

1 **Disorder-tuned soliton nucleation for vibration energy harvesting**  
2 **in discrete nonlinear lattices - Part I : Theoretical framework for**  
3 **metastructure design**

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# Abstract

This study introduces a framework for vibration energy harvesting using stationary solitons as a design guide for nonlinear vibrating metastructures. While previous research primarily focused on propagating solitary waves, the present work investigates localized oscillations enabled by standing solitons, serving as a strategy for vibrational energy management. A mathematical model is developed based on the Disordered Damped Nonlinear Schrödinger equation, derived from a multiphysical nonlinear lattice equipped with piezoelectric transducers. The proposed model elucidates the influence of physical parameters on soliton nucleation, providing practical tools for metastructure engineering beyond the purely mathematical analyses. Numerical simulations demonstrate that localized vibration energy can be efficiently harvested by tuning system parameters, including electrical load, coupling stiffness, nonlinearity, and chain stiffness. This work offers a scalable design approach for soliton-based metastructures, paving the way not only for nonlinear energy harvesting but also for sensing technologies. It establishes the theoretical foundations that is experimentally validated in a subsequent article. The preliminary numerical results demonstrate the predictive capability of the developed models, enabling the application of heuristic optimization strategies in the design of multiphysical nonlinear arrays.

## 9 I. INTRODUCTION

10 After more than 20 years of research since the first works on vibration energy harvesting  
11 (VEH) [1], the scientific community remains focused on advancing this field. Glynne-Jones  
12 *et al.* [2], in one of the pioneering studies in the area, examined a prototype that, while  
13 unable to deliver power levels suitable for practical applications, identified promising direc-  
14 tions for future research. The pursuit of clean energy sources, the development of low-power  
15 hardwares, and advancements in smart structures have driven the growth of VEH tech-  
16 nologies. Moreover, a range of current challenges has employed harvesters as engineering  
17 strategies, demonstrating the versatility of VEH in areas such as MEMS [3–5], quasi-zero  
18 stiffness and compliant mechanisms [6, 7], wind energy harvesting [8–10], wireless geophones  
19 [11], railways [12, 13], vibration control [14, 15], biomedical devices [16, 17], footwear [18],

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20 water purification [19], bridge vibration [20], self-sustained sensors [21, 22], and mechanical  
21 motion rectification [23].

22 Not only the extracted power level but also the conditions for optimal operation are  
23 common topics of investigation in the field. As shown by Erturk and Inman [24], although  
24 linear devices perform well when the excitation frequency of the external source matches the  
25 resonance frequency of the device, they exhibit a significant drop in performance outside their  
26 bandwidth (a frequent occurrence in realistic scenarios [25–29]). In order to overcome this  
27 problem, researchers have investigated the feasibility of using nonlinearities in VEH. Tran  
28 *et al.* [30] provided a comprehensive overview of this literature. According to the authors,  
29 for instance, monostable systems can offer increased bandwidth but perform poorly under  
30 random excitations. On the other hand, bistable systems enhance performance for random  
31 excitations through interwell dynamics but require high excitation levels and prior design  
32 knowledge. Meanwhile, tri- and quadstable systems simplify interwell triggering but involve  
33 more complex designs. Despite the advancements enabled by introducing nonlinearities,  
34 problems related to practical implementations remain an active area of research [30–39].

35 A particularly interesting solution to manipulating the bandwidth of oscillators is the  
36 coupling of structures, where the overall dynamics outperform the sum of the individual  
37 components [40–42]. Studies reporting arrays of magnets [43], beams coupled either magnet-  
38 ically [44–48], through elastic supports [49], or via coupling rods [50] illustrate this approach.  
39 The study of the collective dynamics of coupled oscillators leads to another strategy that has  
40 also proven promising in a variety of applications: the development of metastructures. Wang  
41 *et al.* [51] reported a metastructure composed of high-static-low-dynamic-stiffness (HSLDS)  
42 resonators mounted on a thin plate to achieve very low-frequency band gaps. Similarly,  
43 in a later study, Chen *et al.* [52] published a nonlinear metastructure isolator with HSLDS,  
44 which, according to the authors, demonstrated excellent isolation performance and holds sig-  
45 nificant potential for precision equipment applications. Meanwhile, Li *et al.* [53] introduced  
46 sandwich plates with functionally graded auxetic 3D lattice cores, opening up a range of  
47 possible applications for similar structures. More recently, the versatility of smart metama-  
48 terials has been reaffirmed by the work of Schmidt *et al.* [54], who studied a reconfigurable  
49 piezoelectric metamaterial designed for selective noise directivity. Such versatility was again  
50 demonstrated by Mosquera-Sánchez and De Marqui Jr [55], who investigated broadband  
51 and multimode attenuation in Duffing- and NES-type piezoelectric metastructures.

52 In the context of VEH/vibration, metastructures have also garnered attention due to  
 53 their versatile applications [56–60], and tunability [61]. For example, Bukhari and Barry  
 54 [62] introduced a metastructure capable of simultaneous energy harvesting and vibration  
 55 control. This dual-functionality approach, whether targeting vibration control or energy  
 56 harvesting across multiple physical domains, has been explored in subsequent studies [63–72].  
 57 Expanding on this concept, collective/lattice dynamics offer enhanced energy management,  
 58 overcoming the dynamic limitations of conventional harvesters. For instance, Liu *et al.* [73]  
 59 demonstrated energy localization in five coupled beams for sensor development. Similarly,  
 60 Shao *et al.* [74] applied energy localization in lattice phononic crystals to control elastic  
 61 waves for energy harvesting, a concept further developed by Lu *et al.* [75] in their study of  
 62 a topological metamaterial beam. Furthermore, localized resonance in defect modes [76],  
 63 acoustic black holes [77], and Rubik’s cube-inspired systems [78] exemplify unconventional  
 64 applications of metastructures in energy harvesting.

65 Despite the extensive research in the field and, to the best of the authors’ knowledge,  
 66 there is a strategy that has not yet been explored in the literature: energy harvesting from  
 67 stationary solitons. Since the first description of the phenomenon by Russell [79] and the  
 68 initial mathematical formulations [80], solitons have proven to be important across various  
 69 domains of physics. Their behavior has been studied through the nonlinear Schrödinger  
 70 equation (NLS) in numerous physical contexts, including micro-mechanical systems [81, 82],  
 71 plasma physics [83], biology [84], optics [85, 86], photonics [87], and, lately, metamaterials  
 72 [88, 89]. Regardless of the rich literature on the topic, few studies aim to apply NLS equation  
 73 in mechanical engineering. Supported by the conclusions of [81], we corroborate the idea  
 74 that lattice solitons, when stationary, represent intrinsic localized modes (ILMs) in nonlinear  
 75 chains [90], pointing toward an underexplored direction to control the phenomenon [91]. It  
 76 is important to highlight a key difference between this study and the existing literature:  
 77 while some studies investigate the use of solitary waves or breathers propagating through  
 78 space in mechanical and VEH applications [92–99], our focus takes a different perspective  
 79 by considering the standing wave case.

80 Given the current state of the art, this article aims to propose a pathway for designing  
 81 metastructures for VEH, a work part of a series of prior studies addressing two fundamental  
 82 questions about solitons: how can the phenomenon be achieved? and what are its stability  
 83 conditions? [100–104]. We highlight here the study published in [103], in which we demon-

84 strated that the introduction of disorder expands the stability diagrams for externally driven  
85 solitons. Building upon the results reported in these studies, a mathematical model capable  
86 of describing the nonlinear stationary wave in a multiphysical Duffing chain is developed,  
87 where we analytically and numerically demonstrate that the VEH proposed methodology:  
88 (i) ensures non-chaotic stable regimes in harvesters, despite the possibility of strong non-  
89 linearities; (ii) exhibits superior performance over the simple sum of individual oscillators’  
90 contributions; and (iii) is more robust to uncertainties and disorder when compared to its  
91 linear counterpart.

92 This work is divided into two main parts, where the present study, Part I, establishes  
93 the theoretical foundations that is employed in Part II, where the Proof of Concept is de-  
94 picted. The remainder of the paper is structured as follows: Section II derives the Disordered  
95 Damped Nonlinear Schrödinger Equation and explores its physical implications. Section III  
96 investigates energy localization, emphasizing the role of nonlinearity in generating station-  
97 ary solitons. The outcomes from these sections are then applied in Section IV, where the  
98 frequency domain is analyzed as a foundation for a design strategy. Section V presents an  
99 algorithm specifically developed for vibration metastructures. Owing to the broad applica-  
100 bility of the proposed framework, Section VI addresses the theoretical aspects concerning  
101 the influence of disorder, the number of oscillators, and energy localization, underscoring the  
102 benefits of incorporating nonlinearity and solitonic dynamics (elements experimentally vali-  
103 dated in the subsequent work). Finally, Section VII closes the study and outlines directions  
104 for future research.

## 105 **II. DISORDERED DAMPED NONLINEAR SCHRÖDINGER EQUATION** 106 **(DDNLS)**

107 In this section, the nonlinear Schrödinger equation (NLS) is presented within a multi-  
108 physics context. The system addressed in this study consists of a chain of Duffing oscillators  
109 with coupling terms that may include both linear and nonlinear components. Additionally,  
110 piezoelectric transducers are attached along the elements, as illustrated in Figure 1. The  
111 structure can be mathematically characterized by the interplay of forces among the  $N$  ele-  
112 ments, along with the description of the coupled electrical circuit, from which the following

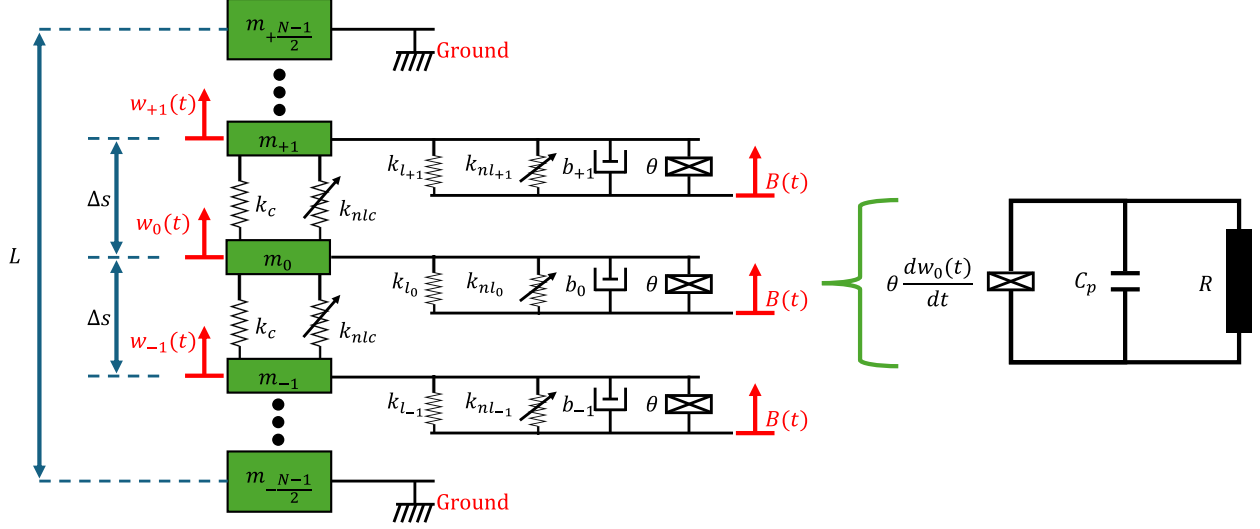


FIG. 1. Equivalent mechanical system describing coupled Duffing Oscillators and their arrangement along the chain. Each unit consists of a mass  $m_i$  connected to the ground via linear  $k_{li}$ , nonlinear  $k_{nli}$  stiffness, and damping  $b_i$ , with adjacent oscillators coupled through linear  $k_c$  and nonlinear  $k_{nlc}$  elements. The system is subjected to external base excitation  $B(t)$ , and energy is extracted from selected nodes via electromechanical coupling  $\theta$ , connected to a resistive-capacitive load  $R$ - $C_p$ .

relationships are derived:

$$\begin{aligned}
 m\ddot{w}_n(t) + (b + q_b(n))\dot{w}_n(t) + (k_l + q_{k_l}(n))w_n(t) + (k_{nl} + q_{k_{nl}}(n))w_n^3(t) \\
 + k_{cl}(2w_n(t) - w_{(n+1)}(t) - w_{(n-1)}(t)) + k_{cnl}\left((w_n(t) - w_{(n+1)}(t))^3 + (w_n(t) - w_{(n-1)}(t))^3\right) \\
 + V_n(t)\theta = -(f_m + q_{f_m}(n))\ddot{B}(t),
 \end{aligned} \tag{1}$$

and

$$C_p \dot{V}_n(t) + \frac{V_n(t)}{R} = \theta \frac{d}{dt}w_n(t), \tag{2}$$

where  $t$ ,  $m$ ,  $w_n(t)$ ,  $b$ ,  $k_l$ ,  $k_{nl}$ ,  $k_{cl}$ ,  $k_{cnl}$ , and  $f_m$  represent, respectively, the time, mass, displacement, damping coefficient, linear stiffness, nonlinear stiffness, linear coupling stiffness, nonlinear coupling stiffness, and the inertial effects<sup>1</sup> associated with the excitation of oscillator  $n$ . Variations, when present, are denoted by  $q$  with a subscripted index linked to the variable. In the electrical domain,  $V_n(t)$  denotes the voltage generated in the piezoelectric

<sup>1</sup>  $f_m = m$  for concentrated masses.

elements as a result of the electromechanical coupling  $\theta$ , capacitance  $C_p$ , and load  $R$ . The excitation of the structure is provided in the form of base acceleration, denoted by  $\ddot{B}(t)$ . For economy of notation,  $\dot{(\ )}$  denotes time derivative and, for algebraic simplicity, variations in mass, mechanical coupling, and electrical parameters are disregarded. With the aim of generalizing the physical parameters, Equation (1) can be normalized by the mass using the relationships

$$\omega_0^2 = \frac{k_l}{m}, \quad 2\zeta\omega_0 = \frac{b}{m}, \quad 2q_\zeta(n)\omega_0 = \frac{q_b(n)}{m}, \quad \phi(n) = \frac{q_{k_l}(n)}{m}, \quad (3)$$

being rewritten as:

$$\begin{aligned} \ddot{w}_n(t) + 2\omega_0(\zeta + q_\zeta(n))\dot{w}_n(t) + (\omega_0^2 + \phi(n))w_n(t) + \left(\frac{k_{nl}}{m} + \frac{q_{k_{nl}}(n)}{m}\right)w_n^3(t) \\ + \frac{k_{cl}}{m}(2w_n(t) - w_{(n+1)}(t) - w_{(n-1)}(t)) + \frac{k_{cnl}}{m}\left((w_n(t) - w_{(n+1)}(t))^3 + (w_n(t) - w_{(n-1)}(t))^3\right) \\ + \frac{\theta}{m}V_n(t) = -\left(\frac{f_m}{m} + \frac{q_{f_m}(n)}{m}\right)\ddot{B}(t). \end{aligned} \quad (4)$$

Equation (4), respecting the integrity of the boundary condition, aims to be general, independent of the network architecture, (whether constituted by pendulums [105], linearly coupled masses [106], or circular structures [107]). It is important to note that lattice periodicity can be disrupted under different philosophies: either unintentionally, as a result of impurities, or intentionally, as part of the design. In the first scenario the analysis is associated with small breaks in periodicity that lead to quasiperiodic systems. In the second case, the analysis is proposed for a more general description, not adhering to the chain's periodicity. In Barbosa *et al.* [104], we demonstrated a hierarchy in the physical origins of disorders, showing that variations in  $\omega_0$  have a greater impact on soliton formation than variations in other physical parameters. The current study does not analyze unintentional breaks but rather focus on intentionally introduced disorders, however, the mathematical derivation in both contexts follows the same formulations.

The steady-state behavior of the chain described by Equation (4) can be formulated through the Disordered Damped Nonlinear Schrödinger Equation (DDNLS), initially re-

ported in [103]:

$$\begin{aligned}
& H \left( 1 + \frac{q_{fm}(\chi)}{f_m} \right) - \left( 2 + \frac{2q_{knl\varepsilon}(\chi)}{k_{nl\varepsilon}} \right) F(\chi, \tau) |F(\chi, \tau)|^2 + i \frac{\partial}{\partial \tau} F(\chi, \tau) + \frac{\partial^2}{\partial \chi^2} F(\chi, \tau) \\
& + \left( -\Gamma_1(\chi) + i(\zeta_\varepsilon + q_{\zeta\varepsilon}(\chi) + \Gamma_2(\chi)) - 1 - \frac{\phi_\varepsilon(\chi)}{2\omega_0^2} \right) F(\chi, \tau) = 0.
\end{aligned} \tag{5}$$

The algebraic procedure leading to the derivation of Equation (5) is detailed in A as well as the definitions of the variables. Initially, the nonlinearity of Equation (1) is assumed to be positive, which contrasts with some studies in literature [101, 105]. We must note that the sign of  $k_{nl}$  determines the polarity of the term  $\left( 2 + \frac{2q_{knl\varepsilon}(\chi)}{k_{nl\varepsilon}} \right) F(\chi, \tau) |F(\chi, \tau)|^2$ . For instance, assuming  $k_{nl} < 0$  while neglecting disorders and transducers, leads to the commonly reported equation in the literature [108]:

$$H + (2|F(\chi, \tau)|^2 + i\zeta_\varepsilon - 1) F(\chi, \tau) + i \frac{\partial}{\partial \tau} F(\chi, \tau) + \frac{\partial^2}{\partial \chi^2} F(\chi, \tau) = 0. \tag{6}$$

During the development process, continuity was adopted to describe the dynamic behavior of the oscillators (Equations (A1)). Many studies do not employ this mathematical transition, instead analyzing solitons from a discrete perspective [109–111]. It is noteworthy that the discretization of Equation (5) in numerical simulation is unavoidable, particularly in the spatial domain, where continuity plays a significant role. In computational terms, if the second-order spatial derivative of  $F(\chi, \tau)$  exists, it can be approximated using the finite difference method:

$$\frac{(-2F_n(\tau) + F_{n+1}(\tau) + F_{n-1}(\tau))}{\Delta \chi^2} \leftrightarrow \frac{\partial^2}{\partial \chi^2} F(\chi, \tau), \tag{7}$$

consequently, the continuity hypothesis becomes more a matter of notation than a fundamental limitation, being associated with the convergence of computation.

## A. Physical interpretations underlying the DDNLS

Equation (5), keeping the attributes of the works in which it is addressed (such as excitation type, absence of impurities, and types of nonlinearities), is frequently studied with an emphasis on its mathematical properties [112–115]. However, in an engineering context, a deeper understanding of the physical terms that compose the NLS is required. The following

162 is a series of observations aimed at clarifying the physical correlations between the DDNLS  
163 and Equation (1).

164 The first interpretation concerns the nonlinear stiffness  $k_{nl}$ . As indicated by the rela-  
165 tionships described in (A14), in a scenario of external excitation, it appears alongside the  
166 acceleration, leading to the conclusion that, from a DDNLS perspective, an increase in the  
167 system's nonlinearity can be interpreted as an increase in its excitation. However, such a  
168 statement does not imply that an increase in  $k_{nl}$  in local regions of the lattice means that  
169 those regions are subjected to a higher excitation. The periodicity break in  $k_{nl}$ , represented  
170 by  $q_{k_{nl}}$ , does not contribute to the term  $H$ .

171 The second interpretation is related to the terms associated with the imaginary part of  
172  $F(\chi, \tau)$ :  $(\zeta_\varepsilon + q_{\zeta_\varepsilon}(\chi) + \Gamma_2(\chi))$ . Considering that  $\zeta = \varepsilon\zeta_\varepsilon$ , it is straightforward to infer that  
173 an increase in the system's damping results in an increase in the damping of the DDNLS.  
174 Albeit, unlike nonlinear stiffness, a local increase in the damping of the oscillators directly  
175 translates into a local increase in the DDNLS damping via  $q_\zeta$ . Furthermore, the introduc-  
176 tion of piezoelectric transducers, as expected, also contributes to damping in Equation (5)  
177 through  $\Gamma_2(\chi)$ , since the lattice energy must also flow into the electrical circuit.

178 The third interpretation focuses on the periodicity break in the natural frequency of the  
179 oscillators, represented by  $-\frac{\phi_\varepsilon(\chi)}{2\omega_0^2}$ . From a general perspective, alongside  $-\Gamma_1(\chi)$ , this  
180 term appears in the DDNLS as a linear potential, where  $-\left(2 + \frac{2q_{k_{nl\varepsilon}}(\chi)}{k_{nl\varepsilon}}\right)$  represents,  
181 analogously, the nonlinear one [116, 117]. The mathematical development does not nec-  
182 essarily assume localized periodicity breaks, which would result in systems with potential  
183 fields characterized by impurities [118].

184 The interpretation focused on the coupling stiffness  $k_{cl}$  requires some algebraic manipu-  
185 lation. Using the relationships from (A17) and the chain dimensions shown in Figure 1, the  
186 spatial domain of Equation (5) is then determined by the following equation:

$$L_\chi = \frac{L}{\delta_s}. \quad (8)$$

187 Considering that the number of oscillators  $N$  must satisfy Equation:  $L = (N - 1)\Delta s$ , the  
188 influence of  $k_{cl}$  is explicitly expressed as:

$$L_\chi = \frac{\sqrt{2}\omega_0\sqrt{\varepsilon}(N - 1)}{\sqrt{k_{cl}}}. \quad (9)$$

189 We should notice that increasing the coupling between the oscillators reduces the spatial  
 190 domain  $L_\chi$ , thereby amplifying the influence of boundary conditions on the formation of  
 191 the DDNLS. Based on the work of Terrones *et al.* [119], which analyzed the damped driven  
 192 NLS on a finite interval, and the subsequent study by Barashenkov and Smirnov [108],  
 193 which examined an expansion of the spatial domain ( $L_\chi \rightarrow \infty$ ), one important point can  
 194 be highlighted: the stability of Equation (5) is sensitive to the boundaries defined by  $L_\chi$ .  
 195 Specifically, reducing  $L_\chi$  allows the emergence of stability windows that are absent when  
 196  $L_\chi \rightarrow \infty$ , due to the instability caused by long-wavelength perturbations, which cannot  
 197 occur on short intervals.

## 198 **B. Multi-Coupled Oscillators**

199 The coupling between oscillators can occur through different approaches, such as springs  
 200 connecting individual components [120], a common base shared by the entire structure  
 201 [121], or the use of permanent magnets [122]. The mathematical development that derives  
 202 Equation (5) from Equation (1) assumes the presence of coupling only between adjacent os-  
 203 cillators. Let us analyze the following expressions representing new coupling terms  $k_{1cl}, k_{2cl},$   
 204  $\dots, k_{1cnl}, k_{2cnl}, \dots$  introduced into Equation (1):

$$\begin{aligned}
 & k_{1cl} (2w_n(t) - w_{(n+1)}(t) - w_{(n-1)}(t)) + k_{1cnl} \left( (w_n(t) - w_{(n+1)}(t))^3 + (w_n(t) - w_{(n-1)}(t))^3 \right) \\
 & + k_{2cl} (2w_n(t) - w_{(n+2)}(t) - w_{(n-2)}(t)) + k_{2cnl} \left( (w_n(t) - w_{(n+2)}(t))^3 + (w_n(t) - w_{(n-2)}(t))^3 \right) \\
 & \quad \vdots
 \end{aligned} \tag{10}$$

205 Applying once again the assumptions and notations that established the continuity of Equa-  
 206 tions (A1), Equation (A3) is extended to include the following terms:

$$\begin{aligned}
& -\frac{\varepsilon k_{1cl} \Delta s_{(\sqrt{\varepsilon})}^2}{m} w_{ss}(s, t) + \frac{k_{1cnl}}{m} \left[ \left( -\frac{\varepsilon \Delta s_{(\sqrt{\varepsilon})}^2}{2} w_{ss}(s, t) - \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \\
& + \frac{k_{1cnl}}{m} \left[ \left( -\frac{\varepsilon \Delta s_{(\sqrt{\varepsilon})}^2}{2} w_{ss}(s, t) + \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \\
& -\frac{\varepsilon 4k_{2cl} \Delta s_{(\sqrt{\varepsilon})}^2}{m} w_{ss}(s, t) + \frac{k_{2cnl}}{m} \left[ \left( -2\varepsilon \Delta s_{(\sqrt{\varepsilon})}^2 w_{ss}(s, t) - 2\Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \cdot \quad (11) \\
& + \frac{k_{2cnl}}{m} \left[ \left( -2\varepsilon \Delta s_{(\sqrt{\varepsilon})}^2 w_{ss}(s, t) + 2\Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \\
& \vdots
\end{aligned}$$

By grouping the linear terms constituting (11), an equivalent coupling term  $k_{cleq}$  can be formulated as follows

$$k_{cleq} = k_{1cl} + 4k_{2cl} + 9k_{3cl} + 16k_{4cl} + \dots + j^2 k_{jcl} = \sum_{n=1}^j i^2 k_{icl}, \quad (12)$$

where  $j$  defines the maximum index distance of between the oscillators.

It should be noted that  $k_{cleq}$  is derived under the same perturbative assumptions with respect to the parameter  $\varepsilon$  as those adopted for systems without multiple coupling. This, however, is not necessarily an assumption that can be considered for all physical architectures. In fact, potential limitation of this formulation concerns the scale of the coupling terms: differing orders of magnitude contradict the assumption that all couplings can be grouped within the same  $\varepsilon$  power. In order to avoid restricting the applicability of the methodology to physical systems that strictly conform to the assumptions used in the derivation of  $k_{cleq}$ , the following generalization is proposed:

$$\begin{aligned}
& k_{cleq} \rightarrow \\
& \left( \frac{k_{1cl}}{\sum_{n=1}^j k_{icl}} \right) k_{1cl} + \left( \frac{k_{2cl}}{k_{1cl}} \right) 4k_{2cl} + \left( \frac{k_{3cl}}{k_{1cl}} \right) 9k_{3cl} + \left( \frac{k_{4cl}}{k_{1cl}} \right) 16k_{4cl} + \dots + \left( \frac{k_{jcl}}{k_{1cl}} \right) j^2 k_{jcl}. \quad (13)
\end{aligned}$$

Through the new formulation presented in (13), large discrepancies in the magnitudes of the coupling stiffnesses are compensated by the ratio between each  $k_{jcl}$  and the largest coupling stiffness, here assumed to be  $k_{1cl}$ . An additional weighting term is applied to  $k_{1cl}$  in order to calibrate the contribution of the strongest coupling relative to the total structural stiffness. In the case of uncoupled systems, the expression for  $k_{cleq}$  reduces to  $k_{1cl}$ , thereby enabling the application of this expression in general cases as a first approximation. In experimental contexts, it is recommended that the value of the equivalent coupling should be determined experimentally, with Equation (13) serving as a numerical starting point for model calibration.

The algebraic manipulation that deduced the NLS from Equation (A3) can be re-applied in a multi-coupling context, allowing us to determine the new spatial domain to which the NLS is subjected:

$$L_\chi = \frac{\sqrt{2}\omega_0\sqrt{\varepsilon}(N-1)}{\sqrt{k_{cl}}} \rightarrow \frac{\sqrt{2}\omega_0\sqrt{\varepsilon}(N-1)}{\sqrt{k_{cleq}}}. \quad (14)$$

It is important to note that the greater the index distance between the coupled oscillators, the higher the value of  $k_{cleq}$  and, consequently, the lower the value of  $L_\chi$ , which could contribute to the emergence of stability windows in Equation (5).

The collapse of information exchange between oscillators into a single term represents a simplification in the description of the metastructure's dynamics. Although a lattice with  $k_{cleq} = 10$  kN/m may exhibit the same macroscopic behavior as a lattice where only  $k_{1cl} = 10$  kN/m is present, the two configurations represent fundamentally distinct systems. However, in a scope of design guidelines, homogenization techniques have proven particularly useful for predicting dynamic trends<sup>2</sup>[123]. From this perspective, although  $k_{cleq}$  does not fully capture the dynamic evolution of multi-coupled systems, it remains a valuable tool within the engineering framework.

### III. STANDING SOLITONS

Achieving stationary waves localized in space for any value of  $\tau$  implies that  $\frac{\partial}{\partial \tau}F(\chi, \tau) = 0$ . Note that this is a necessary but not sufficient condition, as it does not address the stability of the phenomenon. To illustrate this issue, let us consider a simplified case of

<sup>2</sup> By dynamic trends, we refer to the identification of frequencies associated with large amplitudes, hysteretic behaviors, the construction of modal bases, or, for instance, design optimizations.

Equation (5), where the electrical components and part of the disorder are neglected:

$$H - 2F(\chi, \tau)|F(\chi, \tau)|^2 + (i\zeta_\varepsilon - 1 + q\delta_d(\chi))F(\chi, \tau) + i\frac{\partial}{\partial\tau}F(\chi, \tau) + \frac{\partial^2}{\partial\chi^2}F(\chi, \tau) = 0, \quad (15)$$

where  $\delta_d(\chi)$  denotes the Dirac delta function indicating the allocation of the impurity  $q$  at  $\chi = 0$ , becoming a particular case of the linear potential:

$$\left(-\frac{\phi_\varepsilon(\chi)}{2\omega_0^2}\right) = q\delta_d(\chi). \quad (16)$$

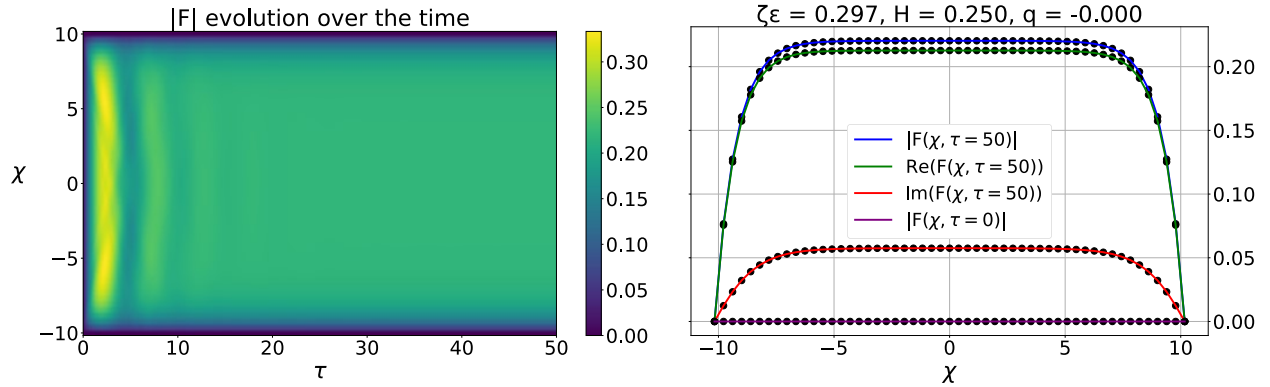


FIG. 2. Numerical integration of Equation (15). Absence of spontaneous nucleation for the chosen parameter configuration.

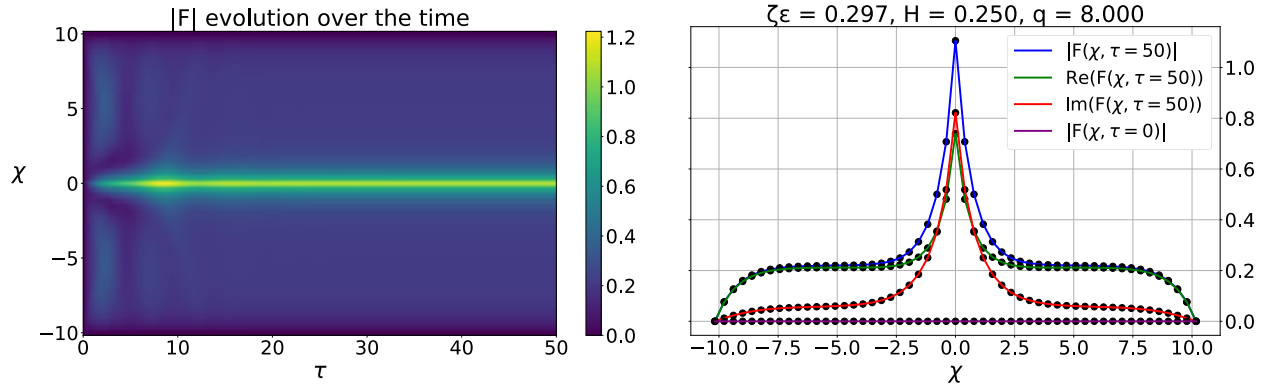


FIG. 3. Numerical integration of Equation (15). Spontaneous nucleation for the chosen parameter configuration.

Figures 2, 3, and 4 evaluate the evolution of  $F(\chi, \tau)$  for different parameter configurations. As shown, a perfectly periodic structure (Figure 2,  $q = 0$ ) does not generate spontaneous nucleation from initially motionless oscillators. In contrast, impurity tuning can not only

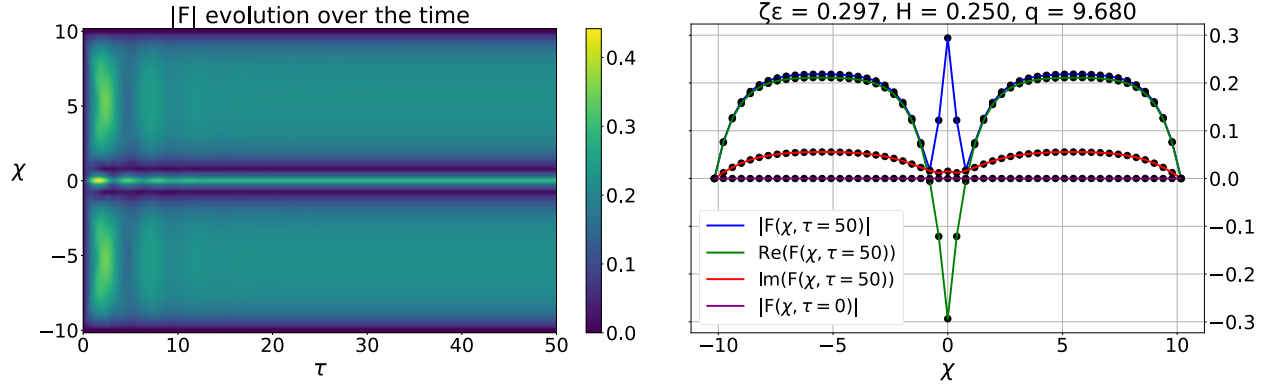


FIG. 4. Numerical integration of Equation (15). Decrease in localization intensity for the chosen parameter configuration.

nucleate the soliton but also localize the energy in the chain at the point where the impurity is allocated (Figure 3,  $q = 8$ ). As seen, an excessive increase in the impurity delocalizes the amplitude that was previously localized by the tuned one (Figure 3,  $q = 9.68$ ).

The stability of stationary solitons has been the focus of numerous studies aiming to identify the parameters that enable the phenomenon. Barashenkov *et al.* [124], for instance, reported the stability diagram of the Parametrically Driven, Damped Nonlinear Schrödinger Equation, and later, in [108], presented the diagram for the externally driven case. Expanding the understanding of the topic, Kenig *et al.* [81] investigated how the introduction of nonlinearities in the damping affects the diagram published in [124]. Interestingly, quasiperiodic systems incorporating localized impurities have been shown to expand the stability diagram for standing solitons in both parametrically excited [125] and externally driven [103] scenarios.

Ensuring the existence of stable solutions does not imply that Equation (1) will exhibit spontaneously localized oscillations, as illustrated in Figure 2. According to [81], initiating soliton dynamics from a stationary array of resonators is a complex task. For this to happen, the system must move beyond the basin of attraction of other stable states, particularly flat ones [126]. One approach involves applying an unstable initial condition to the system and, after a certain number of periods, enforcing stability. Kimura and Hikihara [127] demonstrated this technique numerically and experimentally by modulating the nonlinear stiffness of an oscillator chain to generate ILMs.

More than 65 years ago, Anderson [128] illustrated how energy localization in linear

lattices could spontaneously occur through the introduction of impurities. In contrast, nonlinear chains can exhibit a similar phenomenon via ILMs, eliminating the need to break periodicity [121]. In fact, the relationship between Intrinsic Localized Modes and stationary solitons is documented in the literature [81, 105–107, 129–131]. According to Płodzień and Sacha [132], the center of mass of a soliton may undergo Anderson localization, a result also corroborated by [133]. In experimental setups, it is essential to ensure that an oscillator has an impurity within the nucleation range while impurities in other oscillators remain beyond this range. Analyses of the disorder levels tolerable for Anderson localization in nonlinear regimes can be found in [134] and [135].

In a previous work, where we proposed a design strategy that adhered to the periodicity hypothesis [101], we did not consider the possibility of using impurities for soliton nucleation. In the present study, we incorporate this new guideline, eliminating the need to develop a strategy to displace the basin of attraction for competing stable solutions<sup>3</sup>.

#### IV. FREQUENCY-DOMAIN ANALYSIS OF THE MULTIPHYSICAL DDNLS

Following recent studies from the last decade [43, 45, 48, 136–138], it can be stated that, in the context of energy harvesting, coupled oscillators outperform individual oscillators, particularly when nonlinearities are introduced. We highlight two works on coupled beam systems: Xie *et al.* [47], which considered the clamped-free configuration, and Shim *et al.* [49], which addressed the clamped-clamped case. In both papers, a significant improvement in the frequency response of the harvesters was achieved, prompting us to explore possible strategies for engineering similar systems. Indeed, regarding the design of metastructures for vibration-related applications, various studies have introduced a range of techniques, including post-buckling instability [139], energy absorption [140], topology optimization [141], voxel modeling [142], bio-inspired design [143], wave-based finite element method [144], stochastic modeling [145], inverse design [146], among others. Regardless of the examples found in the literature, to the best of the authors' knowledge, no studies have utilized the DDNLS as a design framework for energy harvesters.

The first step toward designing such structures through the DDNLS in VEH is to verify the ability of Equation (5) to predict the steady-state dynamic behavior of Equation (1)

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<sup>3</sup> The equation studied by Kenig *et al.* [81], for instance, always admits the zero as a solution for the soliton.

from a frequency perspective. It is important to note that all parameters defining Equation (5) are determined based on the choice of the perturbation factor  $\varepsilon$ , which, in turn, is directly related to the excitation frequency  $\Omega$  (Equations (A2)). This observation allows us to structure the analysis in two stages: first, for a fixed frequency, and subsequently, the study of the frequency response.

#### A. Fixed frequency: time domain

TABLE I. Parameters used in the fixed-frequency simulations of Sec. IV A.

| Parameter        | Value                  | Unit              | Parameter           | Value              | Unit              |
|------------------|------------------------|-------------------|---------------------|--------------------|-------------------|
| $N$              | 53                     | dimensionless     | $k_{nl}$            | $1 \times 10^9$    | $\text{N m}^{-3}$ |
| $\Delta s$       | $1.923 \times 10^{-3}$ | m                 | $q_{k_{nl}}(n)$     | 0                  | $\text{N m}^{-3}$ |
| $m$              | 1                      | kg                | $k_{cnl}$           | $-1 \times 10^7$   | $\text{N m}^{-3}$ |
| $k_\ell$         | $1 \times 10^5$        | $\text{N m}^{-1}$ | $f_m$               | 0.9                | kg                |
| $\phi(0)$        | $-1.5 \times 10^4$     | $\text{s}^{-2}$   | $q_{f_m}(n)$        | 0                  | kg                |
| $\phi(n \neq 0)$ | 0                      | $\text{s}^{-2}$   | $C_p$               | $1 \times 10^{-8}$ | F                 |
| $\zeta$          | 0.3%                   | dimensionless     | $\theta(n)$         | $1 \times 10^{-2}$ | $\text{N V}^{-1}$ |
| $q_\zeta(n)$     | 0%                     | dimensionless     | $\varepsilon$       | $1 \times 10^{-2}$ | dimensionless     |
| $A_{cc}$         | 10%                    | g                 | Steady-state regime | 700                | periods           |

Table I presents the parameters employed in the numerical simulations performed in this section, where, for an initial investigation, the only free parameters<sup>4</sup> are the linear coupling stiffness  $k_{cl}$  and the electrical load  $R$ . Considering the fixed physical parameters, Equation 5 reduces to the following form:

$$0.2124 + (-2|F|^2 - \Gamma_1 + i(0.3 + \Gamma_2) - 1 + 8\delta_d(\chi))F + i\frac{\partial}{\partial \tau}F + \frac{\partial^2}{\partial \chi^2}F = 0, \quad (17)$$

where  $F(\chi, \tau)$  is denoted just as  $F$  and  $\Gamma_1$ ,  $\Gamma_2$ , and  $L_\chi$  are given by the Equations:

$$\Gamma_1 = \frac{1 \times 10^{-6} R^2}{2 \times 10^{-7} R^2 + 2} \quad \Gamma_2 = \frac{R}{6.32510^{-5} R^2 + 632.456} \quad L_\chi = \frac{2325.511}{\sqrt{k_{cl}}}. \quad (18)$$

<sup>4</sup> The choice of controllable variables is made according to the feasible control capabilities of the structure.

More general examples are provided in the subsequent sections. For didactic purposes, the linear coupling and electrical resistance are chosen as the variable parameters, having the same order of magnitude as the parameters used in the Proof of Concept reported in the second part of this work [147].

It is important to emphasize that, for the variable  $q$  to remain positive, the periodicity breaking must occur in a way that reduces the stiffness of the oscillator where the impurity is located (see Equation 16), an important nuance demonstrated in [103].

Since the focus is on developing a generalizable methodology, the dimensionless parameters in Equation 5 are the primary main topic of investigation. Notably, all physical parameters can influence the stability of the soliton, making it impractical to establish a general stability diagram. In other words, questioning, for instance, the required acceleration level to generate a stationary wave is only meaningful when the linear stiffness, damping, resistor characteristics, coupling, number of oscillators, and other system properties are predefined.

The predictive capability of the DDNLS is assessed through the numerical integration of Equation (17), conducted in parallel with the numerical integration of Equation (1), where three cases are presented<sup>5</sup>. Notice that the solution of Equation (1) consists of terms involving multiple powers of  $\varepsilon$  (see Equation (A6)), whereas the secular terms that led to the DDNLS are exclusively related to the zeroth-order term of this solution (see Equation (A11)). In this approach, we do not aim to numerically correlate the amplitude obtained from the DDNLS with that of the differential equation governing the chain. Instead, we focus on the qualitative agreement of the oscillation amplitude profiles, which, due to the multiple-scale method, should exhibit the same qualitative behavior. In other words, a nucleation observed in the DDNLS would imply a corresponding nucleation in the lattice. This reasoning can be extended to other physical phenomena, such as multinucleations, chaotic/non-periodic behaviors, or sensitivity to physical parameters.

The first case under investigation, shown in Figure 5, presents a configuration of  $R$  and  $k_{cl}$  capable of nucleating oscillations in the DDNLS as a consequence of the impurity/disorder located at  $\chi = 0$ . The numerical verification of the dynamics governed by Equation (1) confirms this prediction in both the mechanical and electrical domains. The voltage amplitude predicted by the DDNLS can be quantified through the terms  $\frac{R_{\varepsilon^{-1}\omega_0}\theta_\varepsilon(s)\varphi(s, t\varepsilon)e^{i\omega_0 t}}{C_{p\varepsilon}R_{\varepsilon^{-1}\omega_0} - i}$  in Equation (A11).

The second case under analysis, shown in Figure 6, employs the same electrical resistance as in Figure 5, however with a coupling four times weaker. As a consequence of the new coupling an increase in the spatial domain over which the DDNLS operates occurs (see

<sup>5</sup> The search for closed-form solutions to the damped and externally driven NLS remains an open research area. For an introduction to the state of the art, we recommend referring to [102].

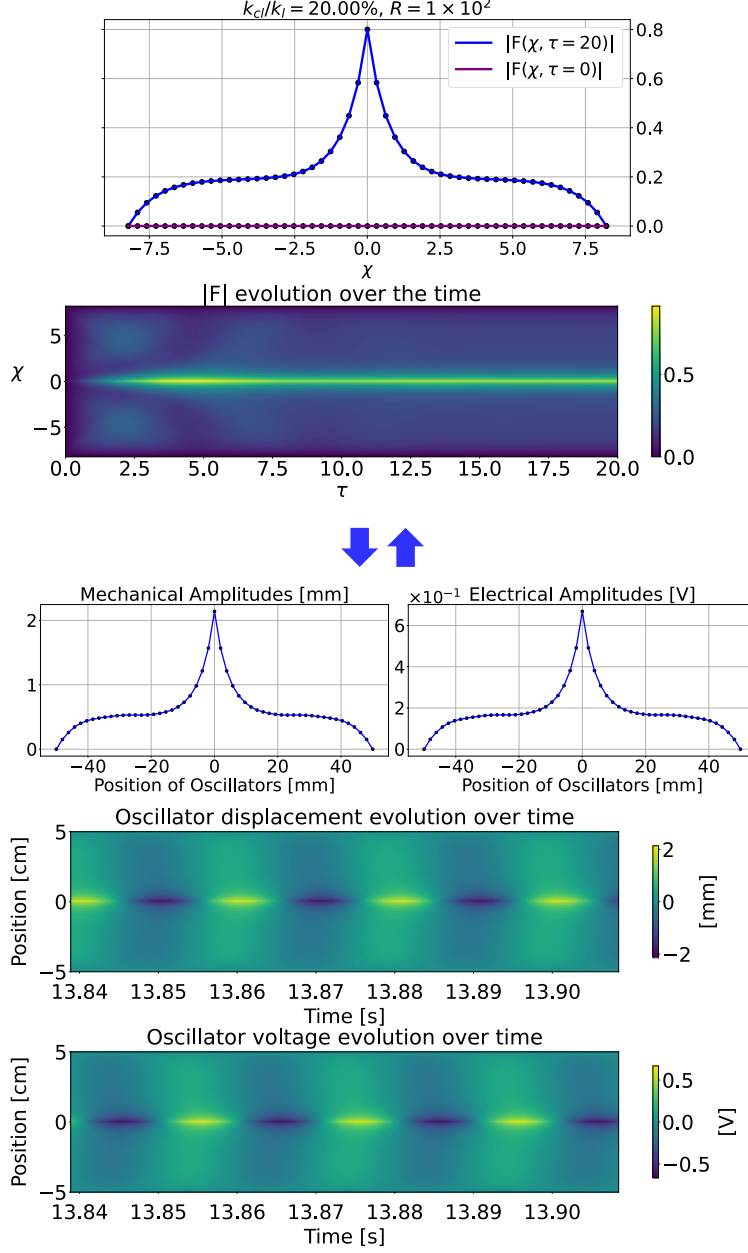


FIG. 5. Comparison between the numerical integration of Equation (17) and the numerical integration of Equation (1). Simulation parameters are shown in Table I. Integration method: Python function DOP 853. Case 1: Configuration of  $R = 100\Omega$  and  $k_{cl} = 20\%k_l$  leading to nucleation.

Equations (18)). Precisely, for a fixed value of  $N$ , the discretization density of Equation (17) becomes sparser, leading to a broader spacing between oscillators in the DDNLS and weakening the continuity assumption [101]. Once again, the evolution of Equation (1) obeys the qualitative behavior of Equation (17), resulting in a delocalization of the nucleated solution observed in Figure 5.

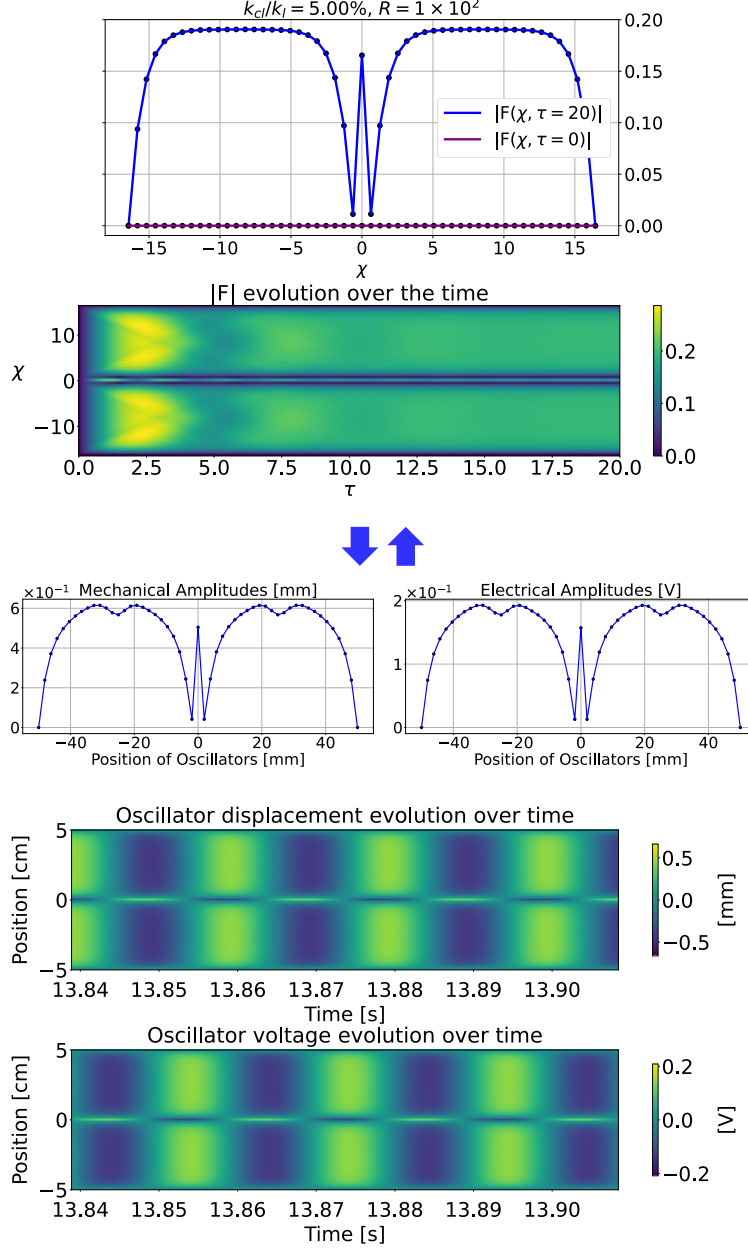


FIG. 6. Comparison between the numerical integration of Equation (17) and the numerical integration of Equation (1). Simulation parameters are shown in Table I. Integration method: Python function DOP 853. Case 2: Configuration of  $R = 100\Omega$  and  $k_{cl} = 5\%k_l$  not leading to nucleation due to low coupling.

347 The third case, depicted in Figure 7, maintains the same coupling as in the first case but  
348 with an increased value of  $R$ . Despite the unchanged coupling (and therefore the same spatial  
349 domain  $L_\chi$ ) the previously observed localization around  $\chi = 0$  begins to fade. Equation  
350 (1) reproduces the delocalization predicted by the DDNLS, exemplifying once more the

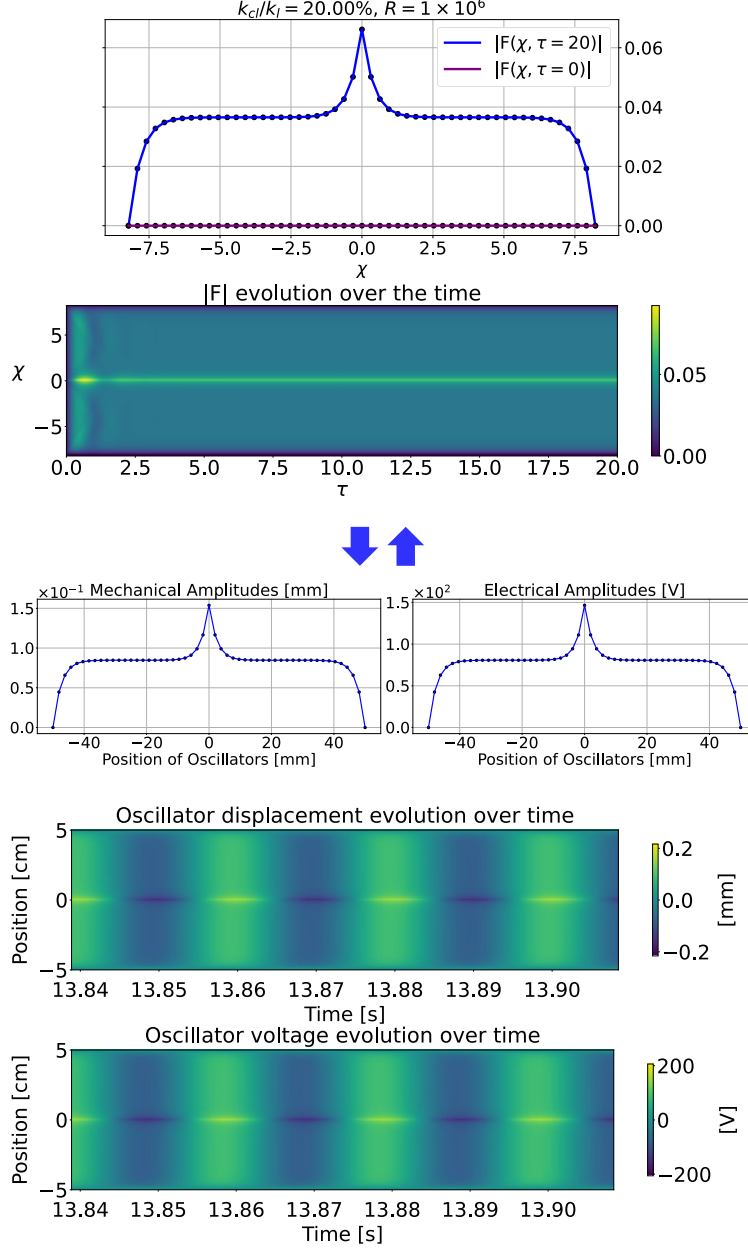


FIG. 7. Comparison between the numerical integration of Equation (17) and the numerical integration of Equation (1). Simulation parameters are shown in Table I. Integration method: Python function DOP 853. Case 3: Configuration of  $R = 1 \times 10^6 \Omega$  and  $k_{cl} = 20\%k_l$  not leading to nucleation due to high electrical resistance.

351 qualitative correlation between the DDNLS and the lattice differential equation.

### 1. Fixed frequency: time domain - Power analysis

An immediate question in the context of energy harvesting concerns the selection of the load  $R$  that maximizes the extracted power. Given the demonstrated influence of electrical resistance, a numerical sweep guided by the DDNLS can be conducted to analyze the relationship:

$$P_n(t) = \frac{[V_n(t)]^2}{R}, \quad (19)$$

where  $P_n(t)$  denotes the instantaneous power associated with the  $n$ -th oscillator. It is assumed that the total power harvested by the system corresponds to the sum of the individual contributions, as defined by the topology depicted in Figure 1. By neglecting the transient terms in the solutions given by Equations (A11) and reversing the normalization introduced in (A17), the numerical integration of Equation (18) for varying values of  $R$  enables the identification of an optimal resistance for a given performance metric related to power extraction.

It is important to note that instantaneous power,  $P_n(t)$ , provides limited insight into the overall performance of the harvester, as the system under consideration is in oscillatory regime. While it is common to evaluate the integral of  $P_n(t)$  over a single oscillation cycle [23, 148], the soliton framework focuses on steady-state amplitudes of individual components. Accordingly, the same approach is adopted for the power metric. Thus, the optimization process presented herein pertains not to the instantaneous power  $P_n(t)$ , but to  $P_n^M$ , which denotes the amplitude of the extracted power associated with oscillator  $n$ .

Figure 8 summarizes the search for the optimal electrical resistance and is sequentially divided into three parts. The first part (color map) illustrates the power distribution of the oscillators for various values of  $R$ , highlighting the wave delocalization as the resistance changes. The second part presents the total power as a function of  $R$ , allowing for the identification of a clear design criterion. Finally, the last part of the figure displays the power profile along the chain for three different values of  $R$ . Such profile is obtained through the numerical integration of Equation (1), rather than from the DDNLS (18), meaning that while the soliton serves as a design guide, its results are independently verified. The resistance value determined by the DDNLS proves to be more effective when compared to deviations around its optimal value.

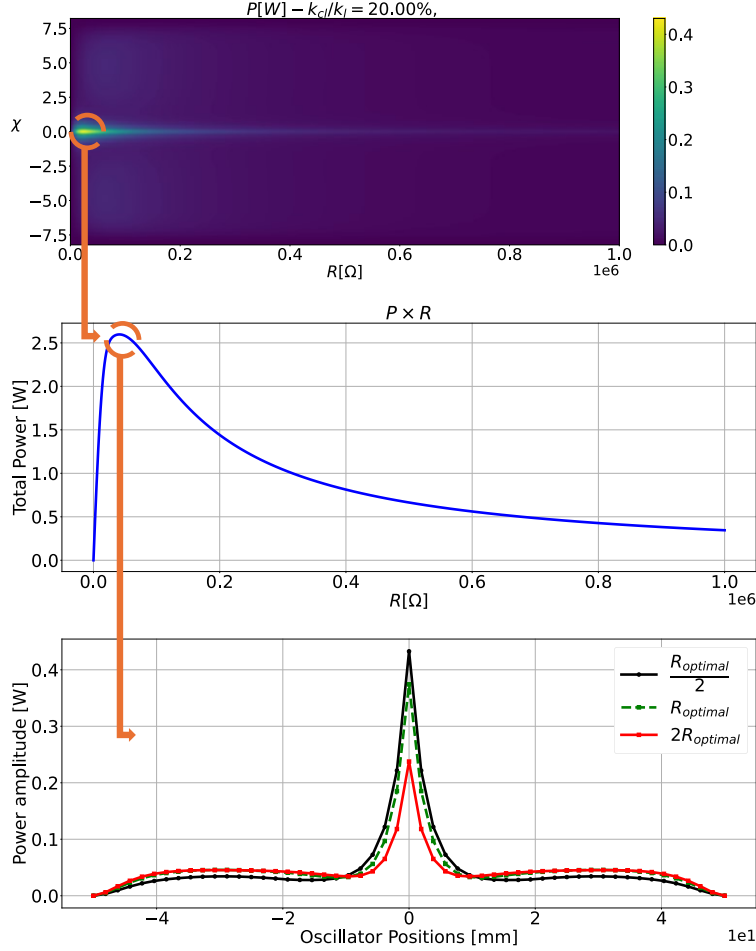


FIG. 8. Numerical search of an optimal  $R$  in Equation (17). Sequentially: oscillator power described by Equation (17) for different values of  $R$ , total power sum of all oscillators for the corresponding  $R$  values, and the power profile of the oscillators for the optimal  $R = 4.13 \times 10^4 \Omega$  indicated by the DDNLS (profile obtained through Equation (1)). The linear coupling of the oscillators is set to 20% of the linear stiffness.

## B. Frequency domain: $\Omega < \omega_0$

Unlike the DDNLS obtained for a given excitation frequency, a change in  $\Omega$ , being correlated with  $\varepsilon$  (see Equation (A2)), necessarily modifies all dimensionless parameters of Equation (5). In the simulations presented in this section, the parameters listed in Table I are maintained, along with a coupling of  $k_{cl} = 10\% k_l$  and a resistance of  $R = 1 \times 10^6 \Omega$ . By varying  $\Omega$  within the range of 98% to 99% of the natural frequency  $\omega_0$  (corresponding to  $0.01 \leq \varepsilon \leq 0.02$ ), the parameters defining Equation (17) evolve as illustrated in Fig-

ure 9. An advantage of using the DDNLS instead of the differential equation governing the lattice is the significant reduction in computational cost for numerical treatment. This characteristic enables the investigation of steady-state amplitude behavior without requiring extensive computation time to reach the permanent regime. The choice of an analysis range within  $98\%\omega_0 \leq \Omega \leq 99\%\omega_0$  is not dictated by computational cost but rather, again, for demonstration purposes.

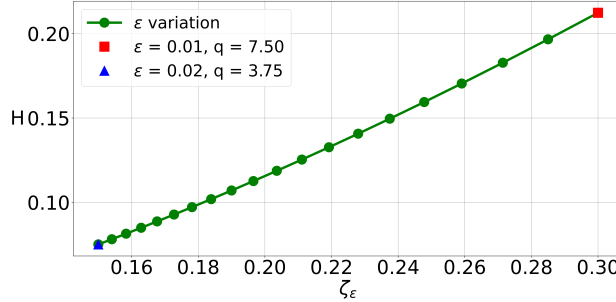


FIG. 9. Variation of  $H$  and  $\zeta_\varepsilon$  as a function of excitation frequency  $\Omega$ .

It is necessary to remember that the factor responsible for external excitation  $H$  depends on the cubic nonlinearity and, as indicated in Equation (6), the sign of  $k_{nl}$  determines the sign of the term accompanying  $|F|^2$ . Based on this information, the system's frequency response<sup>6</sup> is presented in two contexts: hardening and softening. The goal of this analysis is not an exhaustive study of the polarity's effect on  $k_{nl}$  but to demonstrate that, in both cases, the DDNLS can be employed for frequency analysis, providing the designer with greater engineering flexibility.

In agreement with the parametric variation shown in Figure 9, as  $\Omega \rightarrow \omega_0$ , the perturbation factor tends to zero ( $\varepsilon \rightarrow 0$ ), i.e. leading to  $H$  and  $\zeta_\varepsilon \rightarrow \infty$ . Concerning the choice of simulation parameters (Table I), the use of positive nonlinearities exhibits a stable behavior across the entire frequency range when  $\Omega < \omega_0$  (Figure 10), a result supported by Shim *et al.* [49].

Alternatively, by imposing  $k_{nl} < 0$ , a completely different dynamics emerge (Figure 10). For the same simulation parameters, contrary to what is observed in hardening behavior, an abrupt nucleation occurs as the excitation approaches the natural frequency. Worth

<sup>6</sup> We corroborate the notion that the concept of a frequency response curve is strictly applicable to linear systems. Here, we refer to the term “frequency response” as the response of the model in the frequency domain.

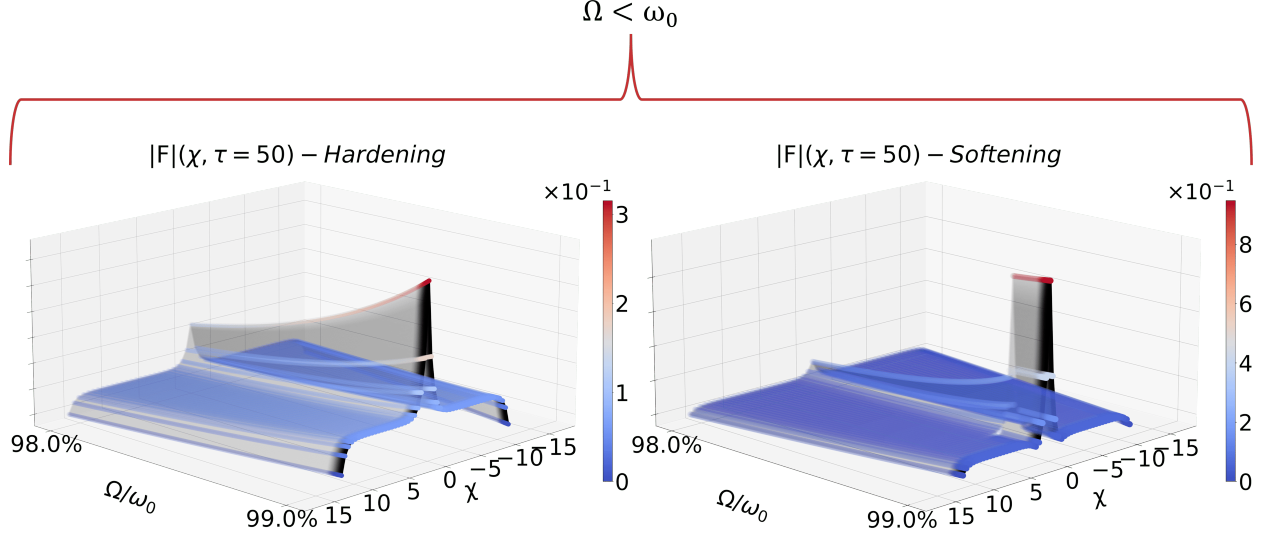


FIG. 10. Spatial configuration evolution of the solitonic envelope amplitude  $|F(\chi, \tau = 50)|$  governed by the DDNLS for different excitation frequencies  $\Omega < \omega_0$ . Left: system with positive nonlinear stiffness (hardening effect); Right: system with negative nonlinear stiffness (softening effect). In both cases, soliton nucleation is observed as the excitation frequency approaches the linear fundamental frequency  $\omega_0$ , confirming the predictive capacity of the DDNLS model. The softening case presents higher localization amplitude and spontaneous nucleation onset compared to the hardening regime.

noting that all simulations performed in this study start from motionless oscillators. As a consequence of this condition, all nucleation events presented are spontaneous, arising not from avoiding basins of attraction of undesired solutions but from verifying the possibility of direct emergence (an important advantage in experimental contexts).

One should observe that both the spatial domain  $L_\chi$  and the dimensionless impurity  $q$  (Equation (16)) are also subject to frequency variation. As shown in Figure 10, as  $\Omega \rightarrow \omega_0$ ,  $L_\chi \rightarrow 0$ . Moreover, from Figure 9, it can be observed that as in  $H$  and  $\zeta_\epsilon$ , also  $q \rightarrow \infty$ . These mathematical relationships impose two simultaneous physical consequences on the frequency response of the DDNLS: the emergence of stability windows near resonance (due to the reduction of the spatial domain [119]) and the expansion of the stability diagram (as a result of the increase in  $q$  [101]), making the derivation of a general chart, again, a numerical difficulty.

### 421 C. Frequency domain: $\Omega > \omega_0$

422 Up to this point in the mathematical development, all analyses have been conducted  
 423 under the premise that  $\Omega < \omega_0$  (see Equations (A2)). In a context of energy harvesting,  
 424 excluding excitations greater than  $\omega_0$  leads to a significant reduction in the bandwidth of  
 425 the harvesting structures. To address this issue, the following algebraic step is taken:

$$\Omega = \omega_0(1 - \varepsilon) \rightarrow \omega_0(1 + \varepsilon). \quad (20)$$

426 Notice that, due to the condition  $0 < \varepsilon$ , it follows that  $\Omega > \omega_0$ . By strictly following the  
 427 same algebraic manipulation that led to Equation (A16) from Equation (1), the following  
 428 term in the equation is modified:

$$H \left( -1 - \frac{q_{fm}(\chi)}{f_m} \right) e^{-i\tau} \rightarrow H \left( -1 - \frac{q_{fm}(\chi)}{f_m} \right) e^{i\tau}, \quad (21)$$

429 leading to a new normalization

$$\Phi(\chi, \tau) = e^{-i\tau} \overline{F(\chi, \tau)} \rightarrow e^{i\tau} \overline{F(\chi, \tau)}, \quad (22)$$

430 which consequently leads us back to the DDNLS:

$$\begin{aligned} & H \left( 1 + \frac{q_{fm}(\chi)}{f_m} \right) - \left( 2 + \frac{2q_{knl\varepsilon}(\chi)}{k_{nl\varepsilon}} \right) F(\chi, \tau) |F(\chi, \tau)|^2 \\ & + \left( -\Gamma_1(\chi) + i(\zeta_\varepsilon + q_{\zeta\varepsilon}(\chi) + \Gamma_2(\chi)) + 1 - \frac{\phi_\varepsilon(\chi)}{2\omega_0^2} \right) F(\chi, \tau) \end{aligned} \quad (23)$$

$$i \frac{\partial}{\partial \tau} F(\chi, \tau) + \frac{\partial^2}{\partial \chi^2} F(\chi, \tau) = 0.$$

431 Let us observe the only difference between Equation (5) and Equation (23): the constant  $-1$   
 432 in (5) becomes positive in (23), while all definitions and relationships of the dimensionless  
 433 variables remain unchanged.

434 The numerical simulation of Equation (23) is presented in Figure 11, considering the  
 435 same physical parameters as in Section IV B. From variable  $L_\chi$  point of view, two symmet-  
 436 ric behaviors are observed as the excitation frequency approaches the natural frequency. As  
 437  $\Omega \rightarrow \omega_0$ , one has  $L_\chi \rightarrow 0$ , which, again according to [101, 119], is expected to enhance the  
 438 stability of the DDNLS. For the case where  $\Omega < \omega_0$  (Figure 10), soliton nucleation becomes

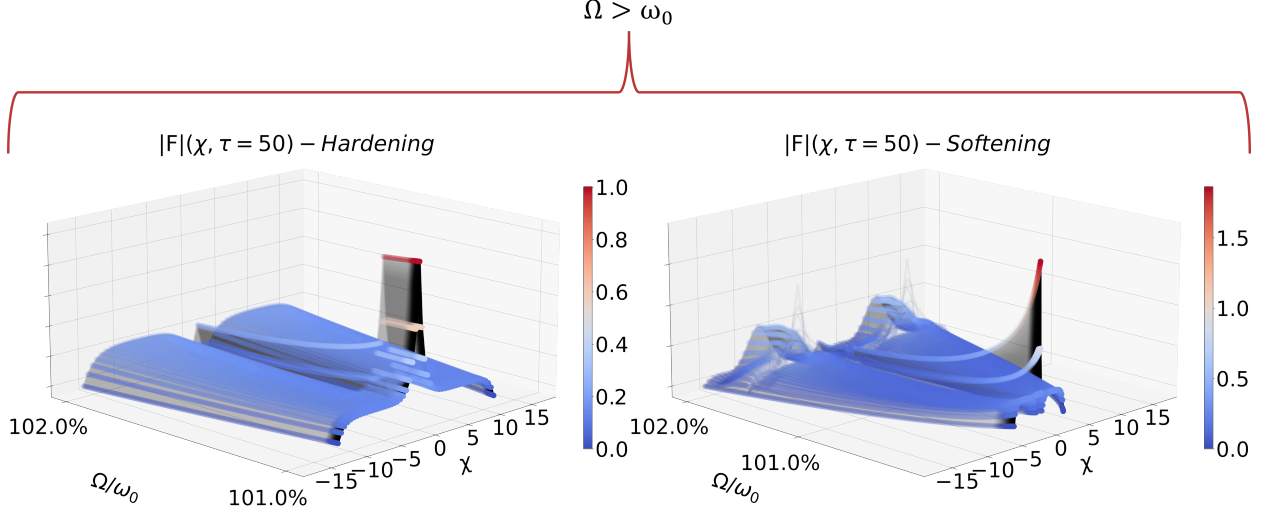


FIG. 11. Spatial configuration evolution of the solitonic envelope amplitude  $|F(\chi, \tau = 50)|$  governed by the DDNLS for excitation frequencies  $\Omega > \omega_0$ . Left: system with positive nonlinear stiffness (hardening effect); Right: system with negative nonlinear stiffness (softening effect). As in the sub-resonant case, soliton nucleation occurs near the natural frequency, but the spatial profiles differ significantly. The hardening regime exhibits localized structures with sharp amplitude decay, while the softening regime shows an opposite behavior.

more pronounced as the excitation approaches resonance, particularly when a negative non-  
linearity is imposed, where an abrupt transition is observed. Conversely, when  $\Omega > \omega_0$ , a  
similar trend is identified, but the abrupt nucleation occurs under a positive nonlinearity  
rather than a negative one. Since nucleation is favored as  $\Omega \rightarrow \omega_0$ , and consequently  $L_\chi \rightarrow 0$ ,  
the conclusions previously reported in [101, 119] are reaffirmed in the frequency-domain.

## V. USE OF THE DDNLS IN THE DESIGN OF SMART STRUCTURES

Based on the analyses presented along sections IV A, IV B and IV C, the development  
of a design algorithm becomes feasible. In [101], a strategy founded on the assumption of  
periodicity, a purely mechanical physical domain, and compliance with the stability diagram  
reported by Barashenkov and Smirnov [108] was employed. This approach is not adopted  
in the present study.

Motivated by the findings reported in [103], the assumption of perfect periodicity is not

considered, acknowledging that structural periodicity is inherently limited by manufacturing tolerances. Figures 2, 3, and 4 highlight the soliton’s sensitivity to impurity values, while Figures 5 and 6 demonstrate its sensitivity to coupling. Figure 8 illustrates the influence of electrical resistance on the phenomenon, Sections IV C and IV B address its dependence on excitation frequency, and works in the literature have reported its sensitivity to damping [108], acceleration/nonlinearity [101], spatial domain [119], and excitation type [102].

In Section IV A, a parametric study is presented, relating two design variables: the electrical load and the coupling between oscillators. It is worth noting that a similar parametric analysis could be conducted with any of the 14 fixed parameters listed in Table I, since all of them influence the phenomenon. Given these multiple dependencies, a heuristic approach becomes a better option for seeking an optimal configuration of the physical parameters corresponding to a desired behavior.

The proposed framework assumes prior knowledge of both the frequency range and the acceleration levels to which the structure is submitted, given that various applications are already documented in the literature [149].<sup>7</sup> Another predefined variable is the number  $N$  of oscillators, as it relates to the maximum number supported by the architecture and design parameters. Referring again to Equation (9), it is trivially observed that, from a spatial domain point of view, the effects of decreasing  $N$  can be compensated by an increase in coupling. Additionally, the type of transducer also imposes constraints on the adjustable physical parameters. The use of coils and permanent magnets allows for tunability of the damping coefficient [150], a degree of freedom not afforded by piezoelectric transducers, which, on the other hand, modify stiffness. In this context, damping is also assumed to be a given/measured property of the individual oscillators. Regarding the coupling stiffness, a greater flexibility in adjustment could be assumed, for instance, by tuning the distance between magnets [151].

#### A. Design Algorithm - Framework

The first step in applying the DDNLS in optimal design of harvesters is the identification of the adjustable physical parameters. For the specific case considered in this section, the

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<sup>7</sup> Liang *et al.* [149] provided a comprehensive figure illustrating this assumption based on applications reported by different authors.

mechanical ones are:  $k_l$ ,  $k_{nl}$ ,  $k_{cl}$ , and  $\phi(n_i)$  (where  $n_i$  represents the intermediate oscillator in the chain). For the electrical domain, the adjustable parameter is  $R^8$ . The disorder functions associated with the variables in Equation (1) are disregarded ( $q_b(n) = q_{k_{nl}} = q_{f_m} = 0$ ), though they could be incorporated depending on the specific application. The parameters presented in Table I are reused, without constraining the variables under design.

The second step of the algorithm involves the mathematical formalization of the DDNLS equation associated with the lattice, summarized as follows:

$$\begin{aligned}
& H(k_{nl}, k_l, \Omega) + \alpha_{k_{nl}} 2F(\chi, \tau) |F(\chi, \tau)|^2 + i \frac{\partial}{\partial \tau} F(\chi, \tau) + \frac{\partial^2}{\partial \chi^2} F(\chi, \tau, k_{cl}, \Omega) \\
& + [-\Gamma_1(\chi, R, k_l, \Omega) + i[\zeta_\varepsilon(\Omega) + \Gamma_2(\chi, R, k_l, \Omega)] + \alpha_\Omega + q(k_l, \phi, \Omega) \delta_d(\chi)] F(\chi, \tau) = 0
\end{aligned} \tag{24}$$

where  $\alpha_{k_{nl}} = 1$  for  $k_{nl} < 0$  and  $\alpha_{k_{nl}} = -1$  for  $k_{nl} > 0$ , while  $\alpha_\Omega = -1$  for  $\Omega < \omega_0$  and  $\alpha_\Omega = 1$  for  $\Omega > \omega_0$ . The variables influencing the dimensionless parameters are explicitly defined by the equations derived in Section II. It is again worth mentioning that questions such like “*What level of nonlinearity should the system be subjected to?*” are only meaningful after selecting  $\Omega$ ,  $k_l$ ,  $R$ ,  $\phi$ , and  $k_{cl}$ .

The search for design parameters can only take place based on efficiency measures that are specific to their respective contexts: Energy localization along the chain? Maximization of the total power? Control of ILMs? etc. Given that our focus is on energy harvesting, functions dependent on the extracted power must be defined. These include the total amplitude of power in permanent regime of the oscillators at a given frequency  $P_T^M(\Omega)$ :

$$P_T^M(\Omega) = \sum_{i=0}^N P_i^M(\Omega), \tag{25}$$

the power within a specific region of the lattice,  $P_{ab}^M(\Omega)$ :

$$P_{ab}^M(\Omega) = \sum_{i=a}^b P_i^M(\Omega), \tag{26}$$

and the power of a single oscillator  $P_c^M(\Omega)$ , where  $a$ ,  $b$  and  $c$  designate the indices interest. From the power definitions, diferentes performance metrics, denoted by  $Opt$ , become feasible, such as:

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<sup>8</sup> The capacitance of the piezoelectric element, as well as its electromechanical coupling, are typically functions of the electrical circuit architecture and the materials used [152].

$$Opt = \int_{\Omega_i}^{\Omega_f} P_T^M(\Omega) d\Omega, \quad (27)$$

for the maximization of the harvested power over the entire excitation frequency range  $[\Omega_i, \Omega_f]$  across all oscillators, or, for instance,

$$Opt = \int_{\Omega_i}^{\Omega_f} \frac{P_{ab}^M(\Omega)}{P_T^M(\Omega)} d\Omega + \int_{\Omega_i}^{\Omega_f} P_T^M(\Omega) d\Omega, \quad (28)$$

for a quantity that considers the energy localization between specified oscillators.

Due to the nonlinear nature of the problem, the absence of analytical solutions, the discontinuities in the wave function, the potentially unstable behavior of Equation (24), and the high number of design variables the development of an optimization algorithm becomes a computational challenge. Under these conditions, the use of heuristic approaches becomes promising, and, in this study, the Bayesian optimization is employed [153].

## B. Implementation Example

The formulation of one possible optimization problem is described by the following relations:

$$\begin{aligned} & \max_{k_l, k_{nl}, k_{cl}, \phi(n_i), R} \int_{\Omega_i}^{\Omega_f} P_T^M(\Omega) d\Omega, \\ & \text{subject to} \\ & 90 \text{ Hz} \leq \Omega \leq 110 \text{ Hz}, \quad A_{cc} = 0.1g, \end{aligned} \quad (29)$$

$$10^{-5} [100(2\pi)]^2 \leq k_{cl} \leq 6 [100(2\pi)]^2, \quad 10^{-1} [100(2\pi)]^2 \leq \phi(n_i) \leq [100(2\pi)]^2,$$

$$2 \cdot 10^9 \leq k_{nl} \leq 2 \cdot 10^{13}, \quad 2 \cdot 10^4 \leq R \leq 2 \cdot 10^6, \quad 10^{-1} [100(2\pi)]^2 \leq k_l \leq [100(2\pi)]^2,$$

where the remaining physical variables are presented in Table I (physical units omitted).

As mentioned, the range delimiting  $\Omega$  and the acceleration amplitude are known. However, the other constraints related to the design variables depend on the project's control possibilities. As shown, the excitation occurs around 100 Hz, indicating the order of magnitude for the constraint conditions of  $k_l$  and  $\phi(n_i)$ . The coupling stiffness, on the other hand, offers greater numerical flexibility, as we do not assume either a low or high coupling

hypothesis. Unlike the linear stiffness terms involved in the optimization,  $k_{nl}$  and  $R$  are not necessarily constrained by the order of magnitude of  $\Omega$ , being instead defined again by the possibilities enabled by the metastructure's architecture<sup>9</sup>. For the optimization, a Bayesian approach with a division of 5000 points in the parameter domain is performed, followed by an iterative evolution over 500 iterations. The steady-state regime for the soliton dynamics in Equation 24 is reached at  $\tau = 35$ .

For low coupling conditions, the value of  $k_l$  lies around  $[100(2\pi)]^2$ , ensuring that the oscillators resonate close to the central excitation frequency. In scenarios with higher coupling, a reduction in  $k_l$  is required; however, in the absence of prior information, an assumption is made by lowering the minimum bound of  $k_l$  by one order of magnitude. The same rationale is applied to define the bounds of  $\phi(n_i)$ , which characterizes disorder rather than localized impurities.

The limits of the remaining parameters are defined by practical feasibility. For instance, certain system architectures may impose constraints on the minimum attainable values of  $k_{cl}$ . A broader design flexibility would allow for wider admissible ranges. Therefore, a two-step optimization strategy is recommended: first, to delimit the feasible design space; second, to search for optimal configurations within that space. In our scenario, the nonlinearity is allowed to vary across four orders of magnitude, whereas the electrical resistance is constrained to two orders, although these ranges can be adjusted depending on the practical limitations of the lattice.

Figure 12 presents the results of the power predicted by the optimal parameters. On the left, the amplitude in [W] is shown as a function of frequency for all oscillators, while on the right, the total power (representing the sum of all individual contributions) is depicted. The energy localization around the center of the chain occurs before the natural frequency, in alignment with the predictions of the DDNLS. The optimization is performed over a bandwidth of  $90 \text{ Hz} \leq \Omega \leq 110 \text{ Hz}$ , indicating that the obtained parameters do not represent a general result, as they are specifically optimized for this range of  $\Omega$ . Consistently, as a validation mechanism for the predictions of Equation (24), the numerical integration of Equations (1) and (2) is performed. Figure 13 presents the numerical evaluation within the mechanical and electrical physical domains, along with the corresponding power. Similarly to

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<sup>9</sup> For an isolated oscillator, the optimal value of  $R$  depends on the capacitance and the excitation frequency[154].

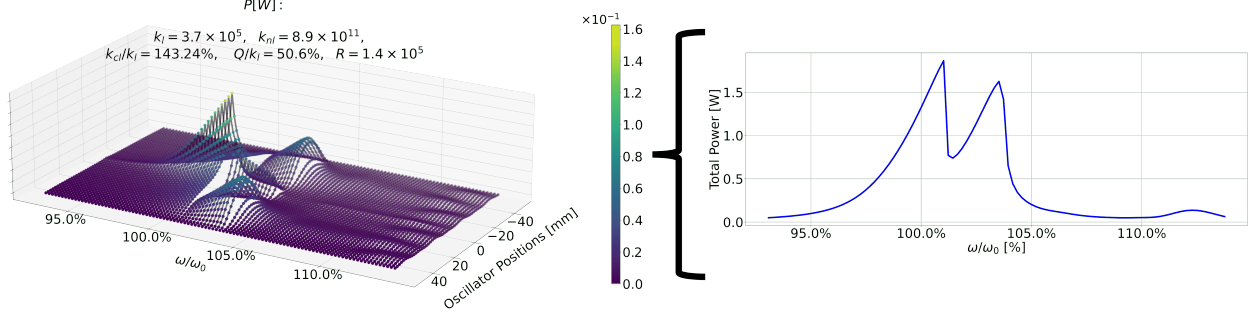


FIG. 12. Soliton power response as a function of normalized excitation frequency  $\omega/\omega_0$ , based on the optimized parameters obtained via the Bayesian optimization algorithm described in Section V B. Left: spatial distribution of the harvested power along the chain for varying excitation frequencies, highlighting frequency regions where energy localization occurs. Right: total power obtained by integrating the power across all oscillators, revealing distinct peaks associated with soliton nucleation. The observed multi-resonant behavior reflects the influence of system nonlinearity and coupling configuration on the emergence of stationary energy-localized states.

Figure 12, energy nucleation around the impurity is observed, followed by another harvesting region.

## VI. THEORETICAL INSIGHTS AND PARTICULARITIES OF SOLITON-BASED VEH

The configuration of physical parameters shown in Figure 13, although originating from the optimization of the DDNLS, does not provide a direct comparison metric with linear systems. To illustrate this issue, the same optimized parameters are employed in a new excitation parametric analysis, progressively increasing the contribution of nonlinearity to the lattice dynamics by means of acceleration increments. Figure 14 presents a comparison of the total power of the structure in two distinct scenarios: with and without nonlinear effects. One should recall that the objective function defined in (29) does not account for energy localization, but rather aims to maximize the sum of the individual powers over a given frequency range. The numerical results demonstrate not only an increase in the total power bandwidth, but also greater power extraction in the domain of interest when the nonlinear regime takes place. It should also be noted that the simulation is not per-

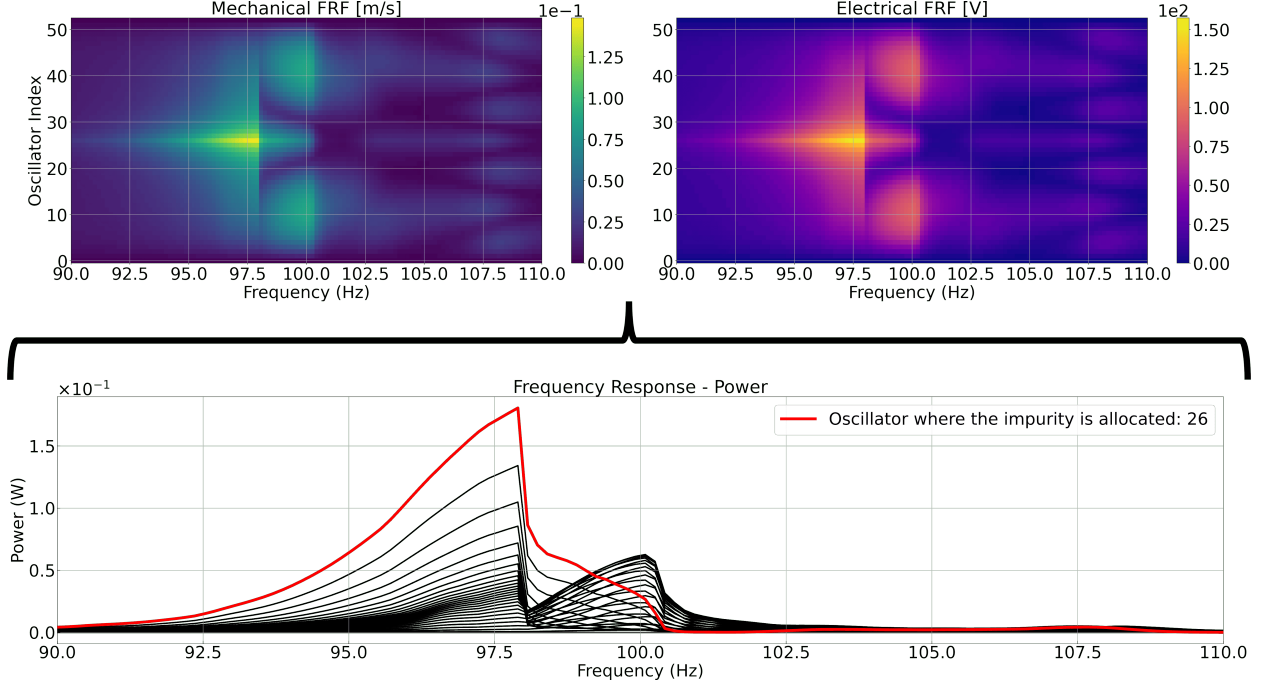


FIG. 13. Numerical integration of Equations (1) and (2) using the optimal parameters obtained via the Bayesian optimization procedure. Top-left: mechanical frequency response (FR) in terms of velocity amplitude across the oscillator chain. Top-right: electrical FR showing the voltage distribution induced by the piezoelectric coupling. Bottom: frequency response of the harvested power for each oscillator; the red curve highlights the oscillator containing the impurity (index 26), which exhibits strong localization and dominates the energy conversion. The results confirm the predictive capability of the DDNLS-based optimization framework and illustrate the effect of spatial impurity on nonlinear energy localization and harvesting.

562 formed via numerical continuation, but through a frequency-by-frequency integration of the  
 563 metastructure’s differential equations, thus eliminating the need for an experimental strat-  
 564 egy to migrate between lower and higher energy branches (a strategy commonly employed  
 565 in nonlinear VEH).

566 In a previous study published by the authors [101], the influence of the number of os-  
 567 cillators was investigated. From a design perspective, it is straightforward to consider the  
 568 minimum number of oscillators required to optimize a power-related metric. In fact, the  
 569 assumption of a minimum number of oscillators implies that the harvested power saturates  
 570 beyond a certain number of components in the metastructure. The Figure 15 follows a  
 571 sequential simulation pipeline to formalize this idea. First, the top panel (“Metric:  $\Sigma P$  in

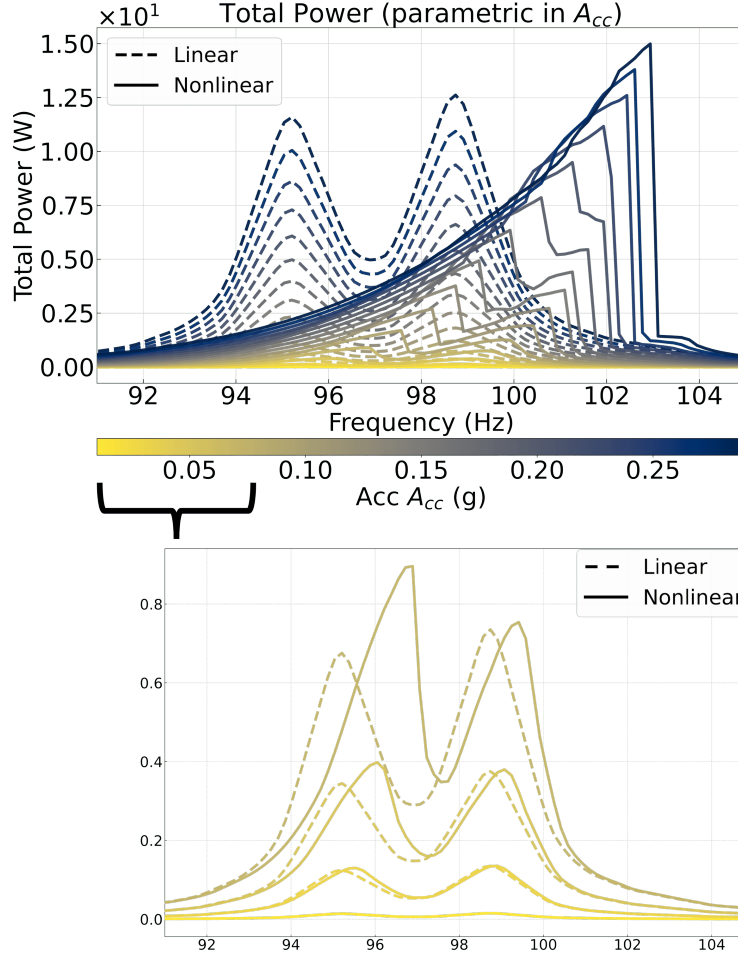


FIG. 14. Total harvested power vs. excitation frequency for the optimized parameters of Figure 13, under an acceleration sweep  $A_{cc}$  (color bar, in  $g$ ). For each  $A_{cc}$ , the metastructure is integrated frequency-by-frequency in time (no numerical continuation) until steady regime; the total power is computed as the sum over oscillators of  $v_n^{\text{peak}2}/R$ . Dashed curves: linear model ( $k_{nl} = k_{cnl} = 0$ ); solid curves: nonlinear model ( $k_{nl} > 0$ ,  $k_{cnl} < 0$ ). The nonlinear case systematically outperforms the linear one in both peak level and bandwidth within the targeted frequency range for the chosen parameters.

central window [W]”) uses the DDNLS to scan  $(k_c/k_l, Q/k_l)$  at a fixed load  $R = 10^{-6}$ ,  $\Omega$  and at  $\Omega = \omega_0(1 + e)$  (here  $e = 0.01$ ). The metric  $\Sigma P$  is defined as the sum of the steady-state electrical power over the seven central oscillators (impurity site and its  $\pm 3$  neighbors). This search identifies an optimum at  $k_c/k_l = 0.796$  and  $Q/k_l = 0.507$ , with  $\Sigma P = 1.01 \times 10^{-10}$ , W. Second, the middle panel (“DNLSE: metric vs R at best  $(k_c, Q)$ ”) keeps  $(k_c, Q)$  fixed and

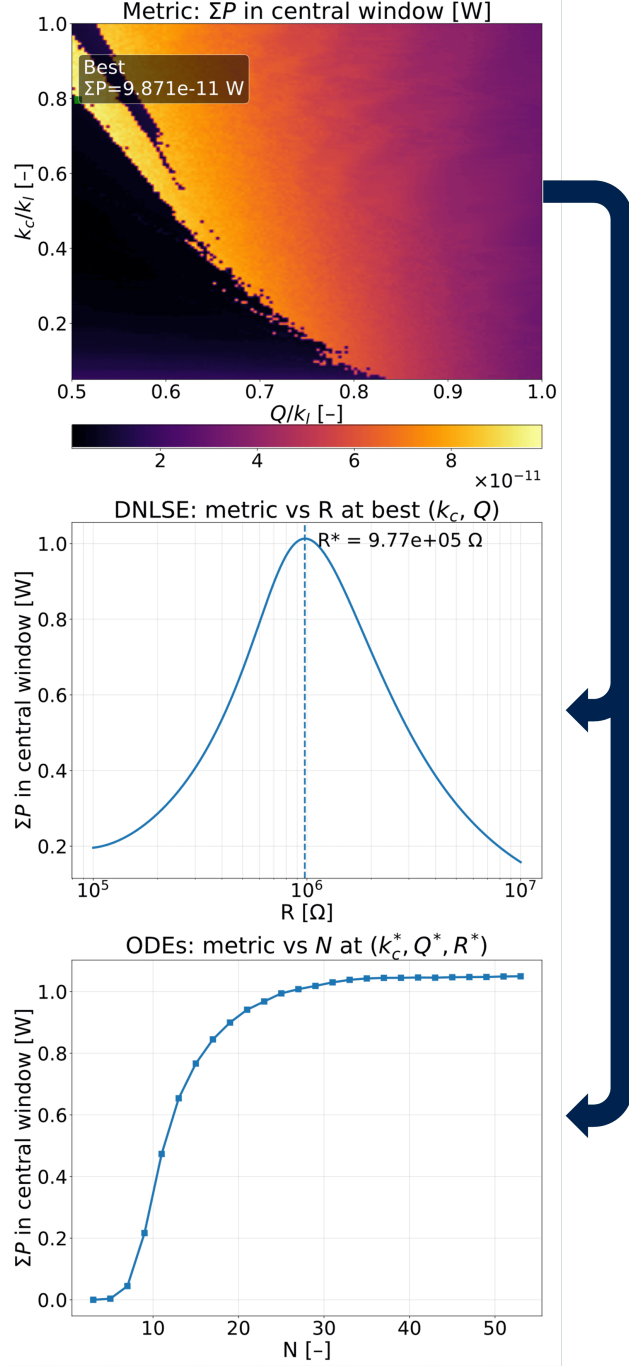


FIG. 15. Sequential simulations from top to bottom (arrows indicate the pipeline). “Metric:  $\Sigma P$  in central window [W]” (DNLSE, fixed  $R = 10^{-6} \Omega$ ,  $\Omega = \omega_0(1 + e)$  with  $e = 0.01$ ) finds  $k_c^*/k_l = 0.796$ ,  $Q^*/k_l = 0.507$ , and  $\Sigma P = 9.871 \times 10^{-11}$  W. “DNLSE: metric vs R at best ( $k_c, Q$ )” gives  $R^* = 9.77 \times 10^5 \Omega$ . “ODEs: metric vs N at ( $k_c^*, Q^*, R^*$ )” shows saturation of  $\Sigma P$  with  $N$ . Starting from 3 with an increment of 2 oscillators per simulation.

sweeps the load to obtain the optimal value  $R = 9.77 \times 10^5, \Omega$ . Third, the bottom panel (“ODEs: metric vs  $N$  at  $(k_c, Q, R)$ ”) integrates the full nonlinear electromechanical ODE chain at  $\Omega = \omega_0(1 + e)$  and shows that  $\Sigma P$  saturates as  $N$  increases (indicating diminishing returns for large arrays and providing an independent validation of the reduced DNLSE trends). It is worth recalling the interpretation associated with Eq. (14): not only the number of oscillators but also the coupling stiffness  $k_c$  influences the spatial domain of the DDNLS. Such correlation implies that saturation with respect to the number of oscillators occurs relative to a given coupling level rather than universally. Numerically, the DDNLS is integrated to steady state for  $N = 53$  with a  $\pm 3$ -node window for the metric, and the ODE chain is excited for 700 cycles with the last 650 cycles used to compute steady-state amplitudes (Hilbert envelope). A relevant observation is that increasing the number of oscillators leads to a superior performance when compared to the multiplication of the individual oscillator performance; that is, the chosen metric for  $N = 7$  is lower than half of the metric obtained for  $N = 13$ .

Enforcing optimized physical parameters that guarantee soliton nucleation enables more robust energy localization, in terms of both frequency bandwidth and range of nucleating disorders, when compared to linear regimes. To address this argument, an understanding of the role of disorder must be developed. For this purpose, a series of numerical integrations is performed for different values of periodicity breaking  $Q$  (relative to the linear stiffness). With the coupling stiffness fixed at 5% of the linear stiffness, a damping ratio of 0.01%,  $R = 1 \times 10^7 \Omega$ , and  $k_{nl} = 1 \times 10^{10} \text{N/m}^3$ , Figure 16 comprises three heat maps detailing: (i) the sum of the power of all oscillators, (ii) the power at the oscillator where the soliton is nucleated, and (iii) the energy localization in this oscillator, for both the linear and nonlinear cases. Focusing on the total power, it is evident that the design region (where significant power levels are achieved) is considerably larger in the nonlinear case, with a substantial increase in the frequency domain. The same conclusion holds for the analysis of the central oscillator’s power. Actually, in the linear case, the contour lines of the central power collapse around a region starting at  $Q = k_l$  and extending toward a resonance predominantly governed by the disorder  $Q$ . Such conclusion does not extend to the nonlinear case, where the contour lines are more widely spaced and, for  $k_{nl} > 0$ , originate to the right of the fundamental frequency  $\omega_0$ . The vibrational energy localization induced by nonlinearity reduces the sensitivity associated with linear localization, in agreement with the enlargement of the

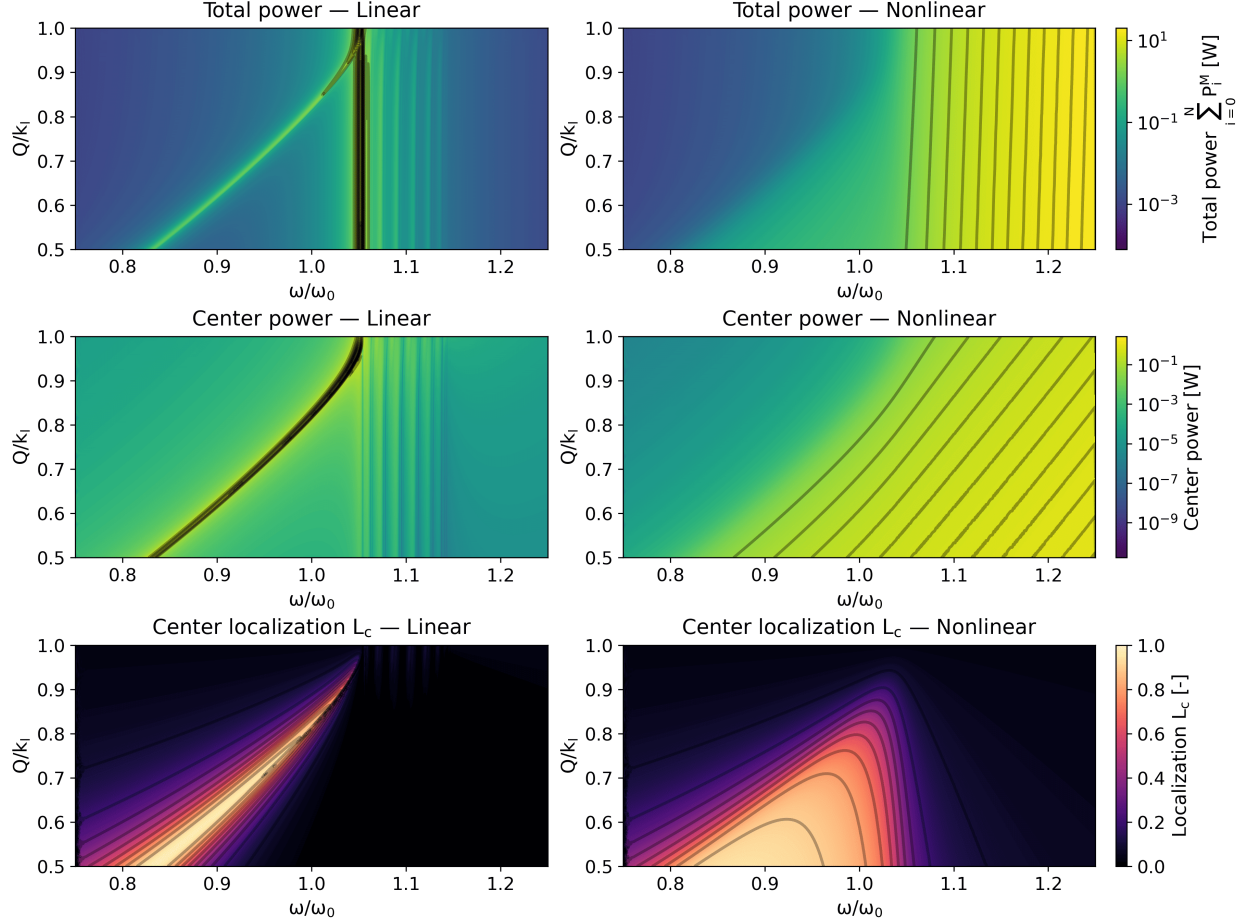


FIG. 16. Sequential numerical simulations under chirp excitation, performed to compare linear and nonlinear energy localization robustness. The maps display, from top to bottom, the total harvested power (sum of all oscillators), the power at the nucleation site, and the localization metric  $L_c = P_{\text{center}} / \sum_i P_i$  (dimensionless). Each column corresponds to a linear or nonlinear chain, with  $k_c = 0.05 k_l$ , damping ratio 0.01%,  $R = 1 \times 10^7 \Omega$ ,  $k_{nl} = 1 \times 10^{10} \text{ N/m}^3$ , and disorder parameter  $Q$  ranging from  $k_l$  to  $0.5 k_l$ . The excitation sweeps the normalized frequency  $\omega/\omega_0$  over a 25% span using a linear chirp. Nonlinear localization exhibits a broader frequency bandwidth and a wider nucleation range compared to the linear case.

area representing the percentage of power confined in the central oscillator, quantified by the metric  $L_c$ .

## VII. CONCLUDING REMARKS

This article presents a approach to vibration energy harvesting by employing standing solitons as a design framework for nonlinear metastructures. Unlike methods relying on propagating waves, the present work explores the potential of localized vibrations for energy extraction. The theoretical foundation is established through the derivation of the Disordered Damped Nonlinear Schrödinger equation, governing the dynamics of a Duffing oscillator chain with piezoelectric transducers coupled to its components. The derivation process employs the method of multiple scales, enabling the identification of secular terms and the reduction of the governing equations to a solvable nonlinear form. This formulation provides a mathematical model for describing soliton dynamics under external excitation, whose predictive capability is confirmed by the numerical integration of the system's differential equation.

A design strategy emerges from the DDNLS equation, guiding parameter tuning to optimize soliton stability and energy extraction. Through a Bayesian optimization, optimal configurations for maximizing harvested power across a specified frequency range are identified. The results indicate that periodicity breaks, introduced as intentional disorder, play a crucial role in soliton nucleation. While perfectly periodic structures fail to support spontaneous soliton formation, localized disorders trigger and stabilize stationary waves, paving the way for experimental implementations.

The frequency-domain analysis investigates the system's response under both sub-resonant ( $\Omega < \omega_0$ ) and super-resonant ( $\Omega > \omega_0$ ) excitation conditions. The perturbation factor  $\epsilon$ , directly linked to the excitation frequency, governs the spatial domain  $L_x$  and the impurity strength  $q$ , both of which are critical parameters for the phenomenon's stability. As the excitation frequency approaches resonance (under both sub-resonant and super-resonant regimes) stability windows begin to form, making the derivation of a general stability diagram a complex numerical challenge. Consequently, the optimization algorithm adopts a heuristic approach, assuming prior knowledge only of the excitation frequency bandwidth and the acceleration level to which the structure will be subjected.

The figure of merit in the present study in the development of a design methodology, which can be addressed to a wide range of applications. Although an example focused on energy harvesting maximization is provided, the methodology also accommodates multi-

642 objective formulations. Applications such as passive vibration control, energy trapping  
643 in specific regions of the chain, and structural sensing are encompassed by the proposed  
644 approach. By ensuring a solitonic dynamic regime, the metastructure gains robustness  
645 against dispersion and, consequently, stability of both mechanical and electrical vibration  
646 amplitudes. Another advantage concerns computational cost: when numerically analyzing  
647 the governing ODE of the lattice, the transient response is inevitably computed. Through  
648 the DDNLS framework a direct access to the steady-state regime is achieved, eliminating  
649 the unnecessary computational expense associated with time integration toward steady-state  
650 solutions.

651 In conclusion, this work establishes the theoretical foundation for soliton-based energy  
652 harvesting, demonstrating its potential to enhance the performance, tunability, and ro-  
653 bustness of metastructures. While the present study focuses on theoretical and numerical  
654 analyses, experimental validation is a crucial next step, which is addressed in the next publi-  
655 cation. Part II presents targeted experiments that verify the three main theoretical findings:  
656 the stability of solitonic regimes for VEH, the superior performance of the metastructure  
657 compared to individual oscillators, its robustness against disorder together with the pre-  
658 dictive capability of the multiphysics DDNLS. Future research aims to validate the use of  
659 solitons in different applications, such as the development of sensors or vibration attenuation  
660 mechanisms.

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## 664 **DECLARATION OF COMPETING INTEREST**

665 The authors declare that they have no known competing financial interests or personal  
666 relationships that could have appeared to influence the work reported in this paper.

668 To solve Equation (4), the method of multiple scales is applied, introducing a perturbation  
 669 parameter  $0 < \varepsilon < 1$ . For an arbitrary variable  $\rho$ , the rescaled variable  $\rho_\varepsilon$  is defined by  
 670 the relation  $\rho = \varepsilon \rho_\varepsilon$ , except for the spacing between oscillators<sup>10</sup>. Simultaneously, two  
 671 assumptions are adopted throughout the derivation of the NLS: the continuity hypothesis,  
 672 expressed through the relations<sup>11</sup>

$$\begin{aligned}
 n &\rightarrow s, \quad \Delta s = \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon}, \quad w_n(t) = w(s, t), \\
 w_{(n+1)}(t) &= \frac{\Delta s_{(\sqrt{\varepsilon})}^2 \varepsilon w_{ss}(s, t)}{2} + \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) + w(s, t), \\
 w_{(n-1)}(t) &= \frac{\Delta s_{(\sqrt{\varepsilon})}^2 \varepsilon w_{ss}(s, t)}{2} - \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) + w(s, t),
 \end{aligned} \tag{A1}$$

673 and the harmonic excitation hypothesis, with a frequency  $\Omega$  and amplitude  $A_{acc}$ :

$$\Omega = \omega_0(1 - \varepsilon), \tag{A2}$$

$$\ddot{B}(t) = A_{acc} \cos(\Omega t) = 0.5 A_{acc} (e^{i\Omega t} + e^{-i\Omega t}).$$

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<sup>10</sup>  $\Delta s = \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon}$

<sup>11</sup> Notation of spatial derivative subscripted to the variable.

By leveraging Equations (A1) and (A2), along with the perturbation factor  $\varepsilon$ , Equation (4) can be rewritten in the form:

$$\begin{aligned}
& \ddot{w}(s, t) + [2\omega_0\varepsilon (\zeta_\varepsilon + q_{\zeta\varepsilon}(s))] \dot{w}(s, t) + [\omega_0^2 + \varepsilon\phi_\varepsilon(s)] w(s, t) + \frac{\varepsilon}{m} V(s, t)\theta_\varepsilon(s) \\
& + \frac{\varepsilon}{m} [k_{nl\varepsilon} + q_{knl\varepsilon}(s)] w^3(s, t) - \frac{\varepsilon k_{cl} \Delta s^2_{(\sqrt{\varepsilon})}}{m} w_{ss}(s, t) \\
& + \frac{k_{cnl}}{m} \left[ \left( -\frac{\varepsilon \Delta s^2_{(\sqrt{\varepsilon})} w_{ss}(s, t)}{2} - \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \\
& + \frac{k_{cnl}}{m} \left[ \left( -\frac{\varepsilon \Delta s^2_{(\sqrt{\varepsilon})} w_{ss}(s, t)}{2} + \Delta s_{(\sqrt{\varepsilon})} \sqrt{\varepsilon} w_s(s, t) \right)^3 \right] \\
& = -\frac{\varepsilon}{m} 0.5 A_{acc\varepsilon} [f_m + q_{fme}(s)] (e^{i\omega_0 t(1-\varepsilon)} + e^{-i\omega_0 t(1-\varepsilon)}) .
\end{aligned} \tag{A3}$$

Before proceeding with the algebraic manipulation, some clarifications must be made regarding the perturbation of physical parameters. The choice of the power of  $\varepsilon$  depends on which terms are intended to compose the final formulation of the NLS. As shown in (A1) and (A3), the coupling is not perturbed, but the spacing between oscillators (see Figure 1) is. The consequence of this choice is the omission of nonlinear coupling terms in the first-order terms of  $\varepsilon$ . Attention must also be given to the excitation frequency  $\Omega$ , which, although close to  $\omega_0$ , requires that  $\Omega < \omega_0$  in a first approach. Lastly, for the sake of algebraic simplification, the mass  $m$  is incorporated into other physical parameters, simplifying the expressions through the following transformation:

$$\frac{k_{nl\varepsilon}}{m} \rightarrow k_{nl\varepsilon}, \quad \frac{q_{knl\varepsilon}}{m} \rightarrow q_{knl\varepsilon}, \quad \frac{k_{cl}}{m} \rightarrow k_{cl}, \quad \frac{k_{cnl}}{m} \rightarrow k_{cnl}, \quad \frac{A_{acc\varepsilon}}{m} \rightarrow A_{acc\varepsilon} \tag{A4}$$

Proceeding with the manipulation of the equations, the electrical circuit (2) is perturbed and rewritten in the form:

$$R_{\varepsilon^{-1}} C_p \dot{V}(s, t) + V(s, t) = R_{\varepsilon^{-1}} \theta_\varepsilon(s) \dot{w}(s, t), \tag{A5}$$

where  $R = \frac{R_{\varepsilon^{-1}}}{\varepsilon}$ . Once again, attention must be given to the choice of perturbation: although it may initially seem arbitrary, it is necessary to ensure that the product  $C_p R$  is properly scaled to the relevant powers of interest.

690 Following the procedure reported in previous studies [81, 100–107, 125, 129, 130], the so-  
 691 lution to the equations can be represented as a series expansion in powers of the perturbation  
 692 parameter<sup>12</sup>:

$$w(s, t) = w_0(s, t, t\varepsilon) + \varepsilon w_1(s, t, t\varepsilon) + O(\varepsilon^2), \quad (\text{A6})$$

$$V(s, t) = V_0(s, t, t\varepsilon) + \varepsilon V_1(s, t, t\varepsilon) + O(\varepsilon^2).$$

693 Notice that the solutions proposed in (A6) are limited to terms up to order  $\varepsilon$ , although  
 694 the mathematical derivation can be extended following the same steps for higher-order ap-  
 695 proximations. By applying the solutions (A6) to Equations (A3) and (A5), the originally  
 696 nonlinear problem, as presented in Equations (1) and (2), can be decomposed into a series of  
 697 linear problems, grouped according to the powers of  $\varepsilon$ , which can then be solved sequentially:

$$\varepsilon^0 \left[ \omega_0^2 w_0(s, t, t\varepsilon) + \frac{\partial^2}{\partial \xi^2} w_0(s, \xi, t\varepsilon) \Big|_{\xi=t} \right] + \varepsilon^1 [F] + O(\varepsilon^2) = 0, \quad (\text{A7})$$

698

$$\varepsilon^0 \left[ C_{p\varepsilon} R_{\varepsilon-1} \frac{\partial}{\partial \xi} V_0(s, \xi, t\varepsilon) \Big|_{\xi=t} - R_{\varepsilon-1} \theta_\varepsilon(s) \frac{\partial}{\partial \xi} w_0(s, \xi, t\varepsilon) \Big|_{\xi=t} \right] \quad (\text{A8})$$

$$+ \varepsilon^0 [V_0(s, t, t\varepsilon)] + \varepsilon^1 [G] + O(\varepsilon^2) = 0.$$

699 where

$$F =$$

$$\begin{aligned} & -k_{cl} \Delta s_{(\sqrt{\varepsilon})}^2 \frac{\partial^2}{\partial s^2} w_0(s, t, t\varepsilon) + (k_{nl\varepsilon} + q_{knl\varepsilon}(s)) w_0^3(s, t, t\varepsilon) + \phi_\varepsilon(s) w_0(s, t, t\varepsilon) \\ & + 2\omega_0 (\zeta_\varepsilon + q_{\zeta\varepsilon}(s)) \frac{\partial}{\partial \xi} w_0(s, \xi, t\varepsilon) \Big|_{\xi=t} + 2 \frac{\partial^2}{\partial t \partial \xi} w_0(s, t, \xi) \Big|_{\xi=t\varepsilon} \\ & + 0.5 A_{acc\varepsilon} (f_m + q_{fm}(s)) (e^{i\omega_0 t(1-\varepsilon)} + e^{-i\omega_0 t(1-\varepsilon)}) \\ & + \omega_0^2 w_1(s, t, t\varepsilon) + \frac{\partial^2}{\partial \xi^2} w_1(s, \xi, t\varepsilon) \Big|_{\xi=t} + \frac{V_0(s, t, t\varepsilon) \theta_\varepsilon(s)}{m}, \end{aligned} \quad (\text{A9})$$

---

<sup>12</sup> The subscript indicates the component of the solution corresponding to each power of  $\varepsilon$ .

700 and

$$\begin{aligned}
G = & \\
& R_{\varepsilon^{-1}} \left( - \frac{\partial}{\partial \xi} w_0(s, t, \xi) \Big|_{\xi=t\varepsilon} - \frac{\partial}{\partial \xi} w_1(s, \xi, t\varepsilon) \Big|_{\xi=t} \right) \theta_\varepsilon(s) \\
& + C_{p\varepsilon} R_{\varepsilon^{-1}} \left( \frac{\partial}{\partial \xi} V_0(s, t, \xi) \Big|_{\xi=t\varepsilon} + \frac{\partial}{\partial \xi} V_1(s, \xi, t\varepsilon) \Big|_{\xi=t} \right) + V_1(s, t, t\varepsilon).
\end{aligned} \tag{A10}$$

701 By solving the terms corresponding to the power  $\varepsilon^0$ , the following mathematical relation-  
702 ships are obtained:

$$\begin{aligned}
w_0(s, t, t\varepsilon) &= \varphi(s, t\varepsilon) e^{i\omega_0 t} + \text{c.c.}, \\
V_0(s, t, t\varepsilon) &= \frac{R_{\varepsilon^{-1}} \omega_0 \theta_\varepsilon(s) \varphi(s, t\varepsilon) e^{i\omega_0 t}}{C_{p\varepsilon} R_{\varepsilon^{-1}} \omega_0 - i} + \iota(s, t\varepsilon) e^{-\frac{t}{C_{p\varepsilon} R_{\varepsilon^{-1}}}} + \text{c.c.}
\end{aligned} \tag{A11}$$

703 where c.c. refers to the complex conjugate, and the variables  $\varphi(s, t\varepsilon)$  and  $\iota(s, t\varepsilon)$  represent  
704 the amplitudes of the zero-order solutions. It is evident that the electrical component in  
705 (A11), under steady-state conditions, is governed by the mechanical component  $\varphi(s, t\varepsilon)$ ,  
706 allowing us to disregard the amplitude  $\iota(s, t\varepsilon)$  as the system's dynamics evolve.

707 Based on the solutions presented in (A11), the differential equations  $F = 0$  and  $G = 0$  can  
708 be solved. However, the solution becomes feasible only through the elimination of secular  
709 terms, which are responsible for the divergence of the first-order solutions  $w_1(s, t, t\varepsilon)$  and  
710  $V_1(s, t, t\varepsilon)$  [155]. To satisfy this requirement, a sequence of algebraic operations is performed.  
711 First, the zero-order solutions given in (A11) are substituted into the expressions for  $F$  (A9)  
712 and  $G$  (A10). This substitution yields the governing equations for the first-order terms.  
713 Subsequently, the resulting expressions are analyzed to identify secular terms (those resonant  
714 with the homogeneous solution) associated with exponential functions of the form  $e^{i\omega_0 t} + \text{c.c.}$   
715 or their harmonics. Finally, a solvability condition is imposed by requiring the cancellation  
716 of these secular terms, thereby ensuring boundedness of the first-order solutions and enabling  
717 the construction of a uniformly valid asymptotic expansion.

718 Following the adopted procedure, the extraction of secular terms can be summarized for

719 the mechanical terms as

$$\begin{aligned}
& \frac{0.25 A_{acc\varepsilon} (f_m + q_{fm}(s) e^{-i\varepsilon t \omega_0})}{\omega_0^2} + \left( \frac{3k_{nl\varepsilon}}{2\omega_0^2} + \frac{3q_{knl\varepsilon}(s)}{2\omega_0^2} \right) \varphi^2(s, \varepsilon t) \overline{\varphi(s, \varepsilon t)} \\
& \frac{C_{p\varepsilon} (R_{\varepsilon^{-1}})^2 \theta_\varepsilon^2(s)}{2m (C_{p\varepsilon}^2 (R_{\varepsilon^{-1}})^2 \omega_0^2 + 1)} \varphi(s, \varepsilon t) + i \left( \frac{R_{\varepsilon^{-1}} \theta_\varepsilon^2(s)}{2m \omega_0 (C_{p\varepsilon}^2 (R_{\varepsilon^{-1}})^2 \omega_0^2 + 1)} + \zeta_\varepsilon + q_{\zeta\varepsilon}(s) \right) \varphi(s, \varepsilon t) \\
& + \frac{\phi_\varepsilon(s)}{2\omega_0^2} \varphi(s, \varepsilon t) + \frac{i \frac{\partial}{\partial \xi} \varphi(s, \xi) \Big|_{\xi=\varepsilon t}}{\omega_0} - \frac{k_{cl} \Delta s^2_{(\sqrt{\varepsilon})} \frac{\partial^2}{\partial s^2} \varphi(s, \varepsilon t)}{2\omega_0^2} = 0,
\end{aligned} \tag{A12}$$

720 where the overbar denotes the complex conjugate of the variable. Equation (A12), driven by  
721 the number of parameters and their interactions, makes its direct application less practical,  
722 requiring normalization. To address this strategy, the following variables are introduced:

$$\tau = \varepsilon t \omega_0, \quad \delta_s = \frac{\sqrt{2} \sqrt{k_{cl}} \Delta s_{(\sqrt{\varepsilon})}}{2\omega_0}, \quad \chi = \frac{s}{\delta_s}, \tag{A13}$$

$$\Phi(\chi, \tau) = -\frac{\sqrt{3} \sqrt{k_{nl\varepsilon}} \varphi(s, \varepsilon t)}{2\omega_0}, \quad H = \frac{0.125 \sqrt{3} A_{acc\varepsilon} f_m \sqrt{k_{nl\varepsilon}}}{\omega_0^3}, \tag{A14}$$

$$\Gamma_1(\chi) = \frac{C_{p\varepsilon} (R_{\varepsilon^{-1}})^2 \theta_\varepsilon^2(\chi)}{2m (C_{p\varepsilon}^2 (R_{\varepsilon^{-1}})^2 \omega_0^2 + 1)}, \quad \Gamma_2(\chi) = \frac{R_{\varepsilon^{-1}} \theta_\varepsilon^2(\chi)}{2m \omega_0 (C_{p\varepsilon}^2 (R_{\varepsilon^{-1}})^2 \omega_0^2 + 1)}, \tag{A15}$$

723 allowing us to rewrite Equation (A12) in a more concise form:

$$\begin{aligned}
& H \left( -1 - \frac{q_{fm}(\chi)}{f_m} \right) e^{-i\tau} + \left( 2 + \frac{2q_{knl\varepsilon}(\chi)}{k_{nl\varepsilon}} \right) \Phi(\chi, \tau) |\Phi(\chi, \tau)|^2 \\
& + \left( \Gamma_1(\chi) + i(\zeta_\varepsilon + q_{\zeta\varepsilon}(\chi) + \Gamma_2(\chi)) + \frac{\phi_\varepsilon(\chi)}{2\omega_0^2} \right) \Phi(\chi, \tau) i \frac{\partial}{\partial \tau} \Phi(\chi, \tau) - \frac{\partial^2}{\partial \chi^2} \Phi(\chi, \tau) = 0.
\end{aligned} \tag{A16}$$

724 An additional algebraic step can be taken to eliminate the oscillatory nature of Equation  
725 (A16) by introducing a new function such that:

$$\Phi(\chi, \tau) = e^{-i\tau} \overline{F(\chi, \tau)}, \tag{A17}$$

726 leading us to the disordered damped nonlinear Schrödinger equation (DDNLS) [103].

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