Observer-based sliding mode boundary control of uncertain Markovian stochastic reaction-diffusion systems

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Abstract

This paper deals with the robust mean square exponential stabilization for uncertain Markovian stochastic reaction-diffusion systems (UMSRDS) via the observer-based sliding mode boundary control (SMBC). First, a suitable boundary-output-based observer is constructed for estimating the unknown system states. Next, to process the impact of Markovian switching, a mode-dependent integral sliding mode surface (SMS) is established, on which the closed-loop system is mean square robust exponentially stable. Furthermore, an observer-based sliding mode boundary controller (SMBCr) is designed to guarantee the almost sure reachability of the predefined SMS. Then, a mode-dependent condition is provided to ensure the robust mean square exponential stability of the closed-loop system. Finally, the proposed method is applied to a CPU thermal model to illustrate the effectiveness of theoretical results.

Keywords: Stochastic reaction-diffusion systems, Markovian switching, sliding mode control, boundary control, mean square exponential stability.

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1. Introduction

The phenomenon of diffusion is prevalent in various real-world engineering applications, such as thermal diffusion, chemical reaction processes, and fluid dynamics [1, 2]. As a result, reaction-diffusion systems have been extensively investigated in both theoretical and practical contexts. Additionally, factors such as component failures or repairs, changes in subsystem interconnections, and abrupt environmental disturbances frequently introduce random and sudden changes in the structure and parameters of hybrid systems. Markovian switching systems have garnered significant attention [3, 4, 5] due to their effectiveness in modeling these complex scenarios. Furthermore, the study of Markovian reaction-diffusion systems with coefficient uncertainty is of significant importance [6]. This is because real-world systems often face inherent variations and unpredictable fluctuations in their parameters, which, if neglected, can lead to models that fail to capture the true dynamics of the system. To more accurately represent practical systems exhibiting diffusion phenomena, it is critical to consider stochastic white noise [7]. Consequently, increasing emphasis has been placed on the study of uncertain Markovian stochastic reaction-diffusion systems (UMSRDS).

Sliding mode control (SMC), a robust control methodology, is used in dynamic systems to achieve stability and precise tracking of desired trajectories, particularly in the presence of uncertainties, disturbances, and nonlinearities [8, 9, 10]. Furthermore, SMC has since gained popularity in various fields, including robotics, aerospace, and automotive control [11, 12, 13, 14]. The fundamental idea of SMC is to create a sliding mode surface (SMS) within the system state space. The SMC law forces system states to reach the predefined SMS within a finite time and then maintain their trajectory on it. Once system states reach the SMS, it exhibits a unique sliding behavior where the dynamics become much simpler, and easier to manage. One of the key advantages of SMC is its robustness against uncertainties and disturbances [15, 16, 17]. Therefore, the SMC is devoted to study uncertain Markov switching systems [18, 19]. For example, Niu et al. have achieved significant results for SMC of Markov jump systems [20, 21, 22]. Besides, SMC is also employed in stochastic Markovian switching systems [23, 24, 25]. It is worth noting that Y. Orlov and others have contributed to the research on SMC of PDEs and promoted the development of SMC of PDEs [26, 27]. Since boundary control has the advantages of cost-saving and easy engineering implementation compared to distributed control, sliding mode boundary control (SMBC), which combines boundary control with SMC, has attracted widespread attention from experts and scholars [28, 29]. The SMBC of one-dimensional PDEs is studied, such as the control of heat equations, wave equations, and Euler-Bernoulli beam equations [30, 31, 32, 33, 34]. However, up to now, the literature on the SMBC of UMSRDS is still limited, which cannot meet the needs of actual engineering application. The SMBC of UMSRDS remains an open challenge to be solved.

In practical applications, not all system states are directly accessible through sensors or measurements, making it difficult to construct the state feedback controller. To overcome this difficulty, it is necessary to design an observer to estimate the unmeasured system states based on available measurements and system dynamics. Therefore, a growing interest is arising in the observer-based SMC [35, 36, 37]. For example, [38] studies the robust observer-based SMC for stochastic Markov jump systems affected by packet loss. An observer-based adaptive sliding mode control strategy was proposed to address nonlinear stochastic Markov jump systems with uncertain time-varying delays in [39]. The aforementioned literature primarily focuses on the research of sliding mode control for stochastic ODE systems. However, to the best of our knowledge, observer-based SMBC for UMSRDS has been rarely addressed. This paper contributes to the field by expanding the system of SMBC and filling a theoretical gap in the existing literature.

Motivated by the above discussions, this paper investigates the mean square robust stabilization for UMSRDS via observer-based SMBC. First, an observer, based on boundary output information, is designed to estimate the unmeasured states of the system. Next, to address the influence of Markov switching, a novel mode-dependent SMS is established in the estimation space. Then, a sliding mode boundary controller (SMBCr) based on the observed states is proposed to ensure the almost sure finite time reachability of the pre-designed SMS. By using the Lyapunov functional method, stochastic analysis and inequalities techniques, a condition to ensure that UMSRDS achieves mean square stabilization is provided. Finally, an example of a CPU thermal model is provided to illustrate the effectiveness of the theoretical results.

The main contributions of this work are summarized as follows

- A novel observer-based mode-dependent SMS and SMBCr are proposed to mitigate the impact of Markovian switching on SMBC and ensure the sliding mode stabilization of the UMSRDS under mode switching. Compared with designing a common SMS for all modes, the mode-dependent SMS we designed is easier to design and has higher application flexibility.
- Since the observer states are influenced by stochastic disturbances, the time at which the observer states first reach the SMS is a stopping time. To ensure that the observer-based SMBC functions properly, we have addressed the key issue of almost sure finite-time reachability. However, many existing references fail to fully explain this problem. For instance, reference [40] overlooks this issue and continues to employ the reachability proof method used for deterministic systems, which is not sufficiently rigorous.
- For UMSRDS, the SMBC based on the boundary output observer is presented, providing a framework for observer-based SMBC studies of partial differential equations (PDEs) with Markovian switching.

Notations: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, in which Ω is the sample space, $\mathcal{F}$ is σ-algebra of subsets of the sample space, and $\mathbb{P}$ is the probability measure. For a matrix/vector A, A^{T} represents the transposition of A. I and $\mathbf{0}$ denote the identity matrix and null vector(matrix) with the appropriate dimensions, respectively. Notation $*$ is represented as an ellipsis for the terms of symmetric block matrices. $A < 0$ (≤ 0) means A is a real symmetric negative definite (negative semi-definite) matrix and $A > 0$ means $-A < 0$. $B < \hat{B}$ ($\leq \hat{B}$) means $B - \hat{B} < 0$ (≤ 0). $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of matrix A, respectively. $\mathrm{sym}(B) = B + B^{\mathrm{T}}$. $\mathcal{C}^2(\mathbb{R}^n \times \mathcal{H} \times \mathbb{R}^+; \mathbb{R}^+)$ denotes the family of all nonnegative functions $V(\eta(\tau), i, \tau)$ on $\mathbb{R}^n \times \mathcal{H} \times \mathbb{R}^+$ such that they are continuously twice differentiable in η; κ denotes the set of all functions: $R^+ \to R^+$, which are continuous, strictly increasing and vanish at zero; κ_∞ denotes the set of all functions which are of class κ and unbounded. We take $W^{m,2}([0,l]; \mathbb{R}^n)$ be a Sobolev space that contains absolutely continuous n-dimensional vector functions $\omega(\theta) : [0,l] \to \mathbb{R}^n$ with square integrable derivatives $\frac{\mathrm{d}^k \omega(\theta)}{\mathrm{d}\theta^k}$ of the order $0 \leq k \leq m$. Denote $\|w(\cdot,\tau)\| = (\int_0^l \sum_{i=1}^n w_i^{\mathrm{T}}(\theta,\tau) w_i(\theta,\tau) \mathrm{d}\theta)^{\frac{1}{2}}$, where $w(\theta,\tau) \in \mathbb{R}^n$. $\|\varsigma(\tau)\|_1 = \sum_{i=1}^n \varsigma_i(\tau) \mathrm{sgn}(\varsigma_i(\tau))$ and $\overrightarrow{\mathrm{sgn}}(\varsigma(\tau)) = (\mathrm{sgn}(\varsigma_1(\tau)), ..., \mathrm{sgn}(\varsigma_n(\tau)))^{\mathrm{T}}$, where $\varsigma(\tau) = (\varsigma_1(\tau), ..., \varsigma_n(\tau))^{\mathrm{T}} \in \mathbb{R}^n$ and

$$\mathrm{sgn}(\varsigma_i(\tau)) \in \begin{cases} \dfrac{\varsigma_i(\tau)}{|\varsigma_i(\tau)|}, & |\varsigma_i(\tau)| \neq 0 \\ [-1,1], & |\varsigma_i(\tau)| = 0. \end{cases}$$

2. Preliminaries

In this paper, $\{\gamma(\tau), \tau \geq 0\}$ is a continuous time Markov chain with discrete states $\mathcal{H} = \{1, 2, ..., \mathcal{N}\}$ and $\mathcal{W}(\tau)$ is a 1-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. And $\gamma(\tau)$ is independent of $\mathcal{W}(\tau)$. The transfer probability of the Markov chain $\gamma(\tau)$ is defined by

$$\mathbb{P}\{\gamma(\tau+\Delta) = j | \gamma(\tau) = i\} = \begin{cases} \delta_{ij}\Delta + \mathrm{o}(\Delta) & i \neq j, \\ 1 + \delta_{ii}\Delta + \mathrm{o}(\Delta) & i = j, \end{cases} \tag{1}$$

where $\Delta > 0$, $\lim\limits_{\Delta \to 0}(\mathrm{o}(\Delta)/\Delta) = 0$. Here $\delta_{ij} \geq 0$ ($i \neq j$) is the transition rate from mode i to mode j, while $\delta_{ii} = -\sum_{j=1, i\neq j} \delta_{ij} < 0$.

The defined form of the transition rate matrix $\Psi = (\delta_{ij})_{\mathcal{N}\times\mathcal{N}}(i, j \in \mathcal{H})$ is

$$\Psi = \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1\mathcal{N}} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\mathcal{N}1} & \delta_{\mathcal{N}2} & \cdots & \delta_{\mathcal{N}\mathcal{N}} \end{pmatrix}. \tag{2}$$

We consider the following uncertain Markovian stochastic reaction-diffusion system (UMSRDS)

$$\begin{aligned} \mathrm{d}w(\theta,\tau) = & \Big[B(\gamma(\tau))\frac{\partial^2 w(\theta,\tau)}{\partial\theta^2} + [A(\gamma(\tau)) + \Delta A(\theta,\tau,\gamma(\tau))]w(\theta,\tau)\Big]\mathrm{d}\tau \\ & + [C(\gamma(\tau)) + \Delta C(\theta,\tau,\gamma(\tau))]w(\theta,\tau)\mathrm{d}\mathcal{W}(\tau), \end{aligned} \tag{3}$$

where $\theta \in (0, 1)$ is the spatial variable, $\tau > 0$ is the time variable. $w(\theta, \tau) \in \mathbb{R}^n$ is the state of the system. The positive definite matrix $B(\gamma(\tau)) \in \mathbb{R}^{n\times n}$ represents the diffusion coefficient. $A(\gamma(\tau)), C(\gamma(\tau)) \in \mathbb{R}^{n\times n}$ are known constant matrixes. $\Delta A(\theta, \tau, \gamma(\tau)), \Delta C(\theta, \tau, \gamma(\tau)) \in \mathbb{R}^{n\times n}$ denote the uncertain parameters and satisfy

$$\Delta A(\theta,\tau,\gamma(\tau)) = D_1(\gamma(\tau))T_1(\theta,\tau)W_1(\gamma(\tau))$$
$$\Delta C(\theta,\tau,\gamma(\tau)) = D_2(\gamma(\tau))T_2(\theta,\tau)W_2(\gamma(\tau))$$

where $T_i^{\mathrm{T}}(\theta,\tau)T_i(\theta,\tau) \le I(i = 1, 2)$, and $D_1(\gamma(\tau))$, $D_2(\gamma(\tau))$, $W_1(\gamma(\tau))$, $W_2(\gamma(\tau))$ are known constant matrixes.

For the sake of simplicity, when $\gamma(\tau) = i,\ i \in \mathcal{H}$, we denote $A(\gamma(\tau)) = A_i$, $B(\gamma(\tau)) = B_i,\ C(\gamma(\tau)) = C_i,\ \Delta A(\theta,\tau,\gamma(\tau)) = \Delta A_i,\ \Delta C(\theta,\tau,\gamma(\tau)) = \Delta C_i$, $D_1(\gamma(\tau)) = D_{1i}$, $D_2(\gamma(\tau)) = D_{2i}$, $W_1(\gamma(\tau)) = W_{1i}$, $W_2(\gamma(\tau)) = W_{2i}$.

Therefore, UMSRDS (3) can be rewritten as

$$\mathrm{d}w(\theta,\tau) = \Big[B_i\frac{\partial^2 w(\theta,\tau)}{\partial\theta^2} + (A_i + \Delta A_i)w(\theta,\tau)\Big]\mathrm{d}\tau + (C_i + \Delta C_i)w(\theta,\tau)\mathrm{d}\mathcal{W}(\tau), \tag{4}$$

where $\gamma(\tau) = i,\ i \in \mathcal{H}$.

We take

$$w(\theta, 0) = \phi(\theta), \gamma(0) = \gamma_0, \tag{5}$$

where $\phi \in L^2([0,1];\mathbb{R}^n)$ is the initial value, γ_0 is the initial mode.

The following Neumann boundary conditions are adopted

$$\frac{\partial w(\theta,\tau)}{\partial \theta}\Big|_{\theta=0} = \mathbf{0}, \frac{\partial w(\theta,\tau)}{\partial \theta}\Big|_{\theta=1} = u(\tau), \tag{6}$$

where $u(\tau) \in \mathbb{R}^n$ is the boundary control input.

Definition 1. *([41]) UMSRDS (3) is said to be mean square robustly exponentially stable if for all admissible $\Delta A(\theta,\tau,\gamma(\tau))$, $\Delta C(\theta,\tau,\gamma(\tau))$, the following inequality holds*

$$\mathbb{E}\|w(\cdot,\tau)\|^2 \leq \beta\mathbb{E}\|\phi(\cdot)\|^2 e^{-\zeta t}, t \geq 0,$$

where scalars $\zeta > 0$, $\beta \geq 1$.

Definition 2 ([42]). *The solution of stochastic system is said to be globally finite-time stable in probability, if for any initial data $\eta(0)$, system we considered has a solution denoted by $\eta(\tau)$, moreover, the following statements hold:*
(i) Finite-time attractiveness in probability: for every initial value $\eta(0) \in \mathbb{R}^n/\mathbf{0}, \gamma_0 \in \mathcal{H}$, the first hitting time

$$t^* = \inf\{\tau \geq 0 : \eta(\tau) = 0\}, \tag{7}$$

which is called the the stochastic settling time, is finite almost surely, that is $\mathbb{P}\{t^* < \infty\} = 1$;
(ii) Global stability in probability: $\forall \xi \in (0,1)$, there exists a κ function $\varphi(\cdot)$, such that for any $\eta(0) \in \mathbb{R}^n/0, \gamma_0 \in \mathcal{H}$ every $\eta(\tau)$ satisfies

$$\mathbb{P}\{\|\eta(\tau)\| \leq \varphi(\|\eta(0)\|)\} \geq 1 - \xi. \tag{8}$$

Remark 1. *From (i) of Definition 2, we can clearly see that finite-time attractiveness in probability implies that the state trajectories will almost surely (with probability 1) reach the origin in finite time.*

Definition 3. *The sliding mode surface (SMS) (15) is said to achieve almost sure reachability if the probability that the system state reaches SMS (15) within finite time is* 1*. This can be mathematically expressed as* $P(t^* < \infty) = 1$*, where* $t^* = \inf\{\tau \geq 0 : \varsigma(\tau) = 0\}$ *is the stopping time.*

Lemma 1 ([42]). *If exists a Lyapunov functional* $V \in \mathcal{C}^2(\mathbb{R}^n \times \mathcal{H} \times \mathbb{R}^+; \mathbb{R}^+)$ *and* κ_∞ *functionals* κ_1, κ_2*, such that for all* $(\eta(\tau), i, \tau) \in \mathbb{R}^n \times \mathcal{H} \times \mathbb{R}^+$*, the following inequalities hold*

$$\kappa_1(\eta(\tau), \tau) \leq V(\eta(\tau), i, \tau) \leq \kappa_2(\eta(\tau), \tau),$$
$$\mathcal{L}V(\eta(\tau), i, \tau) \leq -\vartheta V^{\varrho}(\eta(\tau), i, \tau),$$

then the solution $\eta(\tau)$ *is finite-time stable in probability, where* $\vartheta > 0$ *and* $0 < \varrho < 1$*.*

Lemma 2 (Wirtinger's inequality [43]). *Let a vector function* $w \in W^{1,2}([0,1]; \mathbb{R}^n)$ *with* $w(0) = 0$ *or* $w(1) = 0$*. Then, for any positive matrix* P*, the following integral inequality holds*

$$\int_0^1 w^{\mathrm{T}}(\xi) P w(\xi) \mathrm{d}\xi \leq \frac{4}{\pi^2} \int_0^1 (\frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi})^{\mathrm{T}} P (\frac{\mathrm{d}w(\xi)}{\mathrm{d}\xi}) \mathrm{d}\xi.$$

Lemma 3 ([44]). *If real matrices* X, Q, S, U *and* T *satisfy* $W > 0$ *and* $T^{\mathrm{T}}T \leq I$*. Then, we have*

(a) $2x^{\mathrm{T}}QTSy \leq \nu^{-1}x^{\mathrm{T}}QQ^{\mathrm{T}}x + \nu y^{\mathrm{T}}S^{\mathrm{T}}Sy$*, where the scalar* $\nu > 0$ *and vectors* $x, y \in \mathbb{R}^n$*.*

(b) For any scalar $\epsilon > 0$ *such that* $U - \epsilon QQ^{\mathrm{T}} > 0$*,* $(X+QTS)^{\mathrm{T}}U^{-1}(X+QTS) \leq X^{\mathrm{T}}(U - \epsilon QQ^{\mathrm{T}})^{-1}X + \epsilon^{-1}S^{\mathrm{T}}S$*.*

Lemma 4 ([45]). *Let* $V(w(\cdot,\tau), \gamma(\tau), \tau) = \int_0^1 W(w(\theta,\tau), \gamma(\tau), \tau)\mathrm{d}\theta$ *be an integral-type Lyapunov functional,* $W \in \mathcal{C}^{1,1}(\mathbb{R}^n \times \mathcal{H} \times \mathbb{R}^+; \mathbb{R}^+)$*, then the definition of* *the weak infinitesimal operator is given as follow*

$$\mathcal{L}V(w(\cdot,\tau),i,\tau) = \int_0^1 \left\{ \frac{\partial W(w(\theta,\tau),i,\tau)}{\partial \tau} + \left(\frac{\partial W(w(\theta,\tau),i,\tau)}{\partial w(\theta,\tau)}\right)^{\mathrm{T}} \frac{\partial w(\theta,\tau)}{\partial \tau} + \sum_{j=1}^{\mathcal{N}} \delta_{ij} W(w(\theta,\tau),j,\tau) \right\} \mathrm{d}\theta.$$

3. Design of observer-based SMBC

Due to the uncertainties of parameters and stochastic disturbance in system (3), the states information cannot be accurately measured. Therefore, it is necessary to design an effective observer to estimate the state information of system (3), and then we use SMBC in the observation space to prove the robust mean square exponential stability of system (3).

We take the boundary value $w(1,\tau)$ as the output and design the following boundary-output-based observer for UMSRDS (3) when $\gamma(\tau) = i, i \in \mathcal{H}$

$$\mathrm{d}\hat{w}(\theta,\tau) = \Big[B_i \frac{\partial^2 \hat{w}(\theta,\tau)}{\partial \theta^2} + A_i \hat{w}(\theta,\tau) + L_i(w(1,\tau) - \hat{w}(1,\tau))\Big]\mathrm{d}\tau, \tag{9}$$

where $\tau > 0$, $\theta \in (0,1)$, and $\hat{w}(\theta,\tau) \in \mathbb{R}^n$ is the estimation of $w(\theta,\tau)$ and $L_i \in \mathbb{R}^{n\times n}$ is the observer gain to be designed.

We take the following initial value for observer (9),

$$\hat{w}(\theta,0) = \hat{\phi}(\theta), \gamma(0) = \gamma_0, \tag{10}$$

where $\hat{\phi} \in L^2([0,1];\mathbb{R}^n)$ is the initial value, γ_0 is the initial mode.

The Neumann boundary conditions are

$$\frac{\partial \hat{w}(\theta,\tau)}{\partial \theta}\Big|_{\theta=0} = \mathbf{0}, \frac{\partial \hat{w}(\theta,\tau)}{\partial \theta}\Big|_{\theta=1} = u(\tau), \tag{11}$$

where $u(\tau) \in \mathbb{R}^n$ is the boundary control input.

We define the error variable $e(\theta,\tau) = w(\theta,\tau) - \hat{w}(\theta,\tau)$, then the error system is

$$
\begin{aligned}
\mathrm{d}e(\theta,\tau) &= \Big[B_i\frac{\partial^2 e(\theta,\tau)}{\partial\theta^2} + A_i e(\theta,\tau) + \Delta A_i w(\theta,\tau) - L_i(w(1,\tau)-\hat{w}(1,\tau))\Big]\mathrm{d}\tau \\
&\quad + (C_i + \Delta C_i)w(\theta,\tau)\mathrm{d}\mathcal{W}(\tau) \\
&= \Big[B_i\frac{\partial^2 e(\theta,\tau)}{\partial\theta^2} + A_i e(\theta,\tau) + \Delta A_i w(\theta,\tau) - L_i e(1,\tau)\Big]\mathrm{d}\tau \\
&\quad + (C_i + \Delta C_i)w(\theta,\tau)\mathrm{d}\mathcal{W}(\tau).
\end{aligned} \tag{12}
$$

The initial value for error (12) is

$$
e(\theta,0) = \tilde{\phi}(\theta), \gamma(0) = \gamma_0, \tag{13}
$$

where $\tilde{\phi}^{\mathrm{T}}(\theta) = \phi^{\mathrm{T}}(\theta) - \hat{\phi}^{\mathrm{T}}(\theta)$.

The boundary conditions are

$$
\frac{\partial e(\theta,\tau)}{\partial\theta}\Big|_{\theta=0} = \mathbf{0}, \frac{\partial e(\theta,\tau)}{\partial\theta}\Big|_{\theta=1} = \mathbf{0}. \tag{14}
$$

Based on the designed boundary-output-based observer, we design the following SMS

$$
\varsigma(\tau) = \int_0^1 \hat{w}(\theta,\tau)\mathrm{d}\theta - K_i\int_0^\tau \hat{w}(1,s)\mathrm{d}s = \mathbf{0}, \tag{15}
$$

where $K_i \in \mathbb{R}^{n\times n}(i \in \mathcal{H})$ is control gain, and will be designed in Theorem 2.

Remark 2. *In order to resist the influence of mode switching, we design mode-dependent SMS (15). Compared with the common SMS designed in references [46, 47, 48], mode-dependent SMS (15) is easier to design and offers a higher degree of freedom in its implementation. Especially, the differences between the SMS for different modes are captured in the matrix $K_i, i \in \mathcal{H}$.*

3.1. Reachability

In this subsection, we first design an observer-based SMBCr $u(\tau)$, which drives the states of observer (9) onto the observer-based SMS (15) in finite time almost surely.

Theorem 1. *For any $i \in \mathcal{H}$, if we adopt the following observer-based SMBCr*

$$u(\tau) = B_i^{-1}[-\mu_i(\tau)\overrightarrow{\mathrm{sgn}}(\varsigma(\tau)) + K_i\hat{w}(1,\tau) - L_i e(1,\tau) - \int_0^1 A_i\hat{w}(\theta,\tau)\mathrm{d}\theta], \tag{16}$$

the almost sure reachability of SMS (15) be ensured in finite time, where $\mu_i(\tau) = \frac{q_i}{p_i} + \frac{1}{2p_i}\sum_{j=1}^{\mathcal{H}} \delta_{ij}p_j\|\varsigma(\tau)\|_1$ *and* $q_i > 0$ *is a given constant.*

Proof. To ensure the reachability, let

$$V_1(\varsigma(\tau), i, \tau) = \frac{1}{2}p_i\varsigma^{\mathrm{T}}(\tau)\varsigma(\tau), \tag{17}$$

where $p_i > 0, i \in \mathcal{H}$. To avoid redundancy, $V_1(\varsigma(\tau), i, \tau)$ is simplified to $V_1(\tau, i)$ in the following.

We calculate $\mathcal{L}V_1$ as following

$$\begin{aligned}
\mathcal{L}V_1(\varsigma(\tau), \tau, i) &= p_i\varsigma^{\mathrm{T}}(\tau)\frac{\partial\varsigma(\tau)}{\partial\tau} + \frac{1}{2}\sum_{j=1}^{\mathcal{H}}\delta_{ij}p_j\varsigma^{\mathrm{T}}(\tau)\varsigma(\tau) \\
&= p_i\varsigma^{\mathrm{T}}(\tau)\int_0^1(B_i\frac{\partial^2\hat{w}}{\partial\theta^2} + A_i\hat{w} + L_i e_1)\mathrm{d}\theta - p_i\varsigma^{\mathrm{T}}(\tau)K_i\hat{w}_1 \\
&\quad + \frac{1}{2}\sum_{j=1}^{\mathcal{H}}\delta_{ij}p_j\varsigma^{\mathrm{T}}(\tau)\varsigma(\tau) \\
&= p_i\varsigma^{\mathrm{T}}(\tau)[-\mu_i(\tau)\overrightarrow{\mathrm{sgn}}(\varsigma(\tau)) + K_i\hat{w}_1 - L_i e_1 - \int_0^1 A_i\hat{w}\mathrm{d}\theta] \\
&\quad + p_i\varsigma^{\mathrm{T}}(\tau)\int_0^1 A_i\hat{w}\mathrm{d}\theta + p_i\varsigma^{\mathrm{T}}(\tau)L_i e_1 - p_i\varsigma^{\mathrm{T}}(\tau)K_i\hat{w}_1 \\
&\quad + \frac{1}{2}\sum_{j=1}^{\mathcal{H}}\delta_{ij}p_j\varsigma^{\mathrm{T}}(\tau)\varsigma(\tau) \\
&\leq -p_i\mu_i(\tau)\|\varsigma(\tau)\|_1 + \frac{1}{2}\sum_{j=1}^{\mathcal{H}}\delta_{ij}p_j\|\varsigma(\tau)\|_1^2 \\
&\leq -q_i\|\varsigma(\tau)\|_1 \leq -q_i(2V_1(\tau, i))^{\frac{1}{2}},
\end{aligned} \tag{18}$$

where $\hat{w} = \hat{w}(\theta, \tau)$, $\hat{w}_1 = \hat{w}(1, \tau)$, and $e_1 = e(1, \tau) = w(1, \tau) - \hat{w}(1, \tau)$.

By Lemma 1 and (18), the system states reached SMS (15) in finite time almost surely, and the proof is complete. ∎

Remark 3. *SMBCr (16) depends on modes, which allows our design to have a wider degree of freedom. For any mode* $i, i \in \mathcal{H}$*, SMBCr (16) guides the state trajectories to the* i*-th SMS. When the mode switches from* i *to* $j(i \neq j, j \in \mathcal{H})$ *in the reach phase, SMBCr (16) will also switch to the corresponding mode* j*, and the system states will be driven to the* j*-th SMS. Besides, if the mode switches from* i *to* j *during the sliding phase, the mode-dependent controller (16) drives the state trajectories from the* i*-th SMS to the* j*-th SMS and then slides on the corresponding* j*-th SMS. As time goes by, despite Markov switching and stochastic disturbances, SMBCr (16) always ensures that the system states almost sure reach the SMS (15) in finite time.*

Remark 4. *Since* $\hat{w}(\theta, \tau)$ *is related to the stochastic process* $w(1, \tau)$*,* $\hat{w}(\theta, \tau)$ *is also a stochastic process. Then for each mode* $i, i \in \mathcal{H}$*, the time when* $\hat{w}(\theta, \tau)$ *reaches SMS (15) is a stopping time. Therefore, in order to ensure that the stochastic system states reach SMS (15) in finite time, we need to prove the almost sure reachability. We apply Lemma 1 to prove* $\varsigma(\tau)$ *is finite-time stable in probability. Then, by Definition 2 and Remark 1, the almost sure finite time reachability of SMS (15) is obtained.*

In this subsection, the almost sure finite-time reachability of SMS (15) was established. This result is not only a prerequisite for initiating the sliding motion but also forms the foundation for analyzing the system's stability. The next step is to analyze the mean square robust stability of system (3) during the sliding motion.

3.2. Observer-based mean square robust stabilization

In this subsection, the observer-based mean square robust stabilization is investigated for UMSRDS (3). By analyzing the stability of the composite system $Y(\theta, \tau) = (w^{\mathrm{T}}(\theta, \tau), e^{\mathrm{T}}(\theta, \tau))^{\mathrm{T}}$, the mean square robust stabilization of closed-loop system (3) is obtained, and the mode-dependent stability condition is given.

Obviously, the equivalent controller $u_{eq}(\tau)$ is

$$u_{eq}(\tau) = B_i^{-1}(K_i\hat{w}_1 - \int_0^1 A_i\hat{w}\mathrm{d}\theta - L_ie_1). \tag{19}$$

Then we have

$$\mathrm{d}Y(\theta,\tau) = \Big[\bar{B}_i\frac{\partial^2 Y(\theta,\tau)}{\partial\theta^2} + \bar{A}_iY(\theta,\tau) + \bar{L}_iY(1,\tau)\Big]\mathrm{d}\tau + \bar{C}_iY(\theta,\tau)\mathrm{d}\mathcal{W}(\tau), \tag{20}$$

where $\bar{B}_i = diag\{B_i, B_i\}$, $\bar{L}_i = [\mathbf{0}\ \mathbf{0};\ \mathbf{0}\ -L_i]$, $\bar{A}_i = [A_i{+}\Delta A_i\ \ \mathbf{0};\ \Delta A_i\ \ A_i]$, $\bar{C}_i = [C_i + \Delta C_i\ \ \mathbf{0};\ C_i + \Delta C_i\ \ \mathbf{0}]$.

The initial value of system (20) is

$$Y(\theta,0) = \Phi(\theta), \tag{21}$$

where $\Phi(\theta) = (\phi^{\mathrm{T}}(\theta), \tilde{\phi}^{\mathrm{T}}(\theta))^{\mathrm{T}}$, and the boundary conditions are

$$\frac{\partial Y(\theta,\tau)}{\partial\theta}\Big|_{\theta=0} = \mathbf{0}, \frac{\partial Y(\theta,\tau)}{\partial\theta}\Big|_{\theta=1} = U_{eq}(\tau), \tag{22}$$

where $U_{eq}(\tau) = [u_{eq}^{\mathrm{T}}(\tau)\ \ \mathbf{0}]^{\mathrm{T}}_{2n\times 1}$.

Theorem 2. *If there exist $K_i, L_i \in \mathbb{R}^{n\times n}$ and constant $\alpha_i > 0$, such that*

$$\Xi_{1i} = \begin{pmatrix} \Theta 1i & \alpha_i[\mathrm{sym}(-\tilde{K}_i - \tilde{L}_i) - \tilde{A}_i - \bar{L}_i] \\ * & -\frac{\pi^2}{2}\alpha_i\bar{B}_i + \alpha_i\mathrm{sym}(\tilde{K}_i + \tilde{L}_i) \end{pmatrix} < 0, \tag{23}$$

then system (3) is observer-based mean square robust exponentially stable, where $\Theta 1i = \alpha_i[\mathrm{sym}(\tilde{K}_i + \tilde{L}_i + \tilde{A}_i + \bar{L}_i) + G_{1i} + G_{2i}] + \sum_{j\in\mathcal{H}}\delta_{ij}\alpha_jI_{2n}$, $\tilde{K}_i = [K_i\ \ -K_i; \mathbf{0}\ \ \mathbf{0}]$, $\tilde{A}_i = [-A_i\ \ A_i; \mathbf{0}\ \ \mathbf{0}]$, $\tilde{L}_i = [\mathbf{0}\ \ -L_i;\ \mathbf{0}\ \ \mathbf{0}]$, $G_{2i} = [\Pi_i\ \ \mathbf{0}; \mathbf{0}\ \ \mathbf{0}]$, $\Pi_i = \mathrm{sym}[C_i^{\mathrm{T}}C_i + C_i^{\mathrm{T}}D_{2i}D_{2i}^{\mathrm{T}}C_i + (1 + \lambda_{\max}(D_{2i}^{\mathrm{T}}D_{2i}))W_{2i}^{\mathrm{T}}W_{2i}]$, *and* $G_{1i} = \begin{pmatrix} \mathrm{sym}(A_i + W_{1i}^{\mathrm{T}}W_{1i}) + D_{1i}D_{1i}^{\mathrm{T}} & 0 \\ 0 & \mathrm{sym}(A_i) + D_{1i}D_{1i}^{\mathrm{T}} \end{pmatrix}$.

Proof. Let

$$V_2(\tau,i) = \alpha_i\int_0^1 Y^{\mathrm{T}}(\theta,\tau)Y(\theta,\tau)\mathrm{d}\theta. \tag{24}$$

To avoid redundancy, in the following, we denote $Y = Y(\theta,\tau)$, $Y(1,\tau) = Y_1$, $\bar{Y} = Y - Y(1,\tau) = Y - Y_1$. Calculating $\mathcal{L}V_2$ along system (20) and using integration by parts, we get

$$\begin{aligned}
\mathcal{L}V_2(\tau,i) &= 2\alpha_i \int_0^1 Y^{\mathrm{T}}(\bar{B}_i \frac{\partial^2 Y}{\partial\theta^2} + \bar{A}_i Y + \bar{L}_i Y_1)\mathrm{d}\theta + \alpha_i \int_0^1 tr(Y^{\mathrm{T}}\bar{C_i}^{\mathrm{T}}\bar{C}_i Y)\mathrm{d}\theta \\
&\quad + \sum_{j\in\mathcal{H}} \delta_{ij}\alpha_j \int_0^1 Y^{\mathrm{T}} Y \mathrm{d}\theta \\
&= 2\alpha_i Y_1^{\mathrm{T}} \bar{B}_i U_{eq}(\tau) - 2\alpha_i \int_0^1 \frac{\partial Y^{\mathrm{T}}}{\partial\theta} \bar{B}_i \frac{\partial Y}{\partial\theta} \mathrm{d}\theta + \sum_{j\in\mathcal{H}} \delta_{ij}\alpha_j \int_0^1 Y^{\mathrm{T}} Y \mathrm{d}\theta \\
&\quad + \alpha_i \int_0^1 Y^{\mathrm{T}} \bar{C_i}^{\mathrm{T}} \bar{C}_i Y \mathrm{d}\theta + 2\alpha_i \int_0^1 Y^{\mathrm{T}}(\bar{A}_i Y + \bar{L}_i Y_1)\mathrm{d}\theta,
\end{aligned} \tag{25}$$

From lemma 2, we obtain

$$-2\alpha_i \int_0^1 \frac{\partial Y^{\mathrm{T}}}{\partial\theta} \bar{B}_i \frac{\partial Y}{\partial\theta} \mathrm{d}\theta = -2\alpha_i \int_0^1 \frac{\partial \bar{Y}^{\mathrm{T}}}{\partial\theta} \bar{B}_i \frac{\partial \bar{Y}}{\partial\theta} \mathrm{d}\theta \le -\frac{\pi^2}{2}\alpha_i \int_0^1 \bar{Y}^{\mathrm{T}} \bar{B}_i \bar{Y} \mathrm{d}\theta. \tag{26}$$

Making use of Lemma 3, we get

$$\begin{aligned}
2\alpha_i \int_0^1 Y^{\mathrm{T}} \bar{A}_i Y \mathrm{d}\theta &= 2\alpha_i \int_0^1 \Big[w^{\mathrm{T}}(A_i + \Delta A_i)w + e^{\mathrm{T}}\Delta A_i w + e^{\mathrm{T}} A_i e\Big]\mathrm{d}\theta \\
&\le 2\alpha_i \int_0^1 \Big[w^{\mathrm{T}}(A_i + D_{1i}T_1W_{1i})w + e^{\mathrm{T}} D_{1i}T_1W_{1i}w + e^{\mathrm{T}} A_i e\Big]\mathrm{d}\theta \\
&\le \alpha_i \int_0^1 \Big[w^{\mathrm{T}} 2A_i w + w^{\mathrm{T}} D_{1i}D_{1i}^{\mathrm{T}} w + w^{\mathrm{T}} W_{1i}^{\mathrm{T}} T_1^{\mathrm{T}} T_1 W_{1i} w \\
&\quad + e^{\mathrm{T}} D_{1i} D_{1i}^{\mathrm{T}} e + w^{\mathrm{T}} W_{1i}^{\mathrm{T}} T_1^{\mathrm{T}} T_1 W_{1i} w + e^{\mathrm{T}} 2A_i e\Big]\mathrm{d}\theta \\
&\le \alpha_i \int_0^1 \Big[w^{\mathrm{T}}(2A_i + D_{1i}D_{1i}^{\mathrm{T}} + 2W_{1i}^{\mathrm{T}} W_{1i})w \\
&\quad + e^{\mathrm{T}}(D_{1i}D_{1i}^{\mathrm{T}} + 2A_i)e\Big]\mathrm{d}\theta \\
&= \alpha_i \int_0^1 Y^{\mathrm{T}} G_{1i} Y \mathrm{d}\theta,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\alpha_i \int_0^1 Y^{\mathrm{T}} \bar{C_i}^{\mathrm{T}} \bar{C}_i Y \mathrm{d}\theta \\
&\quad = 2\alpha_i \int_0^1 w^{\mathrm{T}}(C_i + \Delta C_i)^{\mathrm{T}}(C_i + \Delta C_i) w \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&\le 2\alpha_i \int_0^1 \Big(w^{\mathrm{T}} C_i^{\mathrm{T}} C_i w + 2 w^{\mathrm{T}} C_i^{\mathrm{T}} \Delta C_i w + w^{\mathrm{T}} \Delta C_i^{\mathrm{T}} \Delta C_i w \Big) \mathrm{d}\theta \\
&\le 2\alpha_i \int_0^1 w^{\mathrm{T}} \Big(C_i^{\mathrm{T}} C_i + 2 C_i^{\mathrm{T}} D_{2i} T_2 W_{2i} + W_{2i}^{\mathrm{T}} T_2^{\mathrm{T}} D_{2i}^{\mathrm{T}} D_{2i} T_2 W_{2i} \Big) w \mathrm{d}\theta \\
&\le 2\alpha_i \int_0^1 w^{\mathrm{T}} \Big(C_i^{\mathrm{T}} C_i + C_i^{\mathrm{T}} D_{2i} T_2^{\mathrm{T}} T_2 D_{2i}^{\mathrm{T}} C_i + W_{2i}^{\mathrm{T}} W_{2i} + W_{2i}^{\mathrm{T}} T_2^{\mathrm{T}} D_{2i}^{\mathrm{T}} D_{2i} T_2 W_{2i} \Big) w \mathrm{d}\theta \\
&\le 2\alpha_i \int_0^1 w^{\mathrm{T}} \Big[C_i^{\mathrm{T}} C_i + C_i^{\mathrm{T}} D_{2i} D_{2i}^{\mathrm{T}} C_i + (1 + \lambda_{\max}(D_{2i}^{\mathrm{T}} D_{2i})) W_{2i}^{\mathrm{T}} W_{2i} \Big] w \mathrm{d}\theta \\
&= \alpha_i \int_0^1 Y^{\mathrm{T}} G_{2i} Y \mathrm{d}\theta.
\end{aligned} \tag{28}
$$

Bringing (26)-(28) into (25), we get

$$
\begin{aligned}
\mathcal{L}V_2(\tau, i) \le\ & 2\alpha_i \int_0^1 Y_1^{\mathrm{T}} (\tilde{K}_i + \tilde{L}_i) Y_1 \mathrm{d}\theta + 2\alpha_i \int_0^1 Y_1^{\mathrm{T}} \tilde{A}_i Y \mathrm{d}\theta - \frac{\pi^2}{2} \alpha_i \int_0^1 \bar{Y}^{\mathrm{T}} \bar{B}_i \bar{Y} \mathrm{d}\theta \\
& + 2\alpha_i \int_0^1 Y^{\mathrm{T}} \bar{L}_i Y_1 \mathrm{d}\theta + \alpha_i \int_0^1 Y^{\mathrm{T}} G_1 Y \mathrm{d}\theta + \alpha_i \int_0^1 Y^{\mathrm{T}} G_2 Y \mathrm{d}\theta \\
& + \sum_{j \in \mathcal{H}} \delta_{ij} \alpha_j \int_0^1 Y^{\mathrm{T}} Y \mathrm{d}\theta \\
\le\ & 2\alpha_i \int_0^1 (Y - \bar{Y})^{\mathrm{T}} (\tilde{K}_i + \tilde{L}_i)(Y - \bar{Y}) \mathrm{d}\theta + 2\alpha_i \int_0^1 (Y - \bar{Y})^{\mathrm{T}} \tilde{A}_i Y \mathrm{d}\theta \\
& - \frac{\pi^2}{2} \alpha_i \int_0^1 \bar{Y}^{\mathrm{T}} \bar{B}_i \bar{Y} \mathrm{d}\theta + 2\alpha_i \int_0^1 Y^{\mathrm{T}} \bar{L}_i (Y - \bar{Y}) \mathrm{d}\theta \\
& + \alpha_i \int_0^1 Y^{\mathrm{T}} (G_{1i} + G_{2i} + \sum_{j \in \mathcal{H}} \delta_{ij} \alpha_j I_{2n}) Y \mathrm{d}\theta \\
=\ & \int_0^1 \Upsilon^{\mathrm{T}}(\theta, \tau) \Xi_{1i} \Upsilon(\theta, \tau) \mathrm{d}\theta,
\end{aligned} \tag{29}
$$

where $\Upsilon(\theta, \tau) = (Y^{\mathrm{T}}(\theta, \tau), \bar{Y}^{\mathrm{T}}(\theta, \tau))^{\mathrm{T}}$.

Then, combining (29) with inequality (23) $\Xi_1 < 0$ and letting $\lambda_0 = -\min_{i \in \mathcal{H}} \lambda_{\min}(\Xi_{1i})$, we have

$$
\mathbb{E}\mathcal{L}V_2(\tau, i) < -\lambda_0 \mathbb{E} \int_0^1 \Upsilon^{\mathrm{T}}(\theta, \tau) \Upsilon(\theta, \tau) \mathrm{d}\theta < -\lambda_0 \mathbb{E} \int_0^1 Y^{\mathrm{T}}(\theta, \tau) Y(\theta, \tau) \mathrm{d}\theta. \tag{30}
$$

Letting $c_1 = \min_{i \in \mathcal{H}} \alpha_i$ and $c_2 = \max_{i \in \mathcal{H}} \alpha_i$, we have

$$
c_1 \int_0^1 Y^{\mathrm{T}} Y \mathrm{d}\theta \le V_2(\tau, i) \le c_2 \int_0^1 Y^{\mathrm{T}} Y \mathrm{d}\theta. \tag{31}
$$

Letting $h_1 > 0$ be sufficiently small satisfying $h_1 c_2 \leq \lambda_0$ and using Dynkin's formula, we get

$$\begin{aligned}\mathbb{E}e^{h_1\tau}V_2(\tau,i) - \mathbb{E}V_2(0,\gamma_0) &= \mathbb{E}\int_0^\tau \mathcal{L}(e^{h_1 s}V_2(s,i))\mathrm{d}s \\ &\leq \mathbb{E}\int_0^\tau [h_1 e^{h_1 s}c_2 \int_0^1 Y^{\mathrm{T}}Y\mathrm{d}\theta - e^{h_1 s}\lambda_0 \int_0^1 Y^{\mathrm{T}}Y\mathrm{d}\theta]\mathrm{d}s \\ &= \mathbb{E}\int_0^\tau e^{h_1 s}[(h_1 c_2 - \lambda_0)\int_0^1 Y^{\mathrm{T}}Y\mathrm{d}\theta]\mathrm{d}s \\ &\leq 0.\end{aligned} \tag{32}$$

Then, we obtain

$$\mathbb{E}\|Y(\cdot,\tau)\|^2 \leq e^{-h_1\tau}\mathbb{E}\|\Phi(\cdot)\|^2, \tag{33}$$

which explains that system (20) is robust exponentially mean square stable. Then it is easy to get that UMSRDS (3) is observer-based mean square robust exponentially stable. The proof is complete. ■

Obviously, there are coupling terms of unknown terms in inequality (23), which makes it difficult to solve K_i, L_i. The following LMI condition is given for easily finding solutions of K_i, L_i.

Theorem 3. *If there exist $E_i > 0, P_i, Q_i \in \mathbb{R}^{n\times n}$, such that*

$$\Xi_{3i} = \begin{pmatrix} \Gamma_{1i} & \Omega_{1i} & \Omega_{2i} & \Omega_{3i} & \Gamma_{2i} & \mathbf{0} \\ * & \Omega_{4i} & \Omega_{3i}^{\mathrm{T}} & Q_i & \mathbf{0} & \Gamma_{2i} \\ * & * & \Omega_{5i} & -P_i - Q_i & \mathbf{0} & \mathbf{0} \\ * & * & * & -\frac{\pi^2}{2}B_iE_i & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\Gamma_{3i} & \mathbf{0} \\ * & * & * & * & \mathbf{0} & -\Gamma_{3i} \end{pmatrix} < 0, \tag{34}$$

then system (3) is observer-based mean square robust exponentially stable, where $E_i^{-1} = \alpha_i I_n$, $\Gamma_{1i} = \mathrm{sym}[P_i + W_{1i}^{\mathrm{T}}W_{1i}E_i + C_i^{\mathrm{T}}C_iE_i + C_i^{\mathrm{T}}D_{1i}^{\mathrm{T}}D_{1i}C_iE_i + (1 + \lambda_{\max}(D_{2i}^{\mathrm{T}}D_{2i}))W_{2i}^{\mathrm{T}}W_{2i}E_i] + \delta_{ii}E_i$, $\Omega_{1i} = -P_i - Q_i + A_iE_i$, $\Omega_{2i} = -P_i + \frac{1}{2}A_iE_i$, $\Omega_{3i} = \frac{1}{2}(P_i + Q_i) - \frac{1}{4}A_iE_i$, $\Omega_{4i} = \mathrm{sym}(A_iE_i - Q_i) + D_{1i}D_{1i}^{\mathrm{T}}E_i + \delta_{ii}E_i$, $\Omega_{5i} = -\frac{\pi^2}{2}B_iE_i + \mathrm{sym}(P_i)$, $\Gamma_{2i} = [\sqrt{\delta_{i1}}E_i, ..., \sqrt{\delta_{i,i-1}}E_i, \sqrt{\delta_{i,i+1}}E_i, ...\sqrt{\delta_{i\mathcal{N}}}E_i]$, $\Gamma_{3i} =$

diag$\{E_1, ..., E_{i-1}, E_{i+1}, ...E_{\mathcal{N}}\}$. *In this case, suitable control gain and observer gain are given by* $K_i = P_i E_i^{-1}$ *and* $L_i = Q_i E_i^{-1}$*, respectively.*

Proof. Using $K_i = P_i E_i^{-1}$, $L_i = Q_i E_i^{-1}$, $\tilde{P}_i = [P_i E_i^{-1} \quad - P_i E_i^{-1}; \mathbf{0} \quad \mathbf{0}], \tilde{Q}_i = [\mathbf{0} \quad -Q_i E_i^{-1}; \mathbf{0} \quad \mathbf{0}], \bar{Q}_i = [\mathbf{0} \quad \mathbf{0};\ \mathbf{0} \quad -Q_i E_i^{-1}], \bar{E}_i = [E_i \quad \mathbf{0};\ \mathbf{0} \quad E_i]$ and performing the congruence transformation by $\bar{E}_i$ to (23), we get

$$\Xi_{2i} = \begin{pmatrix} \Theta_{2i} + \sum_{j\in\mathcal{H}} \delta_{ij} \bar{E}_i \bar{E}_j^{-1} \bar{E}_i & \Theta_{3i} \\ * & \Theta_{4i} \end{pmatrix} < 0, \tag{35}$$

where $\Theta_{2i} = \mathrm{sym}(\tilde{P}_i\bar{E}_i + \tilde{Q}_i\bar{E}_i + \tilde{A}_i\bar{E}_i + \bar{Q}_i\bar{E}_i) + G_{1i}\bar{E}_i + G_{2i}\bar{E}_i$, $\Theta_{3i} = \mathrm{sym}(-\tilde{P}_i\bar{E}_i - \tilde{Q}_i\bar{E}_i) - \tilde{A}_i\bar{E}_i - \bar{Q}_i\bar{E}_i$, $\Theta_{4i} = -\frac{\pi^2}{2}\bar{B}_i\bar{E}_i + \mathrm{sym}(\tilde{P}_i\bar{E}_i + \tilde{Q}_i\bar{E}_i)$.

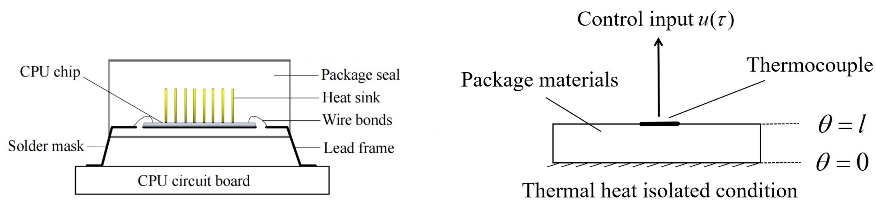
In order to simplify the coupling terms $\bar{E}_i \sum_{j\in\mathcal{H}} \delta_{ij} \bar{E}_j^{-1} \bar{E}_i$, using Schur's complement lemma we have

$$\Xi_{3i} = \begin{pmatrix} \Theta_{2i} + \delta_{ii}\bar{E}_i & \Theta_{3i} & \Theta_{5i} \\ * & \Theta_{4i} & \mathbf{0} \\ * & * & -\Theta_{6i} \end{pmatrix} < 0, \tag{36}$$

where $\Theta_{5i} = \mathrm{diag}\{\Gamma_{2i}, \Gamma_{2i}\}$, $\Theta_{6i} = \mathrm{diag}\{\Gamma_{3i}, \Gamma_{3i}\}$. In order to facilitate calculation, we expand the matrix block in (36), and then we can get LMI (34). ■

4. Simulation Studies

In order to illustrate the effectiveness of the proposed theoretical results, we apply them to the CPU heat dissipation model with a surface package. Fig. 1(a) details a schematic diagram of a surface-mounted package semiconductor within an electronic device. For simplicity, we consider a package that seals the CPU with a uniform thermal conductor of consistent thickness, as shown in Fig. 1(b). In fact, the heat is mainly dissipated through the top of the package seal and the heat sink. To consider the maximum heat dissipation design of the controller, it is assumed that there is a thermal insulation condition at $\theta = 0$ at the bottom of the package seal. For detailed instructions please refer to references [49, 50, 51].



(a) CPU heat dissipation model with surface package (b) A simplified CPU heat dissipation model

Figure 1: CPU heat dissipation model.

Here, we use the forward Euler method for time τ and use the central difference scheme for space θ to construct the following numerical examples.

Example 1. *During the operation of the CPU, due to sudden disturbance of the environment, the structure or parameters of the CPU thermal dissipation modeling system will have stochastic mutations, therefore, system (37) is used to accurately depict the above CPU thermal dissipation model.*

$$\mathrm{d}w(\theta,\tau) = \Big[B_i \frac{\partial^2 w(\theta,\tau)}{\partial \theta^2} + (A_i + \Delta A_i) w(\theta,\tau)\Big]\mathrm{d}\tau + (C_i + \Delta C_i) w(\theta,\tau)\mathrm{d}\mathcal{W}(\tau), \tag{37}$$

where $\theta \in (0,1)$ and $w(\theta,\tau)$ represents the temperature of the CPU. $\mathcal{W}(\tau)$ represents the disturbance of external white noise during the CPU operation. $l = 1$ is the thickness of the homogeneous thermal conductor. $B_i = \frac{k_i}{\rho C_{pi}}$ is the thermal diffusion coefficient determined by the heat conductivity coefficient k_i and the volumetric heat capacity coefficient ρC_{pi}, A_i is the internal temperature effect, and C_i is the stochastic coefficients. ΔA_i and ΔC_i denote uncertainties of coefficient. We consider temperature system (37) with two modes, i.e. $\gamma(\tau) = i,\ i \in \{1,2\}$ and the transition rate matrix is

$$\Psi = \begin{pmatrix} -3.5 & 3.5 \\ 2.4 & -2.4 \end{pmatrix}.$$

System parameters are as follows: $B_1 = 0.9, B_2 = 0.85, A_1 = 0.3, A_2 =$

$0.25, C_1 = 0.4, C_2 = 0.3,$

$$\Delta A_1 = 0.06 \exp(-\frac{1}{3}\theta)\cos(1.9\tau),$$
$$\Delta A_2 = 0.16 \exp(-\frac{1}{3}\theta)\cos(1.9\tau),$$
$$\Delta C_1 = 0.02 \exp(-\frac{1}{2}\theta)\cos(0.9\tau),$$
$$\Delta C_2 = 0.18 \exp(-\frac{1}{2}\theta)\cos(0.9\tau),$$

where a set of uncertain matrices satisfying the specified conditions is used when plotting the graph to intuitively verify the validity of the results.

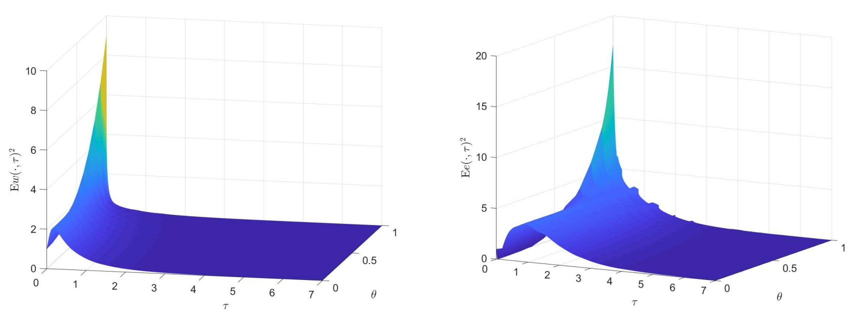
The initial temperature of system (37) and the initial mode are

$$w(\theta, 0) = 1 + 2\theta^2, \quad \gamma_0 = 2,$$

and the boundary conditions are (6).

We adopt boundary-output-based observer (9) with boundary conditions (11) for system (37). The initial value of observer (9) is

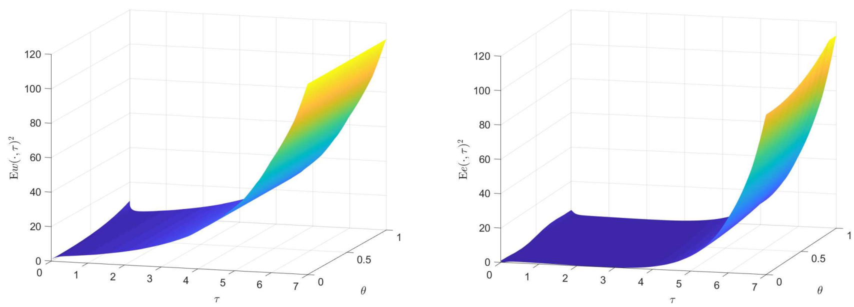
$$\hat{w}(\theta, 0) = 1 + \cos 5\theta.$$



(a) $\mathbb{E}w(\theta,\tau)$ (b) $\mathbb{E}e(\theta,\tau)$

Figure 2: Temperatures of system (37) and error system (12) with boundary controller.

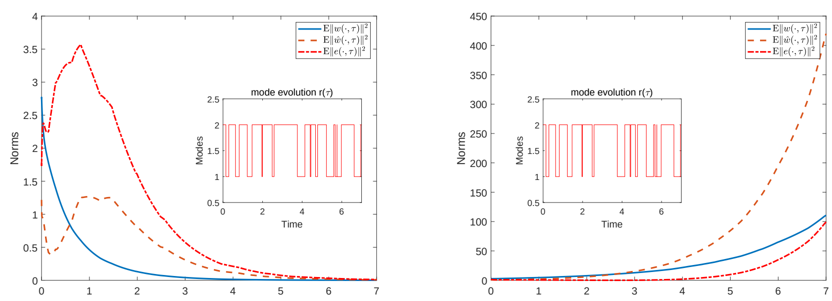
By solving inequality (34) we obtain the control gains and observer gains are $K_1 = -9.3173, K_2 = -10.0677, L_1 = 1.9926, L_2 = 1.8938$. *Then the temperatures of system (37) and error system (12) with the boundary controller are*



(a) $\mathbb{E}w(\theta, \tau)$

(b) $\mathbb{E}e(\theta, \tau)$

Figure 3: Temperatures of system (37) and error system (12) without boundary controller.



(a) Norms with boundary controller

(b) Norms without boundary controller

Figure 4: Norms with and without boundary controller.

obtained, shown as Fig. 2, in which it is seen that temperatures of system (37) and the error system (12) are mean square stable under the SMBCr. This shows that our designed observer (9) can estimate system (37) well. When system (37) is without the SMBCr, we get Fig. 3, in which the temperatures of system (37) are unstable. Furthermore, comparing Fig. 4(a) and Fig. 4(b), the norms of system (37) tend to zero under the SMBCr, but not when without the SMBCr. To sum up, the SMBC strategy we proposed is effective.

5. Conclusion

In this paper, the mean square robust stability is studied for uncertain Markovian switching stochastic reaction-diffusion systems (UMSRDS). Firstly, considering that states of UMSRDS can not be completely accessible in practical applications, an observer based on boundary output is designed to estimate the states of the system. A mode-dependent integral sliding mode surface (SMS) is established based on the observer. Moreover, the observer-based sliding mode boundary controller (SMBCr) is presented, which guarantees the almost sure reachability of the predefined SMS. Then, a mode-dependent criterion is obtained that ensures the mean square robust stability of the resultant system. Finally, theoretical results are applied to a CPU thermal model to further prove its validity. In the future, we will consider the SMBC for semi-Markovian switching reaction-diffusion systems because of the wide existence of semi-Markov switching in applications.

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