. The function $f(T(x,t))\in\mathbb{R}^2$ represents the heat produced within the battery during its operation. Matrix A represents the diffusion coefficient, which describes the rate at which a material responds to temperature changes (A higher thermal diffusivity indicates a better thermal conductivity of the material). The SP $\xi(t)$ represents the power-limited coloured noise existing in the battery with $\mathbb{E}|\xi(t)|^2<M$.

$\xi(t)$ as a class of stationary Gaussian process will be produced by

$$\xi(t)=0.5\cos(1.2t+\aleph)sin(2\pi t)+0.3\mathcal{N}(t),$$

where the random variable $\aleph$ follows a uniform distribution over the interval $[0,2\pi]$, and $\mathcal{N}(t)$ represents Gaussian white noise with $\mathbb{E}\mathcal{N}(t)=0$ and variance 1. It is demonstrated in Fig. 2 that $\xi(t)$ satisfies $\mathbb{E}|\xi(t)|^2<1$. The stationarity of $\xi(t)$ as a Gaussian process is detailed in the Appendix.

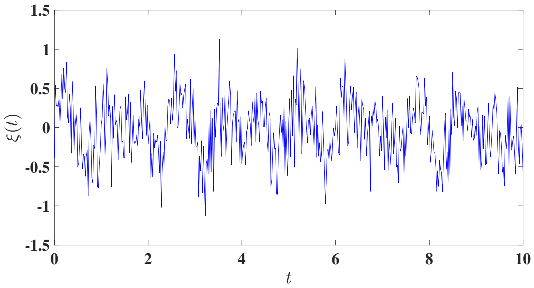


Fig. 2 Random process $\xi(t)$

The diffusion matrix is taken as $A = [0.3 \quad 0; 0 \quad 0.5]$. Two non-linear functions are

$$f(T(x,t)) = [0.7\sin(T_1(x,t)), 0.7\sin(T_2(x,t))]^\top,$$

$$g(T(x,t)) = [0.9\sin(T_1(x,t)), 0.9\sin(T_2(x,t))]^\top.$$

The initial conditions for RRDS (30) are specified as follows

$$T(x,0) = \begin{pmatrix} 0.8(1-\cos(3\pi x)) \\ 0.9(1-\cos(2\pi x)) \end{pmatrix}.$$

The boundary control strategy employed is described by

$$\begin{cases} \left.\dfrac{\partial T(x,t)}{\partial x}\right|_{x=0} = 0, \\ \left.\dfrac{\partial T(x,t)}{\partial x}\right|_{x=R} = U\int_0^R T(x,t)\mathrm{d}x. \end{cases}$$

Set $\varepsilon = 1.5$, $\varepsilon_1 = 0.8$ and $\varsigma(C) = 2C^3 + 3C^2$. A simple calculation shows that $L_1 = 0.49$, $L_2 = 0.81$ and $\varsigma\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right) = 2\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)^3 + 3\left(\varepsilon_1 + \varepsilon_1^{-1}L_2\right)^2 = 21.7642$. By solving inequality (23) in Theorem 7, one can obtain the boundary control gain

$$U = \begin{pmatrix} -4.5837 & 0 \\ 0 & -2.7502 \end{pmatrix}.$$

By implementing the control gain U into the RRDS controller (30), as illustrated in Fig. 3, it is evident that $\mathbb{E}\|T(\cdot,t)\|^2$ diminishes to 0, indicating that system (30) achieves ES-2-M. In other words, the controller can effectively stabilize the system, which is consistent with the result of Theorem 7.

Secondly, situations in probability sense will be discussed.

Still considering system (30) and maintaining the same parameters. $\xi(t)$ will be regenerated by

$$\hat{\xi}(t) = 0.9\cos(1.5t + \aleph)\sin(2\pi t),$$

where $\hat{\xi}(t)$, as a class of mean-ergodic widely stationary process, satisfies $\mathbb{E}|\hat{\xi}(t)|^2 < 1$, which is shown in Fig. 4.

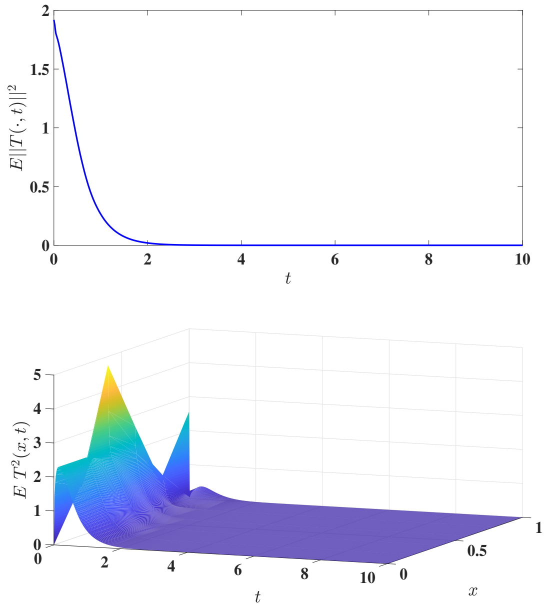


Fig. 3 Response of RRDS (30) with controll in moment

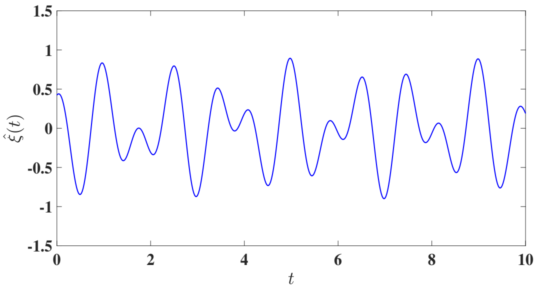


Fig. 4 Random process $\hat{\xi}(t)$

In the Appendix it is verified that the $\xi(t)$ and $\hat{\xi}(t)$ are our desired.

Since to $\varepsilon_1 = 0.8$, $L_2 = 0.81$ and $M = 1$, one gets that $2(\varepsilon_1 + \varepsilon_1^{-1}L_2)\sqrt{M} = 3.6250$. The following boundary control gain can be obtained by solving inequality (28),

$$\hat{U} = \begin{pmatrix} -4.9102 & 0 \\ 0 & -3.5111 \end{pmatrix}.$$

By employing the control gain $\hat{U}$, it follows from Theorem 8 that system (30) is AS-P. $\|T(\cdot,t)\|^2$ is shown in Fig. 5 and we can seen that all sample paths converge to 0, i.e., system (30) is AS-P, which agrees with the theoretical results. Therefore, the validity of Theorem 8 is also verified. (To show the spatio-temporal properties of the system, the $\mathbb{E}T^2(x,t)$ of the system (30) is also shown in the Fig. 5.)

Finally, in order to illustrate the effectiveness of the controller, Fig. 6 depicts the state response of system (30) when the boundary controller disappears (taking

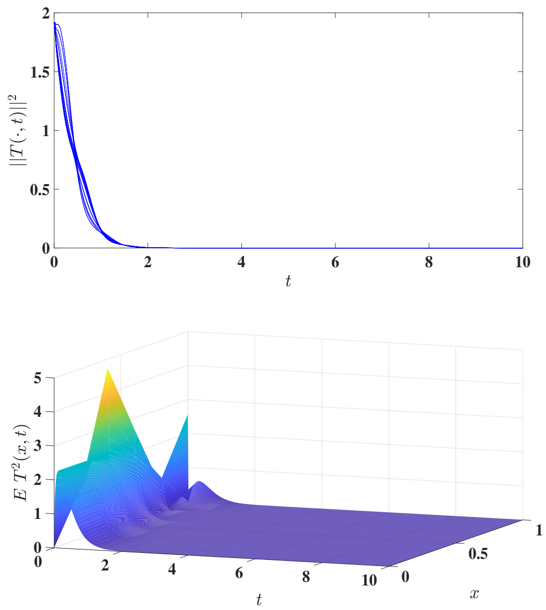
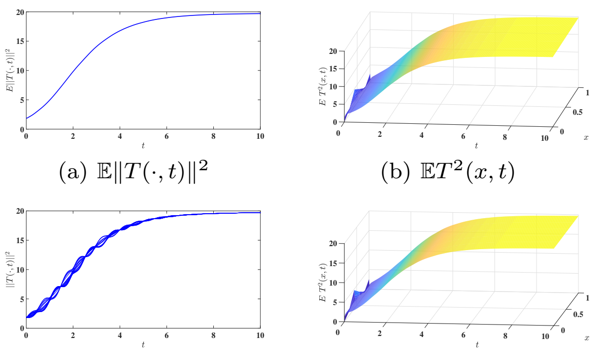


Fig. 5 Response of RRDS (30) with controll in probability

$U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$). It can be seen that neither $\mathbb{E}\|T(\cdot,t)\|^2$ (Fig. 6(a)) nor the sample paths (Fig. 6(c)) converge to 0, which means that the system cannot be stabilized without the controller. In other words, since heat is generated inside the battery during normal operation, the temperature of system cannot be stabilized when no control is applied under the Neumann boundary conditions.



(a) $\mathbb{E}\|T(\cdot,t)\|^2$ (b) $\mathbb{E}T^2(x,t)$

(c) $\|T(\cdot,t)\|^2$ (d) $\mathbb{E}T^2(x,t)$

Fig. 6 Response of RRDS (30) without controll

The performance of boundary controller (22) is verified in above example. Next, a simple example will be used to verify the claims in Remark 6.

Example 2 RRDS (30) with $\xi(t)$ is further analyzed using different values for the diffusion coefficients as detailed

$$A_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$

and other parameters are the same as in Example 1 . The state responses corresponding to different diffusion coefficients are shown in Fig. 7. (For a clearer comparison, only $\mathbb{E}\|T(\cdot,t)\|^2$ is depicted.)

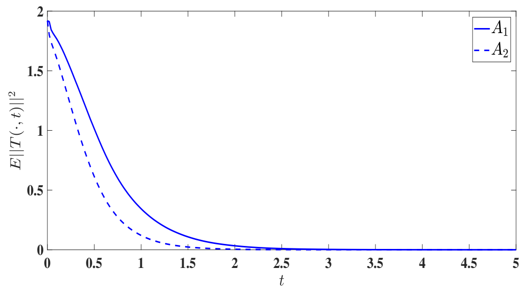


Fig. 7 Response of the RRDS under different diffusion coefficients

It is evident that the system state converges more rapidly with a larger minimum eigenvalue of the diffusion matrix, which verifies the statement in the Remark 6. Physically, better thermal conductivity of the battery material will improve controller performance.

6 Conclusion

In this paper, the asymptotic stability of random reaction-diffusion systems (RRDSs) and its applications to boundary control problems are investigated. This work addresses a gap in research on random systems within the field of partial differential equations (PDEs). Firstly, based on Lyapunov stability theory and statistical methods, sufficient criteria for the asymptotic stability of RRDSs are established in both the second moment and in probability. Next, as an application, the boundary control problem for RRDSs is addressed by designing an integral boundary controller and constructing a Lyapunov functional. Furthermore, the influence of parameters on stability is analyzed using the established criteria. Finally, the performance of the designed boundary controller and the validity of the theoretical results are confirmed through numerical examples involving a battery temperature management system.

In order to establish a more complete theoretical framework for the stability of RRDSs, some related issues, such as the noise-to-state stability of RRDSs, boundary control of impulsive RRDSs and switched RRDSs deserve further thorough investigation.

APPENDIX

In this appendix, we will show that the stochastic processes (SPs) modeled in Section 5 are stationary Gaussian process and mean-ergodic widely stationary process (WSP), respectively.

Proposition I. $\xi(t) = z\cos(ht+\Phi)$ is a mean-ergodic WSP, where z and h are constants, the stochastic variable Φ follows a uniformly distributed over the interval $[0, 2\pi]$.

Proof: It should first be noted that if the SP satisfies

$$\mathbb{E}[\xi(t)] = m(t) = m,$$
$$R_\xi[t_1, t_1] = R_{\xi(t)}(t_1 - t_2),$$
$$\mathbb{E}[\xi^2(t)] < \infty,$$

then $\xi(t)$ is said to be a WSP.

$\mathbb{E}\xi(t)$ and autocorrelation function of $\xi(t)$ are given by the following expressions

$$\begin{aligned}\mathbb{E}[\xi(t)] &= \mathbb{E}[z\cos(ht+\Phi)]\\ &= \int_{-\infty}^{\infty} z\cos(ht+\phi)f_\phi(\phi)\mathrm{d}\phi\\ &= \int_0^{2\pi} z\cos(ht+\phi)\frac{1}{2\pi}\mathrm{d}\phi = 0,\end{aligned}$$

$$\begin{aligned}R_{\xi(t)}(t, t+\tau) &= \mathbb{E}[z\cos(ht+\Phi)z\cos(ht+h\tau+\Phi)]\\ &= \frac{z^2}{2}\mathbb{E}[\cosh\tau + \cos(2ht+2\Phi)],\end{aligned}$$

where

$$\mathbb{E}[\cos(2ht+2\Phi)] = \int_0^{2\pi}\cos(2ht+2\phi)\frac{1}{2\pi}\mathrm{d}\phi = 0.$$

Consequently, we get

$$R_{\xi(t)}(t, t+\tau) = \frac{z^2}{2}\cosh\tau = R_{\xi(t)}(\tau).$$

Combined with the fact that $\mathbb{E}[\xi(t)^2(t)] < \infty$, the SP $\xi(t)$ is a WSP.

The time average of SP $\xi(t)$ is represented as $\overline{\xi(t)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\mathrm{d}t$. $\xi(t)$ is considered a mean-ergodic WSP if $\overline{\xi(t)} = \mathbb{E}[\xi(t)] = m$ holds with probability 1.

Since the sine function is bounded, the time average value of $\xi(t)$ is

$$\begin{aligned}\overline{\xi(t)} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z\cos(ht+\Phi)\mathrm{d}t\\ &= \lim_{T\to\infty}\frac{z}{hT}\cos\Phi\sinh T = 0,\end{aligned}$$

thus there is

$$\overline{\xi(t)} = \mathbb{E}[\xi(t)] = 0.$$

One can finally obtain that $\xi(t)$ is a mean-ergodic WSP.

Proposition II. $\hat{\xi}(t) = z\cos(ht+\Phi) + \mathcal{N}(t)$ is a stationary Gaussian process, where z and h are constants, Φ is the same as in Proposition 1, $\mathcal{N}(t)$ is the Gaussian white noise with $\mathbb{E}\mathcal{N}(t) = 0$ and variance 1.

Proof: It has been demonstrated that $z\cos(ht+\Phi)$ is a stationary process. Combined with the fact that $\mathcal{N}(t)$ is Gaussian white noise, it follows that $\hat{\xi}(t)$ is a stationary Gaussian process.

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