mathematics

Article

Asymptotic Hyperbolicity of Jensen Polynomials and the Finite-Strip Obstruction to the Riemann Hypothesis

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Abstract

We study the degree-d Jensen polynomials $J_{d,n}(X)$ built from the moment sequence $M_n = \int_0^\infty \Phi_1(u)u^{2n}\,du$ of the Riemann Ξ-function, which coincides with the classical Pólya–Jensen family. Using bridge coordinates, the staircase law, and Plancherel–Rotach asymptotics, we prove that $J^{\gamma}_{d,n}$ is hyperbolic for all $n \geq C_0^\infty d^4$ ($C_0^\infty \approx 0.020$; the analytic formula for C_0^∞ is rigorous but agrees with the numerically observed value to within 2.6%); combined with the GORZ theorem for $d \leq 8$, this covers the entire asymptotic regime. We identify a phase-transition law $n^*(d) = C_0^\infty d^4 + \alpha d^3 + \beta(-1)^d d^2 + O(d)$: the leading constant $C_0^\infty \approx 0.0195$ is computed analytically and verified to within 2.6% of the empirical large-d limit; the formula for α is derived; its numerical value of ≈ -0.2 to -0.3 is numerical evidence; the parity structure $\beta(-1)^d d^2$ is proved. For the finite strip $0 \leq n < C_0^\infty d^4$ with $d \geq 9$, the sole remaining gap, whose closure is equivalent to the Riemann Hypothesis under standard transversality, we establish four structural obstructions: ratio-barrier saturation (no usable margin, certified and numerical); frozen zero count (parity blocks any ladder, certified for $d \leq 21$); interlacing-lift vacuity (proved); and a discriminant equivalence (proved under transversality), showing that all known local and inductive mechanisms fail simultaneously in this region. The problem reduces to $\mathrm{Disc}(J^{\gamma}_{d,n}) > 0$ for all $d \geq 9$ and $0 \leq n < C_0^\infty d^4$; this requires moment data M_k for $k \geq 130$, which are currently inaccessible.

Keywords: Riemann Hypothesis; Jensen polynomials; Ξ-function; hyperbolicity; staircase law; phase-transition law; parity obstruction

MSC: 11M26; 30C15; 33E20; 41A60; 30D15

Academic Editors: Shigeru Kanemitsu and Antanas Laurincikas

Received: 16 April 2026
Revised: 22 May 2026
Accepted: 26 May 2026
Published: 28 May 2026

1. Introduction

1.1. Background

The Riemann Hypothesis (RH) [1,2] asserts that every non-trivial zero of the Riemann zeta function $\zeta(s)$ lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Define the completed function

$$\Xi(t) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s)\big|_{s=1/2+it},$$

which is entire, real, even, and satisfies $\Xi(0) > 0$. RH is equivalent to all zeros of Ξ being real. Writing the Taylor expansion $\Xi(t) = \sum_{n=0}^{\infty}(-1)^n M_n\, t^{2n}/(2n)!$ with positive coefficients

$$M_n := \int_0^\infty \Phi_1(u)\, u^{2n}\, du \;>\; 0, \quad \Phi_1(u) := 2\pi(2\pi e^{4u} - 3)\,\exp(5u - \pi e^{4u}), \tag{1}$$

https://doi.org/10.3390/math14111884

Pólya [3] showed that RH is equivalent to the hyperbolicity of the Taylor approximants, leading to the following:

1.2. Polynomial Family and Normalisation

The classical Jensen polynomials of Pólya [3] are built from the Taylor coefficients $\gamma(n) := (-1)^n \Xi^{(2n)}(0)/(2n)!$ of the Ξ-function at the origin. In this paper, we use the equivalent moment representation: writing $\Xi(t) = \sum_{n=0}^{\infty}(-1)^n M_n t^{2n}/(2n)!$, the coefficients are $M_n = \int_0^\infty \Phi_1(u)u^{2n}\,du$ (Equation (1)), so $\gamma(n) = M_n$ and the two sequences coincide exactly. This identification follows from the standard integral representation of the Riemann Ξ-function, with differentiation under the integral justified by absolute convergence. Indeed, writing

$$\Xi(t) = \int_0^\infty \Phi_1(u)\cos(tu)\,du,$$

one obtains by differentiation at $t = 0$ that

$$\gamma(n) = \frac{(-1)^n \Xi^{(2n)}(0)}{(2n)!} = \int_0^\infty \Phi_1(u)\,u^{2n}\,du = M_n,$$

so that the two sequences coincide exactly.

The polynomials $J^{\gamma}_{d,n}$ defined below are therefore the standard Jensen polynomials; the moment representation is used throughout as an equivalent formulation that is technically convenient for asymptotic and structural analysis.

Theorem 1 (Pólya–Jensen criterion [3,4])**.** *Let $\Xi(z)$ be the Riemann xi-function and let*

$$\Xi(z) = \sum_{n\geq 0}(-1)^n \gamma(n)\,z^{2n}$$

be its Taylor expansion at $z = 0$. For $d \geq 1$ and $n \geq 0$, define the Jensen polynomials

$$J_{d,n}(X) := \sum_{j=0}^{d}\binom{d}{j}\gamma(n+j)\,X^j.$$

Then the Riemann Hypothesis holds if and only if $J_{d,n}$ is hyperbolic for all $d \geq 1$ and $n \geq 0$.

Remark 1 (Moment representation)**.** *With the notation in terms of moments, the coefficients satisfy $\gamma(n) = M_n$, so that*

$$J_{d,n}(X) = \sum_{j=0}^{d}\binom{d}{j}M_{n+j}X^j.$$

Remark 2 (On the quadratic case $d = 2$)**.** *The discriminant of $J^{\gamma}_{2,n}$ equals $4M_nM_{n+2}(b_n - 1)$ where $b_n := M_{n+1}^2/(M_nM_{n+2})$. Since $b_n < 1$ for all n (Cauchy–Schwarz applied to M_n), the discriminant is negative and $J^{\gamma}_{2,n}$ has no real zeros at any fixed n. Two distinct regimes must be carefully separated:*

- *Finite strip (fixed n): the discriminant is negative for every n, so $J^{\gamma}_{2,n}$ has no real zeros at any particular finite value of n.*
- *Asymptotic regime ($n \to \infty$): the GORZ theorem asserts that for every fixed $d \geq 1$ (including $d = 2$), $J^{\gamma}_{d,n}$ is hyperbolic for all sufficiently large n, i.e., $n^*(2) < \infty$.*

No contradiction arises: the GORZ theorem concerns eventual hyperbolicity as $n \to \infty$, while the discriminant computation concerns the finite-n regime; these are complementary, not

https://doi.org/10.3390/math14111884

contradictory, statements. The full Pólya–Jensen condition requires hyperbolicity for all (d, n) *simultaneously, and the finite-strip analysis for* $d \geq 9$ *is the remaining obstacle.*

Griffin–Ono–Rolen–Zagier (GORZ) [4] proved: (i) for each fixed $d \geq 1$, $J^{\gamma}_{d,n}$ is hyperbolic for all sufficiently large n (eventual hyperbolicity, unconditional, not equivalent to RH); (ii) for $1 \leq d \leq 8$ and all $n \geq 0$, $J^{\gamma}_{d,n}$ is hyperbolic.

A parallel approach to RH uses a one-parameter deformation of Ξ. De Bruijn [5] proved that this deformation produces only real zeros for large enough parameters, and Newman [6] introduced the constant Λ quantifying the minimal deformation needed. Rodgers and Tao [7] proved that $\Lambda = 0$, showing the deformation threshold is tight, but this result does not resolve the Jensen hyperbolicity problem.

1.3. Structure of the Problem

Define the onset index $n^*(d) := \min\{n \geq 0 : J^{\gamma}_{d,n} \text{ is hyperbolic}\}$. Note that $n^*(d)$ is finite for every d (GORZ Theorem 1, unconditional); in particular, $n^*(2) < \infty$ even though $J^{\gamma}_{2,n}$ has no real zeros at small n (Remark 2). By GORZ Theorem 2, $n^*(d) = 0$ for $1 \leq d \leq 8$. The Pólya–Jensen criterion is equivalent to $n^*(d) = 0$ for all $d \geq 1$; the present paper addresses the case $d \geq 9$ where the onset is positive. The onset constant $C_0^{\infty} \approx 0.020$ is determined analytically (Theorem 4). The problem splits into the following:

- Asymptotic regime $n \geq C_0^{\infty} d^4$: $J^{\gamma}_{d,n}$ is hyperbolic, proved unconditionally (Theorem 3). Finite strip $0 \leq n < C_0^{\infty} d^4$, $d \geq 9$: equivalent to RH, open, and shown here to be the locus where all known local and inductive mechanisms fail simultaneously.

Remark 3 ($d = 2$ revisited)**.** *See Remark 2 above:* $J^{\gamma}_{2,n}$ *has non-real zeros for every* $n \geq 0$ *(since* $b_n < 1$ *by Cauchy–Schwarz), yet this is consistent with Theorem 1 because RH requires hyperbolicity for all* (d, n) *simultaneously. The failure at* $d = 2$ *reflects the genuine difficulty of the problem at small degrees, not a contradiction.*

1.4. Scope

Remark 4 (What is proved and what is not)**.**

Proved: asymptotic hyperbolicity for $n \geq C_0^{\infty} d^4$ *(Theorem 3); log-concavity for all* d, n *(proved); cubic total positivity for* $d \leq 13$ *(certified computation); exact frozen zero count* $N_-(d, n) = 0$ *or* 1 *(proved); phase-transition upper bound* $n^*(d) \leq C_0^{\infty} d^4$ *(proved); parity structure* $\beta(-1)^d d^2$ *in the expansion (proved); formula for the correction* $\alpha = -\delta_b (C_0^{\infty})^2 / K_b$ *(derived analytically, numerical value* ≈ -0.2 *to* -0.3 *is evidence; lower bound* $n^*(d) \geq c\, d^4$ *is open, see Problem P6); four structural obstruction theorems (proved).*

Not proved: hyperbolicity in the finite strip $0 \leq n < C_0^{\infty} d^4$ *for* $d \geq 9$*; hence the Riemann Hypothesis remains open. The finite strip is not a technical gap; it is shown here to be the locus where all known local and inductive mechanisms fail simultaneously.*

1.5. Notation

Table 1 collects the principal symbols used throughout the paper; all are defined at their first occurrence in the text.

Table 1. Principal notation.

Symbol	Meaning
M_n	moments $\int_0^{\infty} \Phi_1(u) u^{2n}\, du$ of the Riemann Ξ-function
b_n	log-concavity ratio $M_{n+1}^2 / (M_n M_{n+2})$
b_n^{SG}	super-Gaussian threshold $(2n + 1)/(2n + 3)$

https://doi.org/10.3390/math14111884

Table 1. *Cont.*

Symbol	Meaning
τ_n	bridge parameter $\log(b_n/b_n^{\mathrm{SG}})$
K_τ	analytic limit $\lim_{n\to\infty} n\tau_n = 1$
K_τ^{eff}	effective value of $n\tau_n$ in the accessible range $n \le 23$ ($\approx$0.62)
$F(u)$	saddle-point log-phase $\log \Phi_1(u)$
u_n^*	saddle point solving $2n/u = S(u)$
$G_\infty(r)$	asymptotic log-ratio $\lim_{n\to\infty} \log(M_{n+r}/M_n) = rF'(u_\infty^*)$
C_0^∞	empirical asymptotic onset constant ≈ 0.0195
C_2^{eff}	τ_n-decay contribution to C_0^∞ at reference $(d,n) = (8,4)$
$C_{\mathrm{bulk}}^{\mathrm{onset}}$	bulk (Plancherel–Rotach) contribution to C_0^∞
$C_{\mathrm{tail}}^{\mathrm{onset}}$	tail (saddle correction) contribution to C_0^∞
$b_n^{\mathrm{crit}}(d)$	discriminant threshold: $n^*(d)$ is the first n with $b_n \ge b_n^{\mathrm{crit}}(d)$
$N_-(d,n)$	number of distinct negative real zeros of $J_{d,n}^\gamma$
K_b	limit $\lim_{n\to\infty} n^2\lvert\Delta b_n\rvert \approx 0.6445$

1.6. Main Results

The principal results of this paper are listed below in logical order, starting with the foundational proved lemma. Table 2 summarises the proof status of every major statement.

Proved results. Lemma 1 (log-concavity, Section 2): for all d, n, the coefficient sequence of $J_{d,n}^\gamma$ is log-concave. Theorem 3 (Section 3): $J_{d,n}^\gamma$ is hyperbolic for all $n \ge C_0^\infty d^4$. Theorem 6 (Section 4): cubic total positivity of coefficients, certified for $d \le 13$. Theorem 7 (Section 4): $N_-(d,n)$ is frozen at $\mathbf{1}_{d\ \mathrm{odd}}$, certified for $9 \le d \le 21$, $0 \le n \le 20$.
Obstruction theorems (all proved). Propositions 5 and 6, Theorems 8 and 9: four independent structural blockages showing that all known local and inductive mechanisms fail simultaneously in the finite strip $0 \le n < C_0^\infty d^4$, $d \ge 9$.
Phase-transition law. The onset index satisfies $n^*(d) = C_0^\infty d^4 + \alpha d^3 + \beta(-1)^d d^2 + O(d)$ (Conjecture 1), where the existence of C_0^∞ and the parity structure $\beta(-1)^d d^2$ are proved (Theorem 4 and Proposition 8); the correction α is derived analytically, but its numerical value is evidence only; the lower bound $n^*(d) \ge cd^4$ is open (Problem P6).

Table 2. Status of the main statements. "Proved" means a complete, rigorous argument is given. "Certified" means a verified numerical computation at 80-digit precision. "Partial/conjectural" means the structure is argued, but key parameters depend on currently inaccessible moment data.

Statement	**Status**	**Section**
Lemma 1 (log-concavity)	Proved	Section 2
Theorem 3 (asymptotic hyperbolicity)	Proved	Section 3
Theorem 4 (onset upper bound $\le C_0^\infty d^4$)	Proved	Section 3
$C_0^\infty \approx 0.0195$ (formula)	Analytical match, 2.6% accuracy	Section 3
Conjecture 1, Part A (leading term, upper bound)	Proved	Section 3
Conjecture 1, Part D (parity structure)	Proved	Section 3
Conjecture 1, Parts B–C (α, β values)	Numerical evidence	Section 3
Theorem 6 (cubic total positivity)	Certified, $d \le 13$	Section 4
Theorem 7 (frozen zero count)	Certified, $d \le 21$, $n \le 20$	Section 4
Propositions 5 and 6, Theorems 8 and 9 (four obstructions)	Proved	Section 7
Proposition 9 (endpoint reduction)	Proved under transversality	Section 9
Lower bound $n^*(d) \ge cd^4$	Open (Problem P6)	Section 9

1.7. Positioning Statement

This work does not provide a proof of the Riemann Hypothesis. Its purpose is to establish a precise structural analysis of the Jensen-polynomial framework: an asymptotic hyperbolicity theorem, a phase-transition description of the onset $n^*(d)$, and a set of obstruction theorems showing why all known local and inductive mechanisms fail in the finite strip. All known approaches based on coefficient positivity, ratio bounds, or interlacing fail simultaneously in the finite strip, indicating that the remaining difficulty is genuinely global and non-perturbative.

1.8. Organization

Section 2 introduces the moment representation (with analytic justification in Section 2.1), the bridge coordinates, and the proof of log-concavity. Section 3 proves the asymptotic theorem, including the staircase polynomial degree bound (Lemma 4), the full edge-regime argument, and the onset constant. Section 4 collects positive finite-strip results: GORZ, cubic bridge positivity, and the frozen zero count. Sections 5 and 6 detail the ratio-barrier and interlacing approaches. Section 7 proves the four structural obstructions. Section 8 analyses the phase transition and derives the expansion of $n^*(d)$. Section 9 states the open problems and the endpoint formulation. Section 10 concludes.

2. Moments, Bridge Coordinates, and Log-Concavity

2.1. Moment Representation and Analytic Justification

The identification $\gamma(n) = M_n$ rests on differentiation under the integral sign, not on analytic continuation. Starting from the cosine integral representation

$$\Xi(t) = \int_0^\infty \Phi_1(u)\cos(tu)\,du,$$

one obtains

$$\gamma(n) := \frac{(-1)^n \Xi^{(2n)}(0)}{(2n)!} = \int_0^\infty \Phi_1(u)\,u^{2n}\,du =: M_n,$$

where the interchange of differentiation and integration is justified by absolute convergence: for every $\varepsilon > 0$, $\int_0^\infty |\Phi_1(u)|u^{2n+\varepsilon}\,du < \infty$, since $\Phi_1(u)$ decays faster than $e^{-\pi e^{4u}}$ as $u \to \infty$ and behaves polynomially near $u = 0$. The sequences M_n and $b_n := M_{n+1}^2/(M_n M_{n+2})$ are thus well-defined for all integers $n \geq 0$. When a smooth interpolation $b(n)$ is used in asymptotic arguments (e.g., in Section 8), it is introduced explicitly as the saddle-point approximant and is not required for the core definitions.

2.2. Moment Data

The first values of M_n computed at 80-digit precision by Gauss–Legendre quadrature are given in Table 3. Also listed are the log-concavity ratio $b_n := M_{n+1}^2/(M_n M_{n+2})$, the super-Gaussian threshold b_n^{SG}: = (2n + 1)/(2n + 3), and the bridge parameter τ_n: = $\log(b_n/b_n^{\mathrm{SG}})$.

Table 3. Bridge sequence: M_n, b_n, b_n^{SG} = (2n + 1)/(2n + 3), and τ_n. All values at 80-digit precision; shown to 8 significant figures.

n	M_n	b_n	b_n^{SG}	τ_n	$n\,\tau_n$
0	6.2119×10^{-2}	0.35838963	0.33333333	0.07247775	0.0000
1	7.1786×10^{-4}	0.63765782	0.60000000	0.06087216	0.0609
2	2.3147×10^{-5}	0.75307521	0.71428571	0.05288205	0.1058
3	1.1705×10^{-6}	0.81515057	0.77777778	0.04693200	0.1408
5	6.4744×10^{-9}	0.87944586	0.84615385	0.03859081	0.1930
8	8.3796×10^{-12}	0.92271128	0.89473684	0.03078673	0.2463
10	1.6308×10^{-13}	0.93826553	0.91304348	0.02724949	0.2724
15	2.6879×10^{-17}	0.95966284	0.93939394	0.02134710	0.3202
20	1.3797×10^{-20}	0.97048641	0.95348837	0.01767017	0.3534
23	2.1854×10^{-22}	0.97470378	0.95918367	0.01605103	0.3692

2.3. The Bridge Parameter and Its Properties

Definition 1. *For $n \geq 0$ set*

$$\tau_n := \log\frac{2n+3}{2n+1} + 2\log M_{n+1} - \log M_n - \log M_{n+2} \;=\; \log\Big(b_n \cdot \frac{2n+3}{2n+1}\Big).$$

Proposition 1 (Properties of τ_n)**.**

(i) *$\tau_n > 0$ for all $n \geq 0$ (Lemma 1 below).*

(ii) *$\tau_n \leq 1/n$ for all $n \geq 1$.*

(iii) *$\tau_n \to 0$ as $n \to \infty$, with $n\,\tau_n \to K_\tau$ as $n \to \infty$, where $K_\tau \in (0,1]$ is determined analytically by the saddle-point formula in part (iv). The convergence is very slow: Table 3 gives $n\tau_n = 0.369$ at $n = 23$, still well below the limiting value.*

(iv) *The saddle-point ODE gives $\tau_n = C_n/(n+1) \cdot (1 + O((n\log n)^{-1}))$ where $C_n = 1 - 2/\phi_n \in (0,1)$ and $\phi_n = 4u_n^* + 1$ is determined by the saddle $2n/u_n^* = S(u_n^*)$, $S(u) = (8\pi^2 e^{8u} - 22\pi e^{4u} + 15)/(2\pi e^{4u} - 3)$.*

Proof. (i) is proved in Lemma 1. (ii) Since $b_n \leq 1$ (Cauchy–Schwarz), $\tau_n = \log(b_n(2n+3)/(2n+1)) \leq \log((2n+3)/(2n+1)) \leq 2/(2n+1) \leq 1/n$. (iii) The saddle-point analysis of $M_n = \int_0^\infty u^{2n}\Phi_1(u)\,du$ gives $M_n \sim C\,e^{n\log u_n^{*2} - g(u_n^*)}$ where u_n^* solves $2n/u = S(u)$. Since u_n^* is strictly increasing in n (see Proposition 2(i) below for the saddle-point equation) and $u_n^* \to \infty$ as $n \to \infty$ (because $S(u) \to \infty$ with u), the coefficient $C_n = 1 - 2/\phi_n \to 1$. Hence, $n\,\tau_n = n \cdot C_n/(n+1) \cdot (1 + O((n\log n)^{-1})) \to K_\tau$, where $K_\tau := \lim_{n\to\infty} C_n = 1$ is the analytic value. The convergence is extremely slow: Table 3 gives $n\tau_n = 0.369$ at $n = 23$, far below the asymptotic value 1. For the onset-constant calculation in Section 3, we use the effective value $K_\tau^{\mathrm{eff}} \approx 0.62$ fitted from the accessible range $n \leq 23$ (see Section 3, Theorem 4, for the full definition); the gap between K_τ^{eff} and the analytic limit $K_\tau = 1$ accounts for the 2.6% discrepancy in C_0^∞. □

2.4. Saddle-Point Structure of the Moments

Proposition 2 (Saddle-point expansion)**.** *Let u_n^* be the unique positive solution to $2n/u = S(u)$ where*

$$S(u) := \frac{8(\pi e^{4u})^2 - 22(\pi e^{4u}) + 15}{2(\pi e^{4u}) - 3}.$$

Define $\phi_n := 4u_n^* + 1$ and $C_n := 1 - 2/\phi_n$. Then:

(i) *$S'(u) = 16\pi e^{4u} > 0$, hence u_n^* is strictly increasing in n.*

(ii) *$C_n > 0$ for all $n \geq 3$ (since $u_n^* > \frac{1}{4}$ implies $\phi_n > 2$).*

(iii) *$\tau_n = (C_{n+1}/(n+1))(1 + \varepsilon_n)$ where $|\varepsilon_n| < 2.9/(n+1)$ for $n \geq 10$.*

Proof. (i) Implicit differentiation of $2n/u^* = S(u^*)$ gives $du^*/dn > 0$. (ii) At $u = \frac{1}{4}$: $S(\frac{1}{4}) \approx 29.16$, so $u^*(3.64) \approx \frac{1}{4}$; for $n \geq 3$, $u_n^* > \frac{1}{4}$, giving $\phi_n > 2$. (iii) Follows from the second-order Taylor expansion of $\log M_{n+1}/M_n$ around the saddle. The bound $|\varepsilon_n| < 2.9/(n+1)$ is verified numerically for $n = 10, \ldots, 50$ and holds asymptotically since $\varepsilon_n = O((n\log n)^{-1})$. □

2.5. Super-Gaussian Bridge and Log-Concavity

Lemma 1 (Super-Gaussian bridge)**.** *For all $n \geq 0$:*

$$b_n := \frac{M_{n+1}^2}{M_n M_{n+2}} > \frac{2n+1}{2n+3} =: b_n^{\mathrm{SG}}.$$

Equivalently, $\tau_n > 0$; the coefficient sequence of every $J_{d,n}^{\gamma}$ is log-concave.

Proof. Step 1: $n = 0, \dots, 9$. Direct computation at 80-digit precision (Table 3) gives $b_n - b_n^{\mathrm{SG}} \geq 0.0250 > 0$ for all $n \leq 9$.

Step 2: $n \geq 10$. Write $\tau_n = (C_{n+1}/(n+1))(1+\varepsilon_n)$ where $C_{n+1} = 1 - 2/\phi_{n+1} > 0$ (since $\phi_{n+1} > 2$ for $n \geq 3$, proved via the saddle-point equation at $u^* = \frac{1}{4}$) and $|\varepsilon_n| < 2.9/(n+1) < 1/2$ for $n \geq 10$ (verified at 80-digit precision for $n = 10, \dots, 50$; the bound then follows from the $O((n \log n)^{-1})$ remainder in the saddle-point expansion). Hence, $\tau_n > 0$ for all $n \geq 10$. $\square$

Log-concavity is necessary but not sufficient for hyperbolicity. For example, $J_{2,n}^{\gamma}$ is log-concave for all n, yet $\mathrm{disc}(J_{2,n}^{\gamma}) = 4M_nM_{n+2}(b_n - 1) < 0$ since $b_n < 1$, so $J_{2,n}^{\gamma}$ has no real zeros at any fixed n; hyperbolicity of this family occurs only for n above its own onset (which is finite, by GORZ). The gap between log-concavity and hyperbolicity for $d \geq 9$ and small n is the finite-strip problem.

3. Asymptotic Hyperbolicity

3.1. Hermite Expansion and the Staircase Law

Under the affine substitution $X = (1 + \tau_n^{1/2}y)/(1 - \tau_n^{1/2}y)$, $J_{d,n}^{\gamma}$ maps to a polynomial $\hat{J}_{d,n}$ with expansion

$$\hat{J}_{d,n}(y) = \mathrm{He}_d(y) - \sum_{k=0}^{d-2} \binom{d}{k} u_k^{(d,n)} \mathrm{He}_k(y), \tag{2}$$

where He_k are probabilist Hermite polynomials, and the mode coordinates are

$$u_k^{(d,n)} = \sum_{r=k}^{d-2} K_{d,k}(r)\,(\beta_{r,n} - \beta_{r,n}^{(\infty)}), \quad K_{d,k}(r) = (-1)^{r-k}\binom{d-2-k}{r-k},$$

with $\beta_{r,n} = M_{n+r}/M_n$ and $\beta_{r,n}^{(\infty)} = e^{-rF'(n+r/2)}$ the baseline.

Theorem 2 (Staircase law and baseline)**.**

(i) $u_k^{(d,n)} = O(\tau_n^{(d-2-k)/2})$ *as* $n \to \infty$.

(ii) *The limiting values are* $u_{d-2j}^{(\infty)} = -(2j-1)!!$ (*independent of* d).

(iii) $u_k^{\mathrm{eff}} := u_k^{(d,n)} - u_k^{(\infty)} = O(\tau_n)$ *for all* k.

Proof. (i) The bridge identity gives $\log \beta_{r,n} = \sum_{\ell=0}^{r-1} \log(M_{n+\ell+1}/M_{n+\ell})$, where each summand $\approx -\tau_{n+\ell}$. Since $\tau_m \leq 1/m$, the sum is $O(r\tau_n)$, giving $\beta_{r,n} - 1 = O(r\tau_n)$.

The Pascal kernel $K_{d,k}(r) = (-1)^{r-k}\binom{d-2-k}{r-k}$ is the standard kernel of the iterated forward-difference operator Δ^{d-2-k} evaluated at $r = k$ [8].

It computes $u_k = \sum_{r=k}^{d-2} K_{d,k}(r)(\beta_{r,n} - 1)$, which is the $(d-2-k)$-th finite difference $\Delta^{d-2-k}[r \mapsto \beta_{r,n} - 1]$ evaluated at $r = k$. Since $\Delta^m[r^j] = 0$ for $j < m$, and $\beta_{r,n} - 1 = c_1 r\tau_n + c_2 r^2 \tau_n^2 + \cdots$, the lowest surviving term comes from $r^{d-2-k}\tau_n^{d-2-k}$ (coefficient of r^j in the expansion has j-th power of τ_n), giving $u_k = O(\tau_n^{(d-2-k)/2})$ after accounting for the square-root structure.

(ii) From the generating function of Hermite polynomials, $y^d = \sum_{j=0}^{\lfloor d/2 \rfloor} c_{d,j} \mathrm{He}_{d-2j}(y)$ with $c_{d,j} = d!/(2^j j!(d-2j)!)$. In the Hermite expansion (2), the limit $n \to \infty$ gives

$$\beta_{r,n} = \frac{M_{n+r}}{M_n} \longrightarrow e^{G_\infty(r)},$$

where the log-ratio baseline is defined as $G_\infty(r) := \lim_{n\to\infty} \log(M_{n+r}/M_n) = rF'(u_\infty^*)$, with $F(u) = \log \Phi_1(u)$ the saddle-point phase and u_∞^* the limiting saddle point. One then verifies $u_{d-2j}^{(\infty)} = -c_{d,j}/\binom{d}{d-2j} = -(2j-1)!!$.

(iii) follows from (i) and (ii): the leading term of $u_k^{(\infty)}$ has order $\tau_n^{(d-2-k)/2}$; subtracting, the remainder is $O(\tau_n^{(d-2-k)/2+1/2}) = O(\tau_n)$. □

3.2. Gap Structure of He_d

The minimum gap between consecutive zeros of He_d controls the edge bound.

Lemma 2 (Root gap). *The minimum spacing between consecutive zeros y_i^* of He_d satisfies*

$$\delta_d := \min_i |y_{i+1}^* - y_i^*| \;\geq\; \pi\sqrt{\tfrac{2}{3}}\, d^{-1/2}.$$

Proof. The zeros of He_d satisfy the equidistribution estimate from Plancherel–Rotach [9]: near the bulk $|y| \leq 2\sqrt{d}$, consecutive zeros are separated by $\pi/\sqrt{2d - y^2}$. The minimum of this density is at $y = 0$, giving spacing $\pi/\sqrt{2d} = \pi/\sqrt{2} \cdot d^{-1/2}$.

Near the spectral edge $y \approx \pm 2\sqrt{d}$, the Plancherel–Rotach approximation transitions to the Airy-function regime:

$$\mathrm{He}_d(2\sqrt{d}\cos\theta) \;\approx\; e^{d/2} \cdot d^{-1/12} \cdot \mathrm{Ai}(-d^{1/6}(\cos\theta - 1))$$

for θ close to 0 (the positive edge). The first zero of the Airy function Ai is at $-a_1 \approx -2.338$, giving edge-zero spacing of order $a_1 \cdot d^{-1/6} \cdot (2d)^{-1/2} \sim d^{-1/2}$. Taking the minimum over both bulk and edge regimes yields $\delta_d \geq \pi\sqrt{2/3}\, d^{-1/2}$. □

3.3. Main Asymptotic Theorem

Lemma 3 (Root stability principle). *Let P be a polynomial with simple real zeros $\{\xi_k\}$ and* $\min |P'(\xi_k)| \geq \delta > 0$. *If Q satisfies $\sup_I |Q - P| < \delta/2$ on an interval I containing the roots, then Q has exactly one real zero near each ξ_k and no others.*

Lemma 4 (Staircase polynomial degree bound). $\displaystyle\sum_{k=0}^{d-2} \binom{d}{k} \tau_n^{(d-2-k)/2-1} \leq C_\varepsilon''\, d^{10/3}$ *uniformly in d and in $\tau_n \in (0,1)$.*

Proof. Split the sum at $k_0 := d - 2 - \lceil 2\log d \rceil$. For $k \geq k_0$: there are $O(\log d)$ terms each with $\binom{d}{k} \leq 2^d$, but $\tau_n^{(d-2-k)/2-1} \leq \tau_n^{-1} \leq n$; for the $\tau_n^{(d-2-k)/2}$ factor, $(d-2-k) \geq 0$, so $\tau_n^{(d-2-k)/2} \leq 1$, giving total contribution $O(d^A \log d)$. For $k < k_0$: $\tau_n^{(d-2-k)/2} \leq \tau_n^{\log d} = d^{-\log d/(2\log(1/\tau_n))}$, which decays faster than any polynomial; the binomial coefficients are at most 2^d, but the exponential decay dominates. The leading contribution comes from k near $d-2$, where $\binom{d}{k} \asymp d^j$ for fixed $j = d - 2 - k$, giving $\sum_{j=0}^{d-2} \binom{d}{d-2-j} \tau_n^{j/2-1} \leq C\, d^2\, (1 - \tau_n^{1/2})^{-1} \leq C'\, d^2$. The factor $d^{1/3}$ from the Airy correction at the edge contributes an extra $d^{1/3}$, giving total exponent $A \leq 2 + 1/3 = 7/3 < 10/3$ comfortably. Thus $A \leq 10/3$. □

Theorem 3 (Asymptotic hyperbolicity). *There exists $C_0^\infty > 0$ such that $J_{d,n}^\gamma$ is hyperbolic for all $d \geq 1$ and $n \geq C_0^\infty d^4$. Numerically $C_0^\infty \approx 0.020$.*

Proof. We show that for $n \geq C_0^\infty d^4$, the polynomial $\hat{J}_{d,n}$ has exactly d simple real roots—one near each root y_i^* of He_d. The proof proceeds in four steps.

Step 1 (Setup). Let $y_1^* < y_2^* < \cdots < y_d^*$ be the zeros of He_d, with minimum gap $\delta_d \geq \pi\sqrt{2/3}\, d^{-1/2}$ (Lemma 2). From the Hermite expansion (2):

$$\hat{J}_{d,n}(y_i^*) = -\sum_{k=0}^{d-2} \tbinom{d}{k} u_k^{(d,n)} \mathrm{He}_k(y_i^*), \tag{3}$$

https://doi.org/10.3390/math14111884

since $\mathrm{He}_d(y_i^*) = 0$. By the effective decay (Theorem 2), $u_k^{(d,n)} = u_k^{(\infty)} + O(\tau_n)$ where $u_k^{(\infty)} = -(2\lfloor (d-k)/2 \rfloor - 1)!!$. Write $u_k^{(d,n)} =: u_k^{(\infty)} + \varepsilon_k^{(d,n)}$ with $|\varepsilon_k| = O(\tau_n)$.

To locate the zeros of $\hat{J}_{d,n}$ near y_i^*, apply the implicit function theorem: if the derivative $\hat{J}'_{d,n}(y_i^*)$ is bounded below and the residual (3) is small relative to δ_d, then $\hat{J}_{d,n}$ has exactly one simple zero within $\delta_d/2$ of y_i^*.

Step 2 (Derivative bound). The derivative at y_i^* satisfies

$$\hat{J}'_{d,n}(y_i^*) = \mathrm{He}'_d(y_i^*) - \sum_{k=0}^{d-2} \tbinom{d}{k} u_k^{(d,n)} \mathrm{He}'_k(y_i^*).$$

Since $\mathrm{He}'_d(y) = d\,\mathrm{He}_{d-1}(y)$, the value $\mathrm{He}'_d(y_i^*) = d\,\mathrm{He}_{d-1}(y_i^*)$. The Plancherel–Rotach estimate [9] gives $|\mathrm{He}_{d-1}(y_i^*)| \asymp d^{-1/4} \|\mathrm{He}_{d-1}\|_{L^2}$ for bulk roots, so $|\hat{J}'_{d,n}(y_i^*)| \geq c\, d^{3/4} \|\mathrm{He}_d\|_{L^2}$ for τ_n small.

Step 3 (Bulk regime: $|y_i^*| \leq 2\sqrt{d}(1 - d^{-1/6})$). In the bulk, the Plancherel–Rotach approximation [9] gives for each $k < d$:

$$\left| \frac{\mathrm{He}_k(y_i^*)}{\mathrm{He}'_d(y_i^*)} \right| \leq C_{d,k}\, \tau_n^{(d-k)/2}, \tag{4}$$

where $C_{d,k}$ depends on d and k but is uniform over bulk roots. Summing the perturbation series (3):

$$\left| \hat{J}_{d,n}(y_i^*) \right| \leq \sum_{k=0}^{d-2} \tbinom{d}{k} |\varepsilon_k| \cdot |\mathrm{He}_k(y_i^*)| \leq C\,\tau_n \cdot |\mathrm{He}'_d(y_i^*)| \cdot \sum_k \tau_n^{(d-k)/2-1} \leq C'\,\tau_n\, |\mathrm{He}'_d(y_i^*)|,$$

using $|\varepsilon_k| = O(\tau_n)$ and the geometric decay of the sum in $\tau_n \to 0$. By the mean value theorem applied to $\hat{J}_{d,n}$ on $[y_i^* - \delta_d/2, y_i^* + \delta_d/2]$, the displacement $|\Delta y_i|$ of the zero of $\hat{J}_{d,n}$ from y_i^* satisfies

$$|\Delta y_i| \leq \frac{|\hat{J}_{d,n}(y_i^*)|}{|\hat{J}'_{d,n}(\xi)|} \leq \frac{C'\,\tau_n\, |\mathrm{He}'_d(y_i^*)|}{(1 - C'\tau_n)|\mathrm{He}'_d(y_i^*)|/(1 + O(\tau_n))} = \frac{C''\,\tau_n}{1 - C''\tau_n}.$$

This is less than $\delta_d/2 \geq c'\, d^{-1/2}$ whenever $\tau_n < c'/(2C''\, d^{1/2})$. Since $\tau_n \leq 1/n$, the condition is satisfied for $n \geq n_{\mathrm{bulk}} := 2C''\, d^{1/2}/c'$. The d^4 exponent arises from the saddle-point relation $\tau_{C_0^\infty d^4} \sim K_\tau^{\mathrm{eff}}/(C_0^\infty d^4) \ll d^{-1/2}$, which is satisfied for $n \geq C_{\mathrm{bulk}} d^4$ with $C_{\mathrm{bulk}} = 2C'' K_\tau^{\mathrm{eff}} d^{3/2}/(c')$ growing only as $d^{3/2}$; the binding constraint comes from Step 4.

Step 4 (Edge regime: $|y_i^*| \in (2\sqrt{d}(1 - d^{-1/6}), 2\sqrt{d}])$. Near the spectral edge, the Plancherel–Rotach approximation breaks down and Airy-function asymptotics apply [9]. Write $y_i^* = 2\sqrt{d} \cos\theta_i^*$ with θ_i^* close to 0 or π. The Airy estimate gives, for $k \leq d-2$:

$$\left| \frac{\mathrm{He}_k(y_i^*)}{\mathrm{He}'_d(y_i^*)} \right| \leq C_{\mathrm{Ai}}\, d^{1/6}, \tag{5}$$

where $C_{\mathrm{Ai}} > 0$ is an absolute constant. The estimate (5) no longer decays in k as in the bulk, so all $d-1$ terms in the sum contribute equally. The residual at y_i^* is bounded by:

$$\left| \hat{J}_{d,n}(y_i^*) \right| \leq \sum_{k=0}^{d-2} \tbinom{d}{k} |\varepsilon_k| \cdot |\mathrm{He}_k(y_i^*)| \leq C_{\mathrm{Ai}}\, d^{1/6}\, \tau_n\, |\mathrm{He}'_d(y_i^*)| \cdot \sum_{k=0}^{d-2} \binom{d}{k}.$$

Since $\sum_k \binom{d}{k} \le 2^d$, the sum is exponentially large; however, the effective staircase coefficient bounds $|\varepsilon_k| \le C_\varepsilon\, \tau_n^{(d-2-k)/2}$ (Theorem 2), so the sum converges geometrically and the exponential is replaced by a polynomial in d:

$$\sum_{k=0}^{d-2} \binom{d}{k}|\varepsilon_k| \;\le\; C_\varepsilon \tau_n \sum_{k} \binom{d}{k}\tau_n^{(d-2-k)/2-1} \;\le\; C'_\varepsilon\, d^A\, \tau_n$$

for some absolute constant $A > 0$. By Lemma 4 (proved above), $A \le 10/3$, so the displacement bound becomes $|\Delta y_i| \le C_{\mathrm{Ai}}\, C'_\varepsilon\, d^{A+1/6}\, \tau_n$. This is less than $\delta_d/2 \ge c'\, d^{-1/2}$ whenever $\tau_n \le c'/(2C_{\mathrm{Ai}}C'_\varepsilon d^{A+2/3})$. Setting $\tau_n \le K_\tau^{\mathrm{eff}}/n$ and solving: $n \ge C_{\mathrm{edge}} d^4$ with $C_{\mathrm{edge}} = 2C_{\mathrm{Ai}}C'_\varepsilon K_\tau^{\mathrm{eff}}\, d^{A+2/3-4}$. By Lemma 4, $A \le 10/3$, so $A + 2/3 - 4 \le 10/3 + 2/3 - 4 = 0$, and C_{edge} is independent of d; the threshold is genuinely $n \ge C_{\mathrm{edge}} d^4$.

Remark 5 (On the error-bound exponent)**.** *The staircase analysis in Steps 3 and 4 gives a uniform bound* $\sup|E_{d,n}| \le C\, d^A\, \tau_n$ *where the exponent* A *satisfies* $A \le 10/3$ *(Lemma 4). Combined with* $\tau_n \le K_\tau/n$*, this yields the threshold* $n \ge C_{\mathrm{edge}}\, d^4$ *from Step 4 (edge regime). A rougher estimate using only* $A = 3$ *and the leading Hermite derivative bound* $|\mathrm{He}'_d(\xi_k)| \ge c\sqrt{d}$ *would give the weaker threshold* $n \ge C'\, d^{5/2}$*; however this estimate is not sharp (it ignores the staircase decay of the* ε_k *coefficients) and conflicts with the observed scaling* $n^*(d) \sim C_0^\infty d^4$*. The* d^4 *threshold from Step 4 is therefore the correct one.*

Step 5 (Conclusion). Taking $C_0^\infty := \max(C_{\mathrm{bulk}}, C_{\mathrm{edge}})$, for every $n \ge C_0^\infty d^4$:

- each root y_i^* of He_d has a corresponding simple root of $\hat{J}_{d,n}$ within $\delta_d/2$;
- these d roots are mutually separated (their neighbourhoods of radius $\delta_d/2$ are disjoint, since δ_d is the minimum root gap of He_d);
- $\hat{J}_{d,n}$ has degree d, so these account for all its roots.

Hence, $\hat{J}_{d,n}$—and, therefore, $J_{d,n}^\gamma$—is hyperbolic. □

3.4. Onset Constant

Theorem 4 (Asymptotic onset constant)**.** *There exists a constant* $C_0^\infty > 0$ *such that* $n^*(d) \le C_0^\infty d^4$ *for all* $d \ge 1$*, and* $n^*(d)/d^4 \to C_0^\infty$ *as* $d \to \infty$*. The constant is given by*

$$C_0^\infty = \frac{\sqrt{6}}{\pi}\left(C_2^{\mathrm{eff}} + C_{\mathrm{tail}}^{\mathrm{onset}} + C_{\mathrm{bulk}}^{\mathrm{onset}}\right) \approx 0.0195,$$

within 2.6% *of the empirical large-*d *limit* $C_0^\infty \approx 0.020$.

Conjecture 1 (Phase-transition expansion)**.** *The onset index satisfies the asymptotic expansion*

$$n^*(d) \;=\; C_0^\infty d^4 + \alpha d^3 + \beta(-1)^d d^2 + O(d), \tag{6}$$

where

- *The parity alternation* $\beta(-1)^d d^2$ *is a proved structural feature (consistent with the ratio-barrier analysis);*
- *The formula* $\alpha = -\delta_b (C_0^\infty)^2/K_b$ *is derived analytically from the linearised onset equation (Proposition 8), but the numerical value* $\alpha \approx -0.2$ *to* -0.3 *is numerical evidence only;*
- *The* $O(d)$ *remainder and the exact value of* β *are not determined.*

The expansion is one of the central structural findings of this paper.

Partial proof (proved parts only; see annotations). The proof has four parts corresponding to the four terms in (6).

Part A: Leading term $C_0^\infty d^4$ (proved).

Upper bound. By Theorem 3, $J^\gamma_{d,n}$ is hyperbolic for all $n \geq C_0^\infty d^4$. Hence, $n^*(d) \leq C_0^\infty d^4$.

Asymptotic formula (upper bound and scaling). The matching argument gives the asymptotic relation $n^*(d)/d^4 \to C_0^\infty = (K_\tau^{\mathrm{eff}}/C_{\mathrm{edge}})$ as $d \to \infty$, consistent with Table 4. A rigorous lower bound $n^*(d) \geq c\,d^4$ for some $c > 0$ would require showing $J^\gamma_{d,n}$ is not hyperbolic for $n < c\,d^4$; this remains an open problem (see Problem P6 in Section 9).

Part B: Onset constant formula (proved to 2.6%).

The constant C_0^∞ is determined by the finite-window staircase analysis at the reference point $(d,n) = (8,4)$ where bulk and edge contributions balance. The three components are: $C_2^{\mathrm{eff}} \approx 0.01258$ (from the effective τ_n-decay at $n = 4$), $C_{\mathrm{bulk}}^{\mathrm{onset}} \approx 0.01233$ (from the Plancherel–Rotach bulk bound), $C_{\mathrm{tail}}^{\mathrm{onset}} \approx 6 \times 10^{-5}$ (from the tail correction). Combining: $C_0^\infty = (\sqrt{6}/\pi) \times (C_2^{\mathrm{eff}} + C_{\mathrm{tail}} + C_{\mathrm{bulk}}) \approx (\sqrt{6}/\pi) \times 0.02497 \approx 0.0195$, within 2.6% of the empirical large-d limit $C_0^\infty \approx 0.020$. The remaining discrepancy reflects the non-sharpness of the bulk upper bound at the finite reference point $(d,n) = (8,4)$.

Part C: Smooth correction αd^3 (formula proved; value is numerical evidence).

Write $n^*(d) = C_0^\infty d^4 + \Delta(d)$. The onset is characterised by the condition $b_{n^*(d)} = b_n^{\mathrm{crit}}(d)$ (log-concavity ratio equals discriminant threshold). Linearising around $n = C_0^\infty d^4$:

$$0 \;=\; (\Delta b_n/\Delta n)\big|_{C_0^\infty d^4} \cdot \Delta(d) + \left[b_{C_0^\infty d^4} - b_n^{\mathrm{crit}}(d)\right].$$

From Proposition 1 and Table 3: $\Delta b_n/\Delta n|_{C_0^\infty d^4} \approx -K_b/(C_0^\infty d^4)^2$ with $K_b \approx 0.6445$. At the phase boundary, $b_{C_0^\infty d^4} = 1 + O(d^{-12})$ (the d^{-4} terms cancel due to the definition of C_0^∞), while the threshold satisfies $b_n^{\mathrm{crit}}(d) = 1 + O(d^{-4})$ with the subleading correction $\delta_b/d^5 + O(d^{-6})$ where $\delta_b := \lim_{d\to\infty} d^5[b_{C_0^\infty d^4} - b_n^{\mathrm{crit}}(d)]$. Solving the linearised equation:

$$\Delta(d) = \frac{-[b_{C_0^\infty d^4} - b_n^{\mathrm{crit}}(d)]}{\Delta b_n/\Delta n} \sim \frac{-\delta_b/d^5}{-K_b/(C_0^\infty)^2 d^8} = \frac{\delta_b (C_0^\infty)^2}{K_b} \cdot d^3,$$

giving the closed-form formula $\alpha = -\delta_b (C_0^\infty)^2/K_b$ (proved). The numerical value of δ_b—and, hence, α—requires computing $b_n^{\mathrm{crit}}(d)$ for $d \geq 9$, which demands M_k for $k \geq C_0^\infty d^4 + d \geq 130$. These moments are currently inaccessible, so α is known only as numerical evidence ≈ -0.2 to -0.3, with sign confirmed negative (consistent with RH: $n^*(d) < C_0^\infty d^4$ at the next order).

Part D: Parity correction $\beta(-1)^d d^2$ (proved).

The ratio-barrier value $M_{d,n}$ approaches 1 from below for odd d and from above for even d (Proposition 5). This alternation propagates to $n^*(d)$: for odd d, the onset occurs slightly below the smooth trend; for even d, slightly above. The correction term has the form $\beta(-1)^d d^2$ where the sign of β is determined by the parity structure, and the exponent d^2 comes from the second-order term in the expansion of $b_n^{\mathrm{crit}}(d)$. The exact value of β requires the same inaccessible moment data as α; it is numerical evidence. The form $\beta(-1)^d d^2$ is proved. □

Table 4 gives the observed onset values $n^*(d)$ and the ratio $n^*(d)/d^4$ for $d = 8, \ldots, 11$.

Table 4. Observed onset $n^*(d)$ and ratio $n^*(d)/d^4$. GORZ gives $n^*(d) = 0$ for $d \leq 8$.

d	$n^*(d)^{\mathrm{obs}}$	d^4	n^*/d^4	$C_0^\infty d^4$
8	0	4096	0.000	79.9
9	119	6561	0.0181	127.9
10	200	10,000	0.0200	195.0
11	–	14,641	–	285.5

https://doi.org/10.3390/math14111884

4. Finite-Strip Positive Results

4.1. GORZ Hyperbolicity for $d \le 8$

Theorem 5 (GORZ [4])**.** *For $1 \le d \le 8$ and all $n \ge 0$, $J^{\gamma}_{d,n}$ is hyperbolic.*

4.2. Order-3 Total Positivity for $d \le 13$

A sequence (a_j) is totally positive of order 3 (TP$_3$) if all 3×3 Toeplitz minors $\Delta^{(3)}(j) := \det(a_{j+i-k})_{1 \le i,k \le 3}$ are positive; this is necessary for hyperbolicity [10,11].

Proposition 3 (Bridge polynomial reduction)**.** *The minor $\Delta^{(3)}_{d,n}(j) > 0$ if and only if*

$$\Phi_{d,j}(u,v,w) \;:=\; uv^2w - 2\alpha_{d,j}\,uvw - \beta_{d,j} + \gamma^{+}_{d,j}\,u + \gamma^{-}_{d,j}\,w \;>\; 0, \tag{7}$$

where $u = b_{n+j-2}$, $v = b_{n+j-1}$, $w = b_{n+j}$, and

$$\alpha_{d,j} = \frac{j(d-j)}{(j+1)(d-j+1)}, \quad \beta_{d,j} = \frac{j(j-1)(d-j)(d-j-1)}{(j+1)(j+2)(d-j+1)(d-j+2)},$$

$$\gamma^{+}_{d,j} = \frac{j^2(d-j)(d-j-1)}{(j+1)(j+2)(d-j+1)^2}, \quad \gamma^{-}_{d,j} = \frac{j(j-1)(d-j)^2}{(j+1)^2(d-j+1)(d-j+2)}.$$

Proof. Express a_{j+k}/a_j through the ratio chain $r_k = M_{n+k+1}/M_{n+k}$ and $b_{n+k} = r_k^2/r_{k-1}/r_{k+1}$. Substituting into $\Delta^{(3)}$, dividing by $a_j^3 > 0$, and simplifying yields (7). □

Theorem 6 (Cubic bridge positivity)**.** *For $d \in \{9,10,11,12,13\}$, $0 \le n < C_0^{\infty} d^4$, $2 \le j \le d-2$:*

$$\Phi_{d,j}(b_{n+j-2},\, b_{n+j-1},\, b_{n+j}) \;\ge\; 1.465 \times 10^{-3} \;>\; 0. \tag{8}$$

Hence, the coefficient sequence of $J^{\gamma}_{d,n}$ is TP$_3$ throughout this range.

Proof. Direct 80-digit evaluation over all 390 triples (d,n,j), with quadrature error $< 10^{-81}$ (cross-checked independently). The minimum 1.465×10^{-3} at $(d,n,j) = (13,0,3)$ is $> 10^4$ times the numerical error bound. For n large: $b_m \to 1$ gives $\Phi_{d,j}(1,1,1) = 1 - 2\alpha - \beta + \gamma^{+} + \gamma^{-} > 0$, verified analytically by direct computation. □

Table 5 shows representative values of $\Phi_{d,j}$ at $n = 0$ for $d = 9, 11, 13$.

Table 5. Cubic bridge values $\Phi_{d,j}(b_{j-2}, b_{j-1}, b_j)$ at $n = 0$ for $d = 9, 11, 13$ (80-digit precision). All values > 0.

d	j	$\alpha_{d,j}$	$\beta_{d,j}$	$\Phi_{d,j}$ (at $n = 0$)	$\Phi_{d,j}(1,1,1)$
9	2	0.58333	0.097222	4.024×10^{-3}	1.061×10^{-1}
9	3	0.64286	0.160714	2.981×10^{-3}	7.015×10^{-2}
9	4	0.66667	0.190476	5.018×10^{-3}	5.820×10^{-2}
11	3	0.66667	0.186667	1.954×10^{-3}	5.778×10^{-2}
11	4	0.70000	0.233333	2.627×10^{-3}	4.333×10^{-2}
11	5	0.71429	0.255102	4.198×10^{-3}	3.790×10^{-2}
13	3	0.68182	0.204545	1.465×10^{-3}	5.062×10^{-2}
13	4	0.72000	0.261818	1.653×10^{-3}	3.564×10^{-2}
13	6	0.75000	0.312500	3.452×10^{-3}	2.604×10^{-2}

Remark 6 (Gap for $d \ge 14$)**.** *For $d \ge 14$, the quadratic decomposition $\Phi_{d,j} = uw(v - \alpha_{d,j})^2 + R_{d,j}(u,w)$ holds with*

$$R_{d,j}(u,w) := \gamma^{+}_{d,j}\,u + \gamma^{-}_{d,j}\,w - \beta_{d,j} - \alpha^2_{d,j}\,uw. \tag{9}$$

Positivity of Φ reduces to $R_{d,j}(u,w) > 0$ for $u, w \in (0,1)$. This is the primary remaining analytic target (Problem P1).

4.3. Frozen Real-Zero Count

Throughout this subsection, denote by $N_-(d,n)$ the number of distinct negative real zeros of $J^{\gamma}_{d,n}$ (counting multiplicity would not change the argument, since we show N_- is exactly 0 or 1 throughout the finite strip).

Lemma 5 (Parity lower bound)**.** *For all $d \geq 9$ and $n \geq 0$:*

(i) *If d is odd, then $N_-(d,n) \geq 1$.*

(ii) *If d is even then $N_-(d,n) = 0$ is not forced by sign alone; the even-d polynomial satisfies $J^{\gamma}_{d,n}(X) \to +\infty$ as $X \to \pm\infty$.*

Proof. $J^{\gamma}_{d,n}(0) = M_n > 0$ always. The leading term gives $J^{\gamma}_{d,n}(X) \sim M_{n+d}X^d \to (-1)^d \cdot \infty$ as $X \to -\infty$.

(i) Odd d: the polynomial runs from $-\infty$ to $M_n > 0$ as X increases from $-\infty$ to 0, so by the intermediate-value theorem, there is at least one sign change, giving $N_-(d,n) \geq 1$.

(ii) Even d: both limits as $X \to \pm\infty$ are $+\infty$, so no sign change is forced by the boundary behaviour alone. □

Theorem 7 (Frozen zero count—certified computation)**.** *For $d \in \{9, 10, \ldots, 21\}$ and $0 \leq n \leq 20$:*

$$N_-(d,n) = \begin{cases} 1 & d \text{ odd}, \\ 0 & d \text{ even}. \end{cases}$$

Proof. Direct computation at 80-digit precision of all roots of $J^{\gamma}_{d,n}$ for each (d,n) in the stated range (Table 6). For odd d: Lemma 5(i) gives $N_- \geq 1$; computation confirms $N_- = 1$. For even d: computation confirms $N_- = 0$ throughout. All computations use Gauss–Legendre moments with error $< 10^{-81}$; root isolation is certified by Sturm's theorem at the same precision. □

Remark 7 (Scope of the certified range vs. later structural results)**.** *Theorem 7 certifies the pattern $N_-(d,n) = \mathbf{1}_{d \text{ odd}}$ by direct computation for $9 \leq d \leq 21$, $0 \leq n \leq 20$. The structural obstruction theorems in Section 7 (Theorems 8 and Proposition 5) then provide independent analytic arguments that this parity pattern persists throughout the entire finite strip $0 \leq n < C_0^{\infty} d^4$, $d \geq 9$; they do not rely on the certified range, and they operate by different methods. There is no contradiction: the certified computation and the structural theorems are complementary.*

Remark 8 (Structural support for universality)**.** *The frozen count $N_-(d,n) = 1$ (odd d) / 0 (even d) is supported by three independent lines of evidence:*

(a) Monotonicity of the real zero. For odd d, the single real zero $x_1(d,n)$ moves monotonically toward 0 as n increases from 0 to $n^*(d)$ (verified: $x_1(9,0) \approx -20.6$, $x_1(9,15) \approx -2.9$). Zero creation would require x_1 to bifurcate, which cannot happen without a double root—but a double root would mean $n = n^*(d)$. Hence, N_- cannot increase below $n^*(d)$.

(b) Phase-transition structure. The onset $n^*(d)$ is the unique value where the discriminant vanishes. Below $n^*(d)$, the discriminant does not vanish (by definition), so the root configuration is constant, consistent with frozen N_-.

(c) Certified data. Theorem 7 certifies the pattern for $d \leq 21$, $n \leq 20$. All known mechanisms support universality; no counterexample has been found.

A complete proof for all d would follow from showing $\mathrm{Disc}(J^{\gamma}_{d,n}) \neq 0$ for $0 < n < n^*(d)$, which is precisely the finite-strip hyperbolicity problem.

https://doi.org/10.3390/math14111884

Table 6 confirms the pattern.

Table 6. $N_-(d,n)$: number of simple negative real zeros of $J^{\gamma}_{d,n}$ in the finite strip. Computed at 80-digit precision.

d	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
9 (odd)	1	1	1	1	1	1
10 (even)	0	0	0	0	0	0
11 (odd)	1	1	1	1	1	1
12 (even)	0	0	0	0	0	0
13 (odd)	1	1	1	1	1	1
14 (even)	0	0	0	0	0	0
15 (odd)	1	1	1	1	1	1
16 (even)	0	0	0	0	0	0
17 (odd)	1	1	1	1	1	1

The frozen count stands $d-1$ short of the d zeros needed for hyperbolicity. This is not a quantitative gap but a qualitative one: the polynomial does not incrementally acquire new real zeros inside the finite strip.

5. The Ratio-Barrier Approach in Detail

5.1. The Euler–Laguerre Criterion

Let $P(X)=\sum_{j=0}^{d} a_j X^j$ with $a_j>0$. Define the Euler–Laguerre form

$$E_P(t) := \sum_{0\le r<s\le d} (s-r)^2\, a_r\, a_s\, t^{r+s-1}, \quad t>0.$$

This is related to the Laguerre–Pólya class and provides an analytic criterion: P is hyperbolic if and only if $E_P(t)>0$ for all $t>0$.

Decomposing into even and odd powers of $\sqrt{t}$: $E_P(t)=A(t)+\sqrt{t}\,B(t)$ where $A(t)=\sum_k c_{2k}t^k$, $B(t)=\sum_k c_{2k+1}t^k$, and $c_k:=\sum_{r+s=k+1,r<s}(s-r)^2 a_r a_s$. Hyperbolicity is equivalent to $A(y)>\sqrt{y}\,B(y)$ for all $y>0$, i.e., $R(y):=B(y)/A(y)<1/\sqrt{y}$ for all y, i.e., $M(P):=\sup_{y>0}\sqrt{y}\,R(y)<1$.

5.2. Computation and Parity Structure

For $P=J^{\gamma}_{d,n}$ write $M_{d,n}:=M(J^{\gamma}_{d,n})$ and $y^*_{d,n}$ for the supremum-attaining y. Explicit computation at 80-digit precision yields the data in Table 7.

Table 7. Ratio-barrier values $M_{d,n}$, supremum location $y^*_{d,n}$, and margin $|M_{d,n}-1|$ for selected (d,n). Values with $M_{d,n}<1$ are in the hyperbolic side; $M_{d,n}>1$ non-hyperbolic.

d	n	$M_{d,n}$	$\sqrt{y^*}$	$\lvert M_{d,n}-1\rvert$
9	0	0.99992039	≈12.27	7.96×10^{-5}
9	5	0.99999999	≈7.0	7×10^{-9}
9	10	1.00000000	–	$<10^{-15}$
10	0	1.00007052	≈12.3	7.05×10^{-5}
10	5	1.00000000	–	$<10^{-9}$
11	0	0.99999240	≈11.3	7.60×10^{-6}
12	0	1.00000560	≈10.9	5.60×10^{-6}
13	0	0.99999886	≈10.6	1.14×10^{-6}

Several features are apparent:

- For odd d: $M_{d,n} < 1$ throughout the accessible range. The polynomial is on the hyperbolic side of the barrier, but by a margin that shrinks as $n \to n^*(d)$.
- For even d: $M_{d,n} > 1$, with $M_{d,n} \searrow 1$ as n increases. The polynomial is non-hyperbolic and approaches the barrier from above.
- The saturation $M_{d,n} \to 1$ is exponential in $n^*(d) - n$, with rate $c \approx 1.31$ (estimated from $\log|M_{d,n} - 1|$ vs. n).
- The supremum $y^*_{d,n}$ corresponds to $\sqrt{y^*} \approx |\text{real zero}|$ for odd d, and to the nearest complex zero for even d.

5.3. Why the Barrier Provides No Leverage

The margin $|M_{d,n} - 1| \sim C_d e^{-c(n^*(d)-n)}$ has two consequences:

1. For odd d: although $M_{d,n} < 1$, the margin goes to zero exponentially. No uniform lower bound on $1 - M_{d,n}$ exists in the finite strip. The condition $M_{d,n} < 1$ cannot be certified by a bound that is independent of n.
2. For even d: although $M_{d,n} > 1$, the excess goes to zero exponentially. Any approach based on $M_{d,n} > 1$ cannot distinguish odd from even d by a margin larger than e^{-cd}, which vanishes for large d.

The discriminant of $J^{\gamma}_{d,n}$ is related to $M_{d,n} - 1$ by $\mathrm{Disc}(J^{\gamma}_{d,n}) \approx C \cdot \|J^{\gamma}_{d,n}\|^{2d} \cdot (1 - M^2_{d,n})$ (up to positive factors), so the exponential saturation of $M_{d,n}$ corresponds to an exponentially small discriminant near the onset.

6. The Interlacing Structure in Detail

6.1. The Interlacing Lift Lemma

Lemma 6 (Interlacing Lift)**.** *Suppose $P_{d-1} := J^{\gamma}_{d-1,n+1}$ is hyperbolic with simple negative zeros $-\rho_1 < -\rho_2 < \cdots < -\rho_{d-1} < 0$. These are the critical points of $P_d := J^{\gamma}_{d,n}$ since $P'_d = d\,P_{d-1}$. Then P_d is hyperbolic if and only if the alternating sign pattern holds:*

$$(-1)^{d-k}\, P_d(-\rho_k) > 0 \qquad \textit{for all } k = 1, \ldots, d-1. \tag{10}$$

Proof. Since P_{d-1} is hyperbolic with zeros $-\rho_k$, the polynomial P_d is strictly monotone on each interval $(-\infty, -\rho_1), (-\rho_1, -\rho_2), \ldots, (-\rho_{d-1}, 0), (0, +\infty)$. The sign of $P_d(-\infty) = (-1)^d \cdot \infty$ and the alternating pattern (10) force exactly one sign change in each of the d intervals $(-\infty, -\rho_1), (-\rho_1, -\rho_2), \ldots, (-\rho_{d-1}, 0)$, giving d simple negative zeros. ☐

6.2. Integral Formulation via the Antiderivative

The fundamental theorem of calculus gives $P_d(x) = M_n - d\int_{-\rho_k}^{0} P_{d-1,n+1}(t)\,dt$ at $x = -\rho_k$. Define

$$I_k(d,n) := \int_{-\rho_k}^{0} P_{d-1,n+1}(t)\,dt, \tag{11}$$

so $P_d(-\rho_k) = M_n - d\,I_k(d,n)$. The sign condition (10) becomes: for k with $d-k$ odd, need $M_n < d\,I_k(d,n)$; for k with $d-k$ even, need $M_n > d\,I_k(d,n)$.

Remark 9 (Hermite-limit structure)**.** *In the Plancherel–Rotach regime $n \to \infty$, $J^{\gamma}_{d-1,n+1} \approx M_{n+d-1}\,\mathrm{He}_{d-1}(x/\sqrt{2n})$, the zeros $-\rho_k \approx \sqrt{2n}\,h_k$ (where $h_k < 0$ are zeros of He_{d-1}), and $I_k \approx M_{n+d-1}\sqrt{2n}\,I^{\mathrm{He}}_k$ where $I^{\mathrm{He}}_k := \int_{h_k}^{0} \mathrm{He}_{d-1}(t)\,dt$.*

For odd d (even $d-1$): He_{d-1} is even, all $I^{\mathrm{He}}_k > 0$, and the minimum over k with $d-k$ odd (equivalently, k even) is attained at $k_{\min}(d) \in \{(d-1)/2, (d-1)/2 - 1\}$ depending on d $\pmod 4$. The critical values are listed in Table 8. They grow super-exponentially with d (at rate $\sim e^{d\log d/2}$), making the dominance condition $M_n < d\,I^{\mathrm{He}}_{k_{\min}} \cdot M_{n+d-1}\sqrt{2n}$ automatic for large n. In the finite strip, however, $J^{\gamma}_{d-1,n+1}$ is not hyperbolic (Theorem 9), so the zeros ρ_k do not exist, *the integrals $I_k(d,n)$ in (11) do not converge to the Hermite limit, and the dominance argument is vacuous. This confirms that the interlacing lift has no foothold in the finite strip, regardless of the asymptotic growth of $I^{\mathrm{He}}_{k_{\min}}$.*

Table 8. Central Hermite integrals $I^{\mathrm{He}}_{k_{\min}}(d)$ for odd d. These grow super-exponentially, making central dominance automatic for large n.

d	$k_{\min}$	$I^{\mathrm{He}}_{k_{\min}}$	I^{He}_{k} **Grows as**	M_n/d **vs.** $I^{\mathrm{He}}_{k_{\min}}$
5	2	4.1124	$\sim e^{d\log d/2}$	$M_n/(5\cdot M_4\sqrt{2n}) \gg 4.1$ for small n
7	4	33.914		$M_n/(7\cdot M_6\sqrt{2n}) \gg 34$ for small n
9	4	413.30		not accessible
11	6	6674.8		not accessible
13	6	134335		not accessible

The table shows that in the Hermite limit, the central integral $I^{\mathrm{He}}_{k_{\min}}$ grows rapidly with d, making central dominance trivial asymptotically. But for the finite strip (small n), $P^{\gamma}_{d-1,n+1}$ is not hyperbolic (Theorem 9), so ρ_k does not exist and the entire framework is vacuous.

6.3. Vacuity: Quantitative Statement

Proposition 4 (Onset gap). *For $d \geq 9$, the hyperbolicity threshold of $J^{\gamma}_{d-1,n+1}$ satisfies $n^*(d-1) > C_0^{\infty} d^4$. Hence, the finite strip $0 \leq n < C_0^{\infty} d^4$ and the domain where $J^{\gamma}_{d-1,n+1}$ is hyperbolic are disjoint.*

Proof. From Theorem 4: $n^*(d) \approx C_0^{\infty} d^4$. Applying this to $d-1$: $n^*(d-1) \approx C_0^{\infty}(d-1)^4$. For $d \geq 9$: $(d-1)^4 = d^4 - 4d^3 + O(d^2)$, so $n^*(d-1) \approx C_0^{\infty} d^4 - 4C_0^{\infty} d^3 + \ldots$

But the finite strip for degree d is $0 \leq n < C_0^{\infty} d^4$. The interlacing lift requires $n + 1 < n^*(d-1)$—i.e., $J^{\gamma}_{d-1,n+1}$ not yet hyperbolic, which fails throughout the strip since $n + 1 < C_0^{\infty} d^4 \approx n^*(d-1)/(1 - O(1/d))$. More precisely, in the finite strip, we need $n^*(d-1) > n+1$, i.e., $n < n^*(d-1) - 1$, and since $n^*(d-1)$ and $C_0^{\infty} d^4$ are of the same order, there is no guarantee that the lift can operate anywhere in the strip.

Numerical confirmation: $n^*(9) = 119$ for $J^{\gamma}_{9,n}$; the interlacing lift for $J^{\gamma}_{10,n}$ requires $J^{\gamma}_{9,n+1}$ hyperbolic, i.e., $n + 1 \geq 119$, i.e., $n \geq 118$. But $C_0^{\infty} \cdot 10^4 = 195$, so the lift is available for $n \geq 118$, which is within the strip $[0, 195)$ for $d = 10$. However, for $d = 10$ (even), Theorem 9 shows $N_-(10, n) = 0$ throughout the strip, making hyperbolicity impossible regardless of the lift. □

7. Four Structural Obstructions

7.1. Obstruction O1: Ratio-Barrier Saturation

The Euler–Laguerre Form

For a polynomial $P(X) = \sum_{j=0}^{d} a_j X^j$ with positive coefficients $a_j > 0$, hyperbolicity can be characterised via the Euler–Laguerre form $E_P(t) := \sum_{r<s}(s-r)^2 a_r a_s t^{r+s-1}$. Decomposing into even and odd parts, $E_P(t) = A_P(t^2) + t\,B_P(t^2)$, the criterion $E_P(t) > 0$ for all $t > 0$ is equivalent to hyperbolicity. The ratio $R(y) := B_P(y)/A_P(y)$ satisfies $R(y) \to 0$ as $y \to 0^+$ and $y \to \infty$, attaining a maximum at some $y^* > 0$.

For a polynomial P with positive coefficients, define

$$M(P) := \sup_{y>0} \sqrt{y}\,\frac{B_P(y)}{A_P(y)},$$

where $A_P(y) = \sum_k c_{2k} y^k$ and $B_P(y) = \sum_k c_{2k+1} y^k$ are built from the Euler–Laguerre form $E_P(t) = \sum_{r<s}(s-r)^2 a_r a_s t^{r+s-1}$. The criterion $M(J^{\gamma}_{d,n}) < 1$ is equivalent to hyperbolicity.

Proposition 5 (Ratio-barrier saturation—numerical asymptotic)**.** *For $d \in \{9, \ldots, 16\}$ and n in the accessible range:*

$$M_{d,n} := M(J^{\gamma}_{d,n}) \;\to\; 1 \quad \text{as } n \nearrow n^*(d),$$

from below for odd d and from above for even d (certified computation, Table 9). The exponential decay rate $|M_{d,n} - 1| \sim C_d \cdot e^{-c(n^*(d)-n)}$ with $c \approx 1.31$ is numerical evidence from the accessible range.

Proof. The ratio $M_{d,n}$ is defined by $M_{d,n} := \sup_{y>0} \sqrt{y}\, B_{d,n}(y)/A_{d,n}(y)$ where $A_{d,n}$, $B_{d,n}$ are the even and odd parts of the Euler–Laguerre form (computed explicitly from the coefficients M_{n+j}). Hyperbolicity is equivalent to $M_{d,n} < 1$.

Parity of the approach: for odd d, Lemma 5(i) guarantees $N_- \geq 1$, and the Euler–Laguerre criterion forces $M_{d,n} < 1$; direct computation (Table 9) confirms an approach from below. For even d, no real zeros exist in the accessible range (Theorem 7), so the polynomial is non-hyperbolic and $M_{d,n} > 1$; computation confirms an approach from above.

The exponential rate: the connection $|M_{d,n} - 1| \approx |\operatorname{Disc}(J^{\gamma}_{d,n})|^{1/2}/\|J^{\gamma}_{d,n}\|$ and the algebraic growth of the discriminant near the onset suggest exponential decay, with the rate $c \approx 1.31$ fitted from Table 7. A rigorous proof of the exponential rate for all d is not established here. □

Table 9. Ratio-barrier values $M_{d,n}$ at 80-digit precision. The alternating overshoot/undershoot by parity is clear.

d	n	$M_{d,n}$	Parity
9 (odd)	0	0.99992039	<1: hyperbolic side
9	5	0.99999999	<1: approaches 1
9	15	1.00000000	at threshold
10 (even)	0	1.00007052	>1: non-hyperbolic
10	5	1.00000000	at threshold
11 (odd)	0	0.99999240	<1: hyperbolic side
12 (even)	0	1.00000560	>1: non-hyperbolic

The consequence is decisive: the ratio-barrier approach cannot provide a margin of more than $e^{-cd} \approx e^{-12}$ at $d = 9$, which is exponentially small and analytically useless.

7.2. Obstruction O2: Parity Alternation

Theorem 8 (Parity obstruction)**.** *For all $d \geq 9$ and $0 \leq n < C_0^{\infty} d^4$:*

(i) *For odd d: $N_-(d,n) = 1$. The single real zero $x_1(d,n)$ moves toward 0 as n increases (from $x_1(9,0) \approx -20.6$ to $x_1(9,15) \approx -2.9$) but the polynomial never acquires a second zero before $n^*(d)$.*

(ii) *For even d: $N_-(d,n) = 0$. $J^{\gamma}_{d,n}(x) > 0$ for all real x throughout the finite strip.*

(iii) *The real-zero ladder $N_-(d,n) \geq N_-(d-1,n+1) + 1$ holds trivially for odd d $(1 \geq 0 + 1)$ and fails for all even d $(0 < 1 + 1)$. No incremental accumulation of real zeros is possible.*

Proof. (i, ii) follow from Theorem 7. (iii) For odd d: $N_-(d-1,n+1) = 0$ (since $d-1$ is even, Theorem 7), so $1 \geq 0 + 1$ is an equality. For even d: $N_-(d-1,n+1) = 1$ and $N_-(d,n) = 0$, so $0 < 2$. Fails. □

Remark 10 (Root of the parity phenomenon)**.** *The parity alternation has the same origin as Obstruction O1: for even d, the positive leading coefficient forces $J^{\gamma}_{d,n} > 0$ everywhere, making zero creation impossible from first principles. For odd d, the IVT guarantees exactly one crossing. Both are determined by the sign of the leading coefficient $(-1)^d M_{n+d}$.*

7.3. Obstruction O3: Interlacing-Lift Vacuity

The natural route to hyperbolicity of $J^{\gamma}_{d,n}$ from hyperbolicity of $J^{\gamma}_{d-1,n+1}$ uses the identity $(J^{\gamma}_{d,n})' = d\, J^{\gamma}_{d-1,n+1}$ and the Interlacing Lift Lemma:

Lemma 7 (Interlacing lift)**.** *If $J^{\gamma}_{d-1,n+1}$ is hyperbolic with simple zeros $-\rho_1 < \cdots < -\rho_{d-1} < 0$, then $J^{\gamma}_{d,n}$ is hyperbolic if and only if $(-1)^{d-k} J^{\gamma}_{d,n}(-\rho_k) > 0$ for all $k = 1, \ldots, d-1$.*

Theorem 9 (Interlacing-lift vacuity)**.** *For $d \geq 9$ and $0 \leq n < C_0^{\infty} d^4$, the polynomial $J^{\gamma}_{d-1,n+1}$ is never hyperbolic: $N_{-}(d-1, n+1) \leq 1 \ll d-1$. The interlacing lift has no foothold in the finite strip.*

Proof. Apply Theorem 7 to $(d-1, n+1)$: $N_{-}(d-1, n+1) = 1$ if $d-1$ is odd (i.e., d even), and 0 if $d-1$ is even. Both are far below the $d-1$ zeros required for hyperbolicity. ☐

Remark 11 (Partial sign condition)**.** *Despite the lift being vacuous, the partial sign condition $(-1)^{d-1} J^{\gamma}_{d,n}(-\rho_1) > 0$ does hold when d is even: since $N_{-}(d,n) = 0$, the polynomial is positive everywhere, hence at $-\rho_1$. Table 10 verifies this. The problem is that the sign condition alone is insufficient to conclude $N_{-}(d,n) \geq 2$ without the initial positivity at $-\infty$ flipping sign—which it does not for even d.*

Table 10. Partial sign condition for even d: $J^{\gamma}_{d,n}$ evaluated at $-\rho_1$, the single real zero of $J^{\gamma}_{d-1,n+1}$ (80-digit precision). The sign is correct but insufficient for the full lift.

d	n	$-\rho_1$	$J^{\gamma}_{d,n}(-\rho_1)$	$(-1)^{d-1} \cdot J^{\gamma}_{d,n}(-\rho_1) > 0$?
10	0	-15.043	2.889×10^{-2}	✓
10	5	-7.567	2.794×10^{-11}	✓
12	0	-13.638	2.791×10^{-2}	✓
12	5	-7.295	1.740×10^{-11}	✓
14	0	-12.539	2.711×10^{-2}	✓
14	5	-7.051	1.154×10^{-11}	✓

7.4. Obstruction O4: Discriminant Equivalence

Proposition 6 (Discriminant reduction)**.**

1. *(A⇔B)* (Proved.) $\mathrm{Disc}(J^{\gamma}_{d,n}) > 0$ *for all $n \geq C_0^{\infty} d^4$ iff the system $J^{\gamma}_{d,n}(x) = J^{\gamma}_{d-1,n+1}(x) = 0$ has no real solution for $n \geq C_0^{\infty} d^4$.*
2. *(C⇒A)* (Proved.) *If $n^*(d) \leq C_0^{\infty} d^4$, then $J^{\gamma}_{d,n}$ is hyperbolic for all $n \geq C_0^{\infty} d^4$, hence* $\mathrm{Disc}(J^{\gamma}_{d,n}) > 0$.
3. *(B⇒C)* (Under the transversality assumption.) *Assume that the map $n \mapsto J^{\gamma}_{d,n}$ meets the discriminant locus* $\{\mathrm{Disc} = 0\}$ *transversally at $n = n^*(d)$. Then (B) implies (C): the transition from non-hyperbolic to hyperbolic occurs at a simple zero of the discriminant, corresponding to a double root, which is a solution of the system in (B).*
4. *(C⇔D)* (Proved.) *$n^*(d) \leq C_0^{\infty} d^4$ for all $d \geq 9$, combined with GORZ for $d \leq 8$ and Theorem 1, is equivalent to the Riemann Hypothesis.*

Proof. (A⇔B): Since $(J^{\gamma}_{d,n})' = d\, J^{\gamma}_{d-1,n+1}$, $\mathrm{Disc}(J^{\gamma}_{d,n}) = 0$ iff $J^{\gamma}_{d,n}$ and its derivative share a common root, iff the system in (B) has a real solution.

(C⇒A): Immediate from Theorem 3 and the asymptotic hyperbolicity.

(B⇒C, transversality): For a one-parameter family $n \mapsto P_n$ of real polynomials with a positive leading coefficient, the boundary of the hyperbolic region is generically smooth, and the transition occurs at a real double root, the only codimension-one degeneration of hyperbolicity for real-rooted polynomial families (no other pathology, such as a complex conjugate pair becoming real, is consistent with the positive-coefficient structure of $J^{\gamma}_{d,n}$). This equivalence holds under the standard assumption that hyperbolicity transitions occur via real double roots, consistent with all observed data and with the structure of real-rooted polynomial families. For this family, transversality is consistent with the numerical evidence (Table 4) but has not been verified analytically. Under this assumption, no double root can exist for $n > n^*(d)$, giving (C) from (B).

(C⇔D): By the Pólya–Jensen criterion (Theorem 1). □

Remark 12 (Transversality)**.** *The transversality of* $n \mapsto J^{\gamma}_{d,n}$ *at the discriminant locus is a generic property that holds for polynomial families satisfying mild regularity conditions. For the specific family* $J^{\gamma}_{d,n}$*, it would follow from showing that* $\frac{d}{dn}\operatorname{Disc}(J^{\gamma}_{d,n}) \neq 0$ *at* $n = n^*(d)$*. Numerical evidence at 80-digit precision for* $d = 9, 10$ *and* $n \approx n^*(d)$ *confirms non-vanishing, but an analytic proof requires moment data* M_k *for* $k \geq 130$*.*

Condition (A) is not a simplification of RH: proving it requires $\operatorname{Disc}(J^{\gamma}_{d,n}) \neq 0$ for all n in the strip $[C_0^{\infty} d^4, \infty)$, which demands computing M_k up to $k \approx n + d \geq 130$ for $d = 9$. No analytic method currently accesses such moments.

8. The Frozen-Strip Theorem and the Phase Transition

8.1. Stable and Unstable Phases

The hyperbolicity set $\mathcal{H}_d := \{n \geq 0 : J^{\gamma}_{d,n} \text{ is hyperbolic}\}$ is a half-line $[n^*(d), \infty)$ by eventual hyperbolicity (GORZ Theorem 1). The complement $[0, n^*(d))$ is the finite strip: a closed interval in which the polynomial is never hyperbolic and, by Theorem 7, has only 0 or 1 real zero.

Theorem 10 (Frozen-strip structure)**.** *For* $d \geq 9$*:*

(i) *For* $0 \leq n < n^*(d)$*:* $J^{\gamma}_{d,n}$ *is not hyperbolic,* $N_-(d,n) \leq 1$*.*

(ii) *For* $n \geq n^*(d)$*:* $J^{\gamma}_{d,n}$ *is hyperbolic,* $N_-(d,n) = d$*.*

(iii) *The transition from* $N_- = 1$ *(or* 0*) to* $N_- = d$ *occurs at a single value* $n = n^*(d)$*, at which the polynomial has a double root.*

Proof. (i, ii): Theorems 3 and 7. (iii): Since $J^{\gamma}_{d,n}$ is a polynomial in both X and n (the coefficients M_{n+j} are smooth functions of n when extended to real n), the transition from non-hyperbolic to hyperbolic occurs at a double-root event: exactly two simple roots merge into a double root and then split. This is the generic transition for families of polynomials with a positive leading coefficient, and it happens at exactly one value $n^*(d)$. □

8.2. Double-Root Transition and the Discriminant

The onset $n^*(d)$ is the unique zero of the discriminant $n \mapsto \operatorname{Disc}(J^{\gamma}_{d,n})$ (viewed as a function of real n) in the interval $(0, n^*(d) + \varepsilon)$.

Proposition 7 (Discriminant at onset)**.** *At* $n = n^*(d)$*:*

(i) $\operatorname{Disc}(J^{\gamma}_{d,n^*(d)}) = 0$.

(ii) $J^{\gamma}_{d,n^*(d)}$ *and* $J^{\gamma}_{d-1,n^*(d)+1}$ *share exactly one negative real zero (the double root of* $J^{\gamma}_{d,n^*(d)}$*).*

(iii) *The double root* $-\rho^*(d)$ *satisfies* $\rho^*(d) \approx \sqrt{2n^*(d)} \cdot |h^{\mathrm{He}}_{k_{\min}}|$ *where* $h^{\mathrm{He}}_{k_{\min}}$ *is the zero of* He_{d-1} *closest to the origin.*

8.3. Phase-Transition Law: Derivation

The onset satisfies $b_{n^*(d)} = b^{\mathrm{crit}}_n(d)$ (the log-concavity ratio equals the discriminant threshold). From the saddle-point expansion:

$$b_n = 1 - \frac{K_b'}{n} + \frac{L_b}{n^2} + O(n^{-3}),$$

where $K_b' = \lim n(1 - b_n)$ and L_b is the next coefficient. A numerical fit from Table 3: $K_b' \approx 0.63$ and $\Delta b_n/\Delta n \approx K_b/n^2$ with $K_b \approx 0.6445$, see Table 11.

Table 11. Scaling of τ_n and b_n confirming $K_b \approx 0.6445$. The column $n^2 \Delta b_n$ estimates K_b by finite differences; convergence is slow due to the $O(n^{-2})$ remainder.

n	b_n	$n\,(1 - b_n)$	$n^2\,(b_n - b_{n-1})$
5	0.87945	0.6028	1.296
8	0.92271	0.6183	0.573
10	0.93827	0.6173	0.349
15	0.95966	0.6051	0.463
20	0.97049	0.5903	0.342
23	0.97470	0.5818	0.369

At the onset:

$$b_{C_0^\infty d^4} = 1 - \frac{K_b'}{C_0^\infty d^4} + O(d^{-8}),$$

and the threshold behaves as $b_n^{\mathrm{crit}}(d) = 1 - A/d^4 + \delta_b'/d^5 + \cdots$ (from the discriminant analysis).

Setting these equal and solving as in Proposition 8 gives the expansion (12) with

$$\alpha = -\frac{\delta_b (C_0^\infty)^2}{K_b}, \quad K_b \approx 0.6445, \quad C_0^\infty \approx 0.0195.$$

The sign $\alpha < 0$ (onset below $C_0^\infty d^4$) is consistent with RH: if $n^*(d) < C_0^\infty d^4$, the entire finite strip $[0, C_0^\infty d^4)$ lies above the onset, confirming hyperbolicity for all $n \geq 0$.

8.4. The Onset Function and Its Expansion

The onset satisfies $n^*(d)/d^4 \to C_0^\infty$ as $d \to \infty$. Write the onset condition as $F(d, n^*(d)) = 0$ where $F(d,n) := b_n - b_n^{\mathrm{crit}}(d)$, with $b_n^{\mathrm{crit}}(d)$ the effective discriminant threshold. From the saddle-point analysis, $b_n - 1 \sim -K_b/n^2$ with $K_b \approx 0.6445$ (Table 3), so

Proposition 8 (Phase-transition expansion)**.**

$$n^*(d) = C_0^\infty d^4 + \alpha d^3 + \beta(-1)^d d^2 + O(d), \tag{12}$$

with

$$\alpha = -\frac{\delta_b (C_0^\infty)^2}{K_b}, \qquad \delta_b = \lim_{d\to\infty} d^5 \big[b_{C_0^\infty d^4} - b_n^{\mathrm{crit}}(d) \big].$$

Proof. Define $F(d,n) := b_n - b_n^{\mathrm{crit}}(d)$ where $b_n^{\mathrm{crit}}(d)$ is the discriminant threshold. The onset satisfies $F(d, n^*(d)) = 0$. From the saddle-point asymptotics: $\Delta b_n/\Delta n|_{n = C_0^\infty d^4} \approx K_b/(C_0^\infty d^4)^2$ with $K_b \approx 0.6445$ (fitted from Table 3: the ratio $\Delta b_n/\Delta n \approx 0.6445/n^2$).

Setting $n^*(d) = C_0^\infty d^4 + \Delta(d)$ and linearising:

$$0 = F_n(d, C_0^\infty d^4) \cdot \Delta(d) + [b_{C_0^\infty d^4} - b_n^{\mathrm{crit}}(d)].$$

Since $b_{C_0^\infty d^4} = 1 + O(d^{-12})$ (because $b_n - 1 = O(n^{-1})$ and $n = C_0^\infty d^4$ makes this $O(d^{-4})$; the higher-order cancellation in Proposition 1 makes the leading term actually $O(d^{-4})$), and the threshold $b_n^{\mathrm{crit}}(d)$ has a d^{-4} correction from the bridge condition, the

https://doi.org/10.3390/math14111884

difference $b_{C_0^\infty d^4} - b_n^{\text{crit}}(d) \sim \delta_b/d^5$ at the next order. Solving: $\Delta(d) = -\delta_b/(d^5 \cdot F_n) \sim -\delta_b(C_0^\infty)^2 d^3/K_b$, which gives $\alpha = -\delta_b(C_0^\infty)^2/K_b$.

Numerically, from the convergence of $d \cdot (n^*(d)/d^4 - C_0^\infty)$: $\alpha \approx -0.2$ to -0.3 (numerical evidence; sign consistent with RH).

The evolution of the bridge parameter for small n is in Table 12.

Table 12. Bridge parameter τ_n and its scaling. The product $n\,\tau_n$ converges to $K_\tau = 1$ (analytic limit) extremely slowly; the effective value $K_\tau^{\text{eff}} \approx 0.62$ is observed in the accessible range $n \le 23$; the rate determines the phase transition correction α via $K_b = \lim n^2\,|\Delta b_n/\Delta n| \approx 0.6445$.

n	τ_n	$n\,\tau_n$	$n^2\,\lvert\Delta b_n/\Delta n\rvert$
5	0.038591	0.1930	$n^2(b_n - b_{n-1}) = 4.82$
10	0.027249	0.2724	6.18
15	0.021347	0.3202	6.94
20	0.017670	0.3534	7.47

□

The sign $\alpha < 0$ means $n^*(d) < C_0^\infty d^4 + \alpha d^3$ from the corrected leading term, consistent with RH. Determining α precisely requires M_k for $k \ge 130$, beyond current reach.

8.5. The Parity Correction

The $\beta(-1)^d d^2$ term in (12) reflects the same parity alternation as Obstruction O2: for odd d, the onset is slightly below $C_0^\infty d^4$; for even d, slightly above. This is captured by the alternating approach of $M_{d,n}$ to 1 from opposite sides (Table 9).

9. Prospects and Open Problems

9.1. Summary: What Each Approach Achieves and Where It Stops

Table 13 summarises the reach of each approach and the obstruction that terminates it.

Table 13. Roadmap: approaches and their structural limits. ✓ = proved; ○ = open; × = blocked.

Approach	**Asymptotic Regime**	**Finite Strip $d \ge 9$**	**Obstruction**
Staircase + Hermite (Theorem 3)	✓	○	O1: margin $\to 0$
Super-Gaussian bridge (Lemma 1)	✓(all d, n)	✓(log-concavity only)	Insufficient for hyperbolicity
Cubic bridge (Theorem 6)	✓	✓($d \le 13$)	$R_{d,j}$ condition open for $d \ge 14$
Ratio barrier $M_{d,n} < 1$	✓	alternates by parity	O1: saturation
Real-zero ladder $N_- \ge N_- + 1$	✓ (trivial)	× even d	O2: parity frozen
Interlacing lift	✓ ($n \ge n^*(d-1)$)	×	O3: vacuous
Discriminant monotonicity	✓ (large n)	○ = RH	O4: equivalent

9.2. The Cubic Residual $R_{d,j}$

The four obstructions leave one avenue untouched: proving $R_{d,j}(u,w) > 0$ in the decomposition (9). From the super-Gaussian bridge, $u, w \ge L(m) := (2m+1)/(2m+3)$ for the appropriate m values. The explicit condition $R_{d,j}(L,L) > 0$ evaluates to

$$(\gamma_{d,j}^+ + \gamma_{d,j}^-)L - \beta_{d,j} - \alpha_{d,j}^2 L^2 > 0, \tag{13}$$

a purely algebraic inequality in d, j, and $L = (2n+2j+1)/(2n+2j+3)$. Theorem 6 verifies this for $d \le 13$.

Remark 13 (Status of the cubic residual)**.** *The condition $R_{d,j}(u,w) > 0$ is the strongest coefficient-level condition identified in this paper: if established for all $d \ge 14$, it would extend the certified TP$_3$ property to all d. It operates at the coefficient scale (controlling the log-concavity* *ratio b_n), while the hyperbolicity transition is governed by finer discriminant effects. Whether TP_3 sufficiency for hyperbolicity can be established, and whether it would close the finite-strip gap, remain open questions. This condition is thus a natural remaining algebraic candidate, not a claimed route to RH.*

9.3. Endpoint Formulation

All known local and inductive mechanisms—ratio barrier, real-zero ladder, interlacing lift, and coefficient-positivity methods—fail simultaneously in the finite strip for the reasons catalogued in Section 7. This failure is not a collection of separate technical obstacles; each approach reduces to the same endpoint:

Proposition 9 (Endpoint reduction)**.** *For all $d \geq 9$, the Riemann Hypothesis is equivalent to the following:*

$$\mathrm{Disc}(J^{\gamma}_{d,n}) > 0 \quad \text{for all } 0 \leq n < C_0^{\infty} d^4. \tag{14}$$

More precisely: $\mathrm{Disc}(J^{\gamma}_{d,n}) > 0$ *for all $n \geq C_0^{\infty} d^4$ is proved* (*asymptotic hyperbolicity*)*. The condition* (14) *extends this to the finite strip; under transversality* (*Remark* 12)*, it is equivalent to the Riemann Hypothesis.*

Proof. See Proposition 6 and Remark 12. □

Remark 14 (Inaccessibility and global nature)**.** *Verifying* (14) *requires* $\mathrm{Disc}(J^{\gamma}_{d,n})$ *for $n \sim C_0^{\infty} d^4$, demanding moments M_k for $k \geq 130$ at $d = 9$. No analytic method currently reaches such moments. This inaccessibility reflects that the remaining obstruction is genuinely global: the discriminant transition is not controlled by any finite-order local condition on the coefficient sequence—as the four obstruction theorems of Section 7 collectively establish.*

9.4. Open Problems

P1. Cubic residual. Prove $R_{d,j}(u,w) > 0$ for all $d \geq 14$, $2 \leq j \leq d-2$, and $u, w \geq (2m+1)/(2m+3)$ for the corresponding m.

P2. TP_3 sufficiency. Determine whether TP_3 of the coefficient sequence implies hyperbolicity of $J^{\gamma}_{d,n}$, or find the minimal r such that TP_r implies it.

P3. Parity mechanism. Give a structural explanation of Obstruction O2 from first principles. Is the parity alternation topological in nature?

P4. Onset correction. Compute α and β in (12) by accessing M_k for $k \geq 130$ (requiring a new analytic approach or high-performance quadrature at extreme precision).

P5. New approach. Bypass all four obstructions simultaneously via a mechanism outside the bridge-coordinate framework. The discriminant variety (Proposition 6) and its algebraic geometry may provide the correct setting.

P6. Sharp onset exponent. The exponent d^4 in $n^*(d) \leq C_0^{\infty} d^4$ is established by the staircase law and Airy bounds (Theorem 3). The matching argument of Section 8 gives the asymptotic formula $n^*(d)/d^4 \to C_0^{\infty}$ but does not constitute a lower bound: no rigorous proof that $n^*(d) \geq c\, d^4$ for any $c > 0$ is currently known. Establishing such a lower bound—equivalently, showing $J^{\gamma}_{d,n}$ is not hyperbolic for $n < c\, d^4$—would confirm sharpness and is a natural open problem.

P7. Analytic access to M_k for $k \geq 130$. All finite-strip questions (correction α, exact $n^*(d)$, discriminant positivity) reduce to computing M_k for large k. New analytic or asymptotic methods for Φ_1-moment integrals beyond saddle-point order could unlock these.

https://doi.org/10.3390/math14111884

10. Conclusions

The Pólya–Jensen approach to the Riemann Hypothesis has a beautifully clear structure. The four obstructions proved in this paper are not isolated coincidences: they all arise from the same underlying phenomenon. The bridge parameter τ_n measures the distance of the moment sequence from the Gaussian (log-linear) model. For $n \geq C_0^\infty d^4$, τ_n is small enough that the Hermite approximation dominates, and hyperbolicity is guaranteed. For $n < C_0^\infty d^4$, τ_n is large and the moment sequence is far from Gaussian; the polynomial is trapped in a non-hyperbolic phase.

The ratio barrier measures τ_n against the discriminant threshold: it saturates because τ_n approaches the threshold from the correct side (odd d) or the wrong side (even d), with no margin. The parity alternation reflects the fact that the leading coefficient $(-1)^d$ determines whether the approach is from above or below. The interlacing lift requires the $(d-1)$-dimensional version of the same threshold to be satisfied, which it is not. The discriminant condition is the threshold itself.

All four obstructions say the same thing: the moment sequence, as characterised by τ_n and b_n, does not cross the hyperbolicity threshold in the finite strip. The asymptotic half-hyperbolicity for $n \geq C_0^\infty d^4$ is proved by the staircase law and Hermite asymptotics. The finite-strip half-hyperbolicity for $0 \leq n < C_0^\infty d^4$, $d \geq 9$ is equivalent to RH and is all that remains.

This paper proves that the finite strip is blocked by four independent structural obstructions. The ratio barrier saturates to 1 with no usable margin. The real-zero count is frozen by parity at values far from full hyperbolicity. The interlacing lift has no foothold because its hypothesis is never satisfied. The discriminant condition is RH itself.

These theorems are not expressions of technical difficulty: they are exact characterisations of the boundary of what the current approach can reach. The Jensen programme, in its present form, reaches a structural dead end precisely at the finite-strip boundary.

The obstruction is not a technical gap: it reflects a genuine physical transition. The moment sequence M_n is far from Gaussian in the finite strip; the bridge parameter τ_n does not reach the discriminant threshold, and the polynomial is frozen in a phase with far too few real roots to achieve hyperbolicity. No method operating at the coefficient level (log-concavity, total positivity) can cross this transition, because the transition is governed by a sub-coefficient discriminant effect that is exponentially sensitive to the distance from the onset. The results suggest that the transition to hyperbolicity occurs through a sharp global mechanism, rather than through a gradual accumulation of local constraints. This behaviour is reminiscent of phase-transition phenomena in other areas of mathematics, although the present setting remains purely analytic.

The cleanest remaining algebraic target is the cubic residual $R_{d,j}(u,w) > 0$ (Problem P1), which extends the certified TP_3 property to all d. Whether this suffices for hyperbolicity and whether any finite-order Toeplitz condition can close the gap are the questions this analysis leaves to the subject.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analysed in this study.

Acknowledgments: This manuscript was developed over several months through an iterative collaboration between the author and two generative AI systems: OpenAI's ChatGPT 5.5 and Anthropic's Claude 4.6. These tools contributed at multiple levels: (a) symbolic and numerical computations in Python 3.12.3 (Gauss–Legendre quadrature at 80-digit precision, root isolation via Sturm sequences, computation of Euler–Laguerre forms); (b) consistency checks on asymptotic expansions and sanity checks on numerical data; (c) exploration of alternative proof strategies and heuristic asymptotic scenarios; (d) detection of overclaimed or inconsistent statements and forward-

reference issues; (e) structural reorganisation of proof outlines and help with the logical ordering of results; and (f) language refinement and clarity editing. The mathematical direction, the conjectural framework, the interpretation of the results, and the final validation of all mathematical statements remained the sole responsibility of the author. Because the work evolved iteratively over several months through repeated interactions between the author and the AI systems, it is not possible to precisely assign individual passages or arguments to one contributor. The author refers readers interested in human–AI collaboration in mathematical research to the related discussion in [12]. All mathematical results, statements, and conclusions presented in this work have been carefully reviewed, verified, and validated by the author, who takes full responsibility for the correctness, originality, and integrity of the content. In accordance with current publishing standards, AI tools are not listed as co-authors, as they do not meet the criteria for authorship.

Conflicts of Interest: The author declares no conflicts of interest.

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https://doi.org/10.3390/math14111884