

1 **Disorder-tuned soliton nucleation for vibration energy harvesting**  
2 **in discrete nonlinear lattices - Part II : Proof of Concept and**  
3 **experimental validation on a piezomagnetic metastructure**

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# Abstract

This article presents the experimental validation of a energy harvesting approach based on stationary solitons in nonlinear metastructures. Building upon the theoretical framework introduced in Part I, this work experimentally investigates a reduced-order model governed by the Disordered Damped Nonlinear Schrödinger Equation (DDNLS). The experimental setup consists of a multiphysics lattice composed of clamped-clamped beams, coupled through magnetic interactions and equipped with piezoelectric transducers, replicating the assumptions underlying the derivation of the DDNLS. The experimental results demonstrate that the DDNLS accurately captures the steady-state Space-Frequency Energy Distribution (SFED), thereby validating its use as a predictive and design-oriented tool. To further assess the model's robustness, the influence of disorder and the number of oscillators on soliton stability is investigated, revealing consistent qualitative agreement with theoretical predictions. To the best of the authors' knowledge, this work constitutes the first experimental investigation of frequency-domain soliton-based energy harvesting in mechanical lattices, establishing a robust and scalable framework for the design of smart metastructures aimed at applications involving vibratory phenomena.

## 9 I. INTRODUCTION

10 Over the past two decades, vibration energy harvesting (VEH) has evolved into a versatile  
11 technological strategy for powering low-energy devices across a wide range of engineering  
12 applications, from MEMS and wireless sensors to infrastructure monitoring and biomedical  
13 systems [1–4]. To address the inherent limitations of linear devices (particularly their nar-  
14 row operational bandwidth) nonlinear designs have been explored as a means to enhance  
15 energy conversion under broadband and stochastic excitations [5–7]. Among the various  
16 strategies developed to enhance bandwidth and harvesting power efficiency, coupled oscil-  
17 lator and metastructures have demonstrated significant potential [8–11], particularly when  
18 nonlinearities are exploited to achieve energy localization [12–14].

19 Despite the documented progress, the concept of energy harvesting from stationary  
20 solitons (spatially confined modes governed by nonlinear wave equations) remains unex-

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21 plored in mechanical systems. While traveling solitons and breathers have been explored in  
 22 VEH/vibration applications [15–19], the stationary regime in the absence of disorder, asso-  
 23 ciated with Intrinsic Localized Modes (ILMs) in nonlinear chains [20], may also provide an  
 24 alternative framework for harvesting vibrational energy [21]. In fact, traditionally studied in  
 25 fields such as optics, plasma physics, and photonics [20, 22, 23], solitons have also received  
 26 attention within mechanical engineering contexts [24].

27 Several studies on metastructure design for vibration-related applications have proposed  
 28 different modeling techniques [25–33]. In Part I of this study [34], a design methodology  
 29 for energy harvesting was proposed, grounded on the generation and control of stationary  
 30 solitons in nonlinear metastructures. Through the application of the method of multiple  
 31 scales on a piezoelectrically coupled Duffing oscillator chain, the Disordered Damped Non-  
 32 linear Schrödinger Equation (DDNLS) was derived as a reduced-order model capable of  
 33 predicting the steady-state spatial distribution of localized vibratory energy. The numerical  
 34 analyses demonstrated that key physical parameters (including nonlinear stiffness, mechan-  
 35 ical and electrical coupling, electrical load, and localized disorder) can be tuned to promote  
 36 soliton nucleation and stability, offering a predictive tool for optimizing energy harvesting  
 37 performance.

38 The present article builds directly upon the Part I theoretical foundations and aims to  
 39 experimentally validate the DDNLS model by demonstrating its predictive capability in  
 40 describing how the energy is distributed along nonlinear multiphysical lattices, particularly  
 41 in the description of Space-Frequency Energy Distribution (SFED). To this end, a physical  
 42 system is devised, whose nonlinear dynamics are consistent with the assumptions underlying  
 43 the DDNLS derivation: clamped–clamped beams coupled via magnetic interactions. A  
 44 nuance of these mechanisms is the emergence of cubic nonlinearities due to the mid-plane  
 45 stretching induced by high excitation levels [35]. In fact, the dynamic description of beams  
 46 under nonlinear oscillation regimes has been the subject of numerous studies [36–40].

47 The remainder of the paper is organized as follows. Section II details the structural  
 48 design and derives the lattice equation based on magnetically coupled nonlinear oscilla-  
 49 tors. Section III describes the experimental setup, including the fabrication process, the  
 50 employed methodology, and the identification of physical parameters used in the DDNLS.  
 51 Once the physical parameters are identified, Section IV investigates the theoretical frame-  
 52 work proposed in [34], focusing on four key aspects: the influence of disorder, the number of

oscillators, the energy localization mechanism, and the potential for storing the harvested energy. Finally, Section V concludes the study and suggests directions for future research.

## II. DESIGNING OF A NONLINEAR METASTRUCTURE: DUFFING ARRAY BEAM SYSTEM

Figure 1 details the notation adopted for the problem's geometry, where  $L_p$ ,  $h_p$ ,  $h_b$ ,  $b$ ,  $x_{p_i}$ ,  $x_{p_f}$ , and  $L_b$  denote, respectively, the length of the piezoelectric element, the thickness of the piezoelectric element, the thickness of the beam, the width of both the beam and the piezoelectric element, the initial position of the piezoelectric element along the beam's longitudinal axis  $x$ , the final position of the piezoelectric element along the same axis, and the total length of the beam. Notice also the presence of other components: a permanent neodymium magnet (whose mass is denoted by  $M_M$ , located at  $L_b/2$ ), additional masses  $M_A$  (each positioned at a known location  $x_A$ ), and a metal plate (B) at the end opposite to the piezoelectric element. Only the permanent magnet is considered in the analytical development, as the metal plate and the additional masses are treated as tuning elements in the in the experimental campaigns. The objective is to use them as a means of adjusting or modifying the natural frequency of the oscillators according to experimental requirements.

The mathematical model of the beam can be derived either through force and moment balance using a Newtonian approach [41] or via energy balance within a Lagrangian framework [42]. In both strategies, the following equation is obtained for the relative vertical displacement  $D(x, t)$ :

$$m(x)\ddot{D}(x, t) + E_{eq}I_{eq}D''''(x, t) + c_{eq}\dot{D}(x, t) - \frac{E_{eq}A_{eq}}{2L_b}D''(x, t) \int_0^{L_b} D'^2(x, t) dx + \left[ \frac{d\delta_d(x - x_{p_i})}{dx} - \frac{d\delta_d(x - x_{p_f})}{dx} \right] \vartheta v(t) = -m(x)\ddot{B}(t), \quad (1)$$

where  $\vartheta$ ,  $m(x)$ ,  $E_{eq}$ ,  $I_{eq}$ ,  $c_{eq}$ , and  $A_{eq}$  represent, respectively, the electromechanical coupling, the mass per unit length, equivalent Young's modulus in bending direction, equivalent area moment of inertia, equivalent damping coefficient, and equivalent cross-sectional area of the beam. It is important to note that Equation (1) doesn't explicitly describe the mechanical influence of the piezoelectric component on the beam but accounts for it through homogenized properties. This approach is chosen due to the experimental perspective, meaning



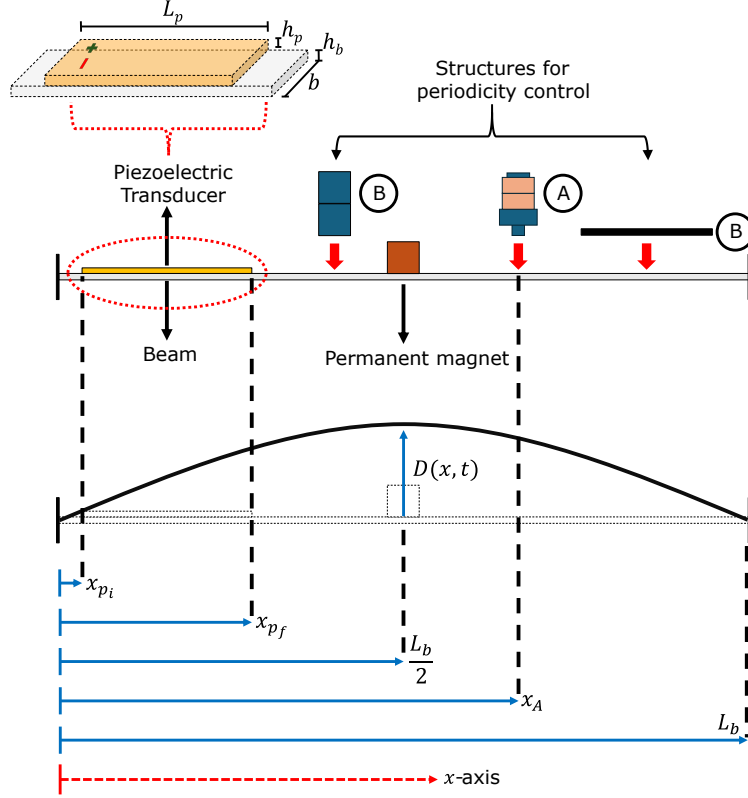


FIG. 1. Geometric description of the clamped-clamped beam adopted in the metastructure design.

that an analytical description is not provided, as the properties are instead measured<sup>1</sup>. It is also relevant to note that the structural damping is represented by the parameter  $c_{eq}$ , which may not be a valid assumption if the adhesive used to bond the piezoelectric element exhibits high viscoelastic characteristics [45, 46].

With respect to the modeling of the piezoelectric component, it can be formulated as follows:

$$C_p \frac{dv(t)}{dt} + \frac{v(t)}{R_l} = \vartheta \int_{x_{pi}}^{x_{pf}} \frac{\partial^3 D(x, t)}{\partial x^2 \partial t} dx. \quad (2)$$

Once again, the variables introduced for capacitance  $C_p$  and electromechanical coupling  $\vartheta$  encapsulate physical parameters that are not explicitly detailed. For instance,  $C_p$  depends on the geometric parameters of the piezoelectric element and its permittivity under constant deformation, while  $\vartheta$  is a function of the piezoelectric stress constant and the geometric relationships that define the neutral axis. Moreover, it is important to note that the electromechanical coupling is assumed to be linear, which is not a universally valid assumption<sup>2</sup>.

<sup>1</sup> For a detailed analysis, we recommend consulting the following works: [5, 39, 43, 44].

<sup>2</sup> We also recommend the following literature: [47].

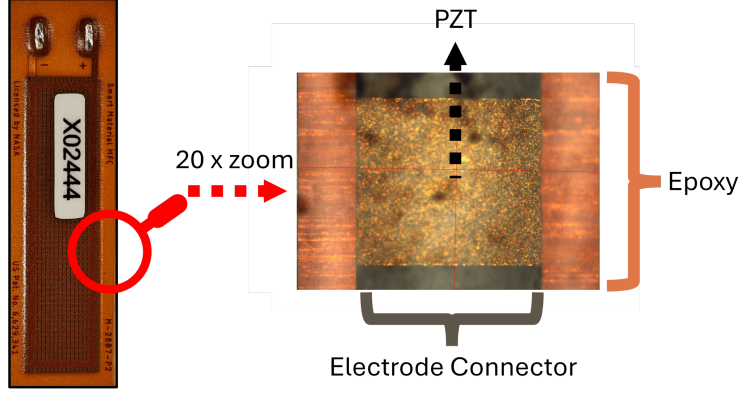


FIG. 2. Representative volume element of the piezoelectric mechanism.

The chosen transducer is the Macro Fiber Composite (MFC), a multi-layered composite featuring an active layer composed of piezoceramic fibers with a micro-scale rectangular cross-section (see Figure 2). The use of this mechanism has been reported in several studies that explored its potential in VEH-related applications [48–52], particularly consolidating its use in flexible systems [53, 54]. The modeling of MFCs is typically performed by analyzing representative volume elements, where, using the mixing rules technique, the equivalent parameters of Equation (2) are derived [55–57].

#### A. Equivalent Duffing oscillator

While neglecting the static displacement, the solution of Equation (1) can be expanded in an infinite basis of orthogonal functions that separate the spatial and temporal domains:

$$D(x, t) = \sum_{r=1}^{\infty} \Xi_r(x) \eta_r(t), \quad (3)$$

where  $\Xi_r(x)$  represents the modal basis and  $\eta_r(t)$  the time-dependent function common to all points of the beam. Since the focus is on the behavior around the first mode, the following approximation is adopted:  $D(x, t) \approx \Xi_1(x) \eta_1(t)$ .

The extraction of a Duffing oscillator from Equation (1) involves the following algebraic procedure: (i) substitute the solution (3) for the displacement  $D(x, t)$ , retaining only the first mode; (ii) multiply the resulting expression by the corresponding mode shape  $\Xi_1(x)$ ; and (iii) integrate over the spatial domain  $[0, L_b]$ . This procedure yields an equation defined solely in the time domain described by:

$$\beta_m \ddot{\eta}_1(t) + \beta_c \dot{\eta}_1(t) + \beta_{k_l} \eta_1(t) + \beta_{k_{nl}} \eta_1^3(t) + \beta_\theta v(t) = -\beta_{f_m} F \cos(\Omega t), \quad (4)$$

109 where

$$\begin{aligned} \beta_m &= \int_0^{L_b} \Xi_1(x) m(x) \Xi_1(x) dx = 1, \quad \beta_c = \int_0^{L_b} \Xi_1(x) c_{eq} \Xi_1(x) dx, \\ \beta_{k_l} &= \int_0^{L_b} \Xi_1(x) E_{eq} I_{eq} \frac{\partial^4}{\partial x^4} \Xi_1(x) dx = \omega_1^2, \quad \beta_{f_m} = \int_0^{L_b} \Xi_1(x) m(x) dx, \\ \beta_{k_{nl}} &= - \int_0^{L_b} \Xi_1(x) \frac{E_{eq} A_{eq}}{2L_b} \left[ \frac{\partial^2}{\partial x^2} \Xi_1(x) \right] \int_0^{L_b} \left[ \frac{\partial}{\partial x} \Xi_1(x) \right]^2 dx, \\ \beta_\theta &= \int_0^{L_b} \Xi_1(x) \left[ \frac{d\delta_d(x - x_{p_i})}{dx} - \frac{d\delta_d(x - x_{p_f})}{dx} \right] \vartheta dx. \end{aligned} \quad (5)$$

110 For the electrical domain, directly substituting the modal solution into the displacement  
111 leads to the governing relation, with no dependence on the spatial domain:

$$C_p \frac{dv(t)}{dt} + \frac{v(t)}{R_l} = \vartheta \int_{x_{p_i}}^{x_{p_f}} \left[ \frac{d^2}{dx^2} \Xi_1(x) \right] dx \quad \eta_1(t). \quad (6)$$

112 Through the properties of the Dirac delta function, in particular that:

$$\int_{-\infty}^{\infty} \frac{d^{(n)} \delta_d(x - x_0)}{dx^{(n)}} f(x) dx = (-1)^n \frac{d^n f(x_0)}{dx^n}, \quad (7)$$

113 it is possible to prove that the electromechanical coupling that links the mechanical domain  
114 to the electrical domain is, in fact, the same coupling that connects the electrical domain to  
115 the mechanical one:

$$\begin{aligned} \beta_\theta &= \int_0^{L_b} \Xi_1(x) \left[ \frac{d\delta_d(x - x_{p_i})}{dx} - \frac{d\delta_d(x - x_{p_f})}{dx} \right] \vartheta dx, \\ &= \vartheta \left[ \frac{d}{dx} \Xi_1(x) \right] \Big|_{x=x_{p_i}}^{x=x_{p_f}} = \vartheta \int_{x_{p_i}}^{x_{p_f}} \left[ \frac{d^2}{dx^2} \Xi_1(x) \right] dx. \end{aligned} \quad (8)$$

116 It can be observed that  $\beta_\theta$  depends on the positioning of the piezoelectric element along the  
117 beam. Maximizing this parameter is of interest for energy harvesting and is achieved by  
118 maximizing the difference between the points of the first derivative of the studied mode. In

other words, the optimal piezoelectric placement depends on the mode under consideration [58]. Figure 3 illustrates this aspect, highlighting the regions partitioned by the  $x$ -axis of the second derivative of the mode shown in 28. The piezo should be placed in partitions of the  $x$ -axis that share the same polarity for the area, ensuring that the transducer does not cancel the green with the red one.

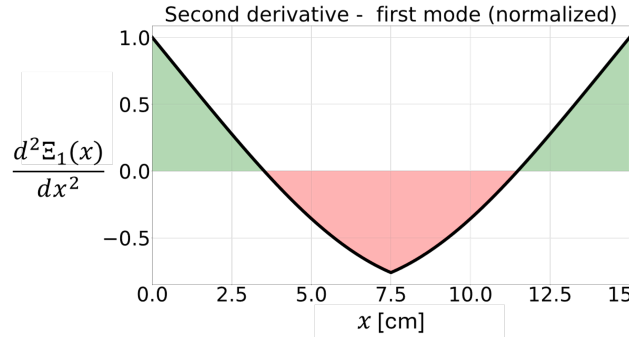


FIG. 3. Second derivative of the mode shown in Figure 28. Green indicates areas above the  $x$ -axis, while red represents areas below the  $x$ -axis.

Equations (4) and (6) describe the mechanical displacement using the variable  $\eta(t)$ . Both equations follow the same structure as their equivalent formulations described in [34], making them suitable for the derivation of the DDNLS. Nevertheless, while  $v(t)$  represents the electrical voltage measured at the piezoelectric terminals,  $\eta(t)$  does not correspond to the physical displacement. Therefore, specifying the measurement point  $p_0$  in the experimental setup is necessary, along with its corresponding displacement  $D(p_0, t)$ , which is given by the relation  $D(p_0, t) = \Xi_1(p_0)\eta_1(t)$ . The derivation of the modal basis, as well as its numerical validation, are presented in appendix A.

## B. Magnetic coupling between oscillators

With the mathematical description of a single oscillator established, the coupling between the  $N$  components of the lattice becomes feasible. Figure 1 introduces the permanent magnet responsible for this coupling, which generates a magnetic field that induces interactions among all oscillators. The mathematical formulation of the magnetic force has been the subject of several studies [59–62], and, despite the specificities of each system, the formulations

138 converge to a description of the following form:

$$\mathbf{F}_{mag(n+i)}(t) = \alpha_{mag} |\mathbf{r}_{n+i}|^{-\beta_{mag}} \mathbf{r}_{n+i}(t), \quad (9)$$

139 where  $\mathbf{F}_{mag(n+i)}(t)$  denotes the magnetic force between adjacent oscillators on the  $\mathbf{r}_{n+i}(t)$   
 140 direction, either to the right ( $i = 1$ ) or to the left ( $i = -1$ ) of the oscillator under analy-  
 141 sis. The constants  $\alpha_{mag}$  and  $\beta_{mag}$  depend on the magnetic field configuration and can be  
 142 identified experimentally.

143 Figure 4 illustrates the spatial arrangement of the oscillator chain. The beams are placed  
 144 side by side to ensure that the vertical oscillation of one oscillator is not obstructed by  
 145 its neighbor, a similar architecture presented by Yamaguchi and Hourri [63]. Due to the  
 146 arrangement of the beams, only the vertical projection of  $\mathbf{F}_{mag(n+i)}(t)$  should be considered,  
 147 which is obtained by analyzing the angles  $\Theta_i$  for oscillators to the right ( $i = 1$ ) and to the  
 148 left ( $i = -1$ ) of the oscillator under analysis.

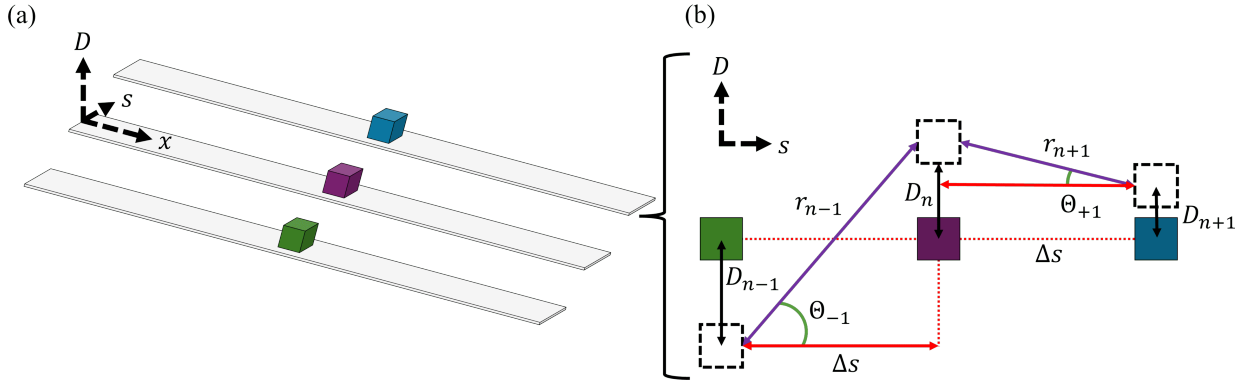


FIG. 4. Spatial arrangement of the oscillator chain and notational convention adopted. (a) Three-dimensional schematic view illustrating the beams placed side by side along the  $s$ -axis, ensuring that the vertical motion of each oscillator (represented by colored magnetic blocks) is not obstructed by its neighbors. (b) Projection in the  $D$ - $s$  plane used for modeling, showing the spatial discretization adopted for computation. The distance between adjacent oscillators is defined as  $\Delta s$ , and the force exerted by the neighboring magnets,  $\mathbf{F}_{mag(n+i)}(t)$ , is projected along the vertical axis through the angle  $\Theta_i$ . The diagram also defines the vector quantities, directions, and notations used in the model.

149 After the trigonometric evaluation, the projection of  $\mathbf{F}_{mag(n+i)}(t)$  onto the direction of  
 150  $D_n$ —denoted by  $F_{mag(n+i)}^{D_n}(t)$ —under the assumption of an attractive interaction ( $\alpha_{mag} > 0$ ),

151 is given by:

$$F_{mag(n+i)}^{D_n}(t) = -\alpha_{mag} \frac{D_{(n+i)}(x, t) - D_n(x, t)}{\left(\Delta s^2 + (D_{(n+i)}(x, t) - D_n(x, t))^2\right)^{0.5\beta_{mag}+0.5}}. \quad (10)$$

152 Note that the beam equation (1) extends to include an additional term on its left side, which  
153 accounts for the magnetic coupling between the oscillators:

$$\begin{aligned} m(x)\ddot{D}(x, t) + E_{eq}I_{eq}D''''(x, t) + c_{eq}\dot{D}(x, t) - \frac{E_{eq}A_{eq}}{2L_b}D''(x, t) \int_0^{L_b} D'^2(x, t) dx + m(x)\ddot{B}(t) \\ + \left[F_{mag(n+1)}^{D_n}(t) + F_{mag(n-1)}^{D_n}(t)\right] \delta_d\left(\frac{L_b}{2}\right) + \left[\frac{d\delta_d(x - x_{p_i})}{dx} - \frac{d\delta_d(x - x_{p_f})}{dx}\right] \vartheta v(t) = 0. \end{aligned} \quad (11)$$

154 Following the approach in [64], a Taylor series expansion is applied to

155  $F_{mag(n+i)}^{D_n}(t)$ , leading to a formulation that resembles the coupling assumed in [34]:

$$\begin{aligned} F_{mag(n+1)}^{D_n}(t) + F_{mag(n-1)}^{D_n}(t) = k_{cl}^{mag} (-D_{(n+1)}(x, t) - D_{(n-1)}(x, t) + 2D_n(x, t)) \\ + k_{cnl}^{mag} \left[(-D_{(n+1)}(x, t) + D_n(x, t))^3\right] + k_{cnl}^{mag} \left[(-D_{(n-1)}(x, t) + D_n(x, t))^3\right] \end{aligned}, \quad (12)$$

156 where

$$k_{cl}^{mag} = \Delta s^{-\beta_{mag}-1} \alpha_{mag}, \quad k_{cnl}^{mag} = \Delta s^{-\beta_{mag}-3} \alpha_{mag} (-0.5\beta_{mag} - 0.5). \quad (13)$$

157 By applying again the procedure used to derive Equation (4), the temporal equation in  $\eta(t)$   
158 broadens to cover the coupling terms between the oscillators, taking the following form:

$$\begin{aligned} \beta_m \ddot{\eta}_1(t) + \beta_c \dot{\eta}_1(t) + \beta_{k_l} \eta_1(t) + \beta_{k_{nl}} \eta_1^3(t) + \beta_\theta v(t) + \beta_{k_{cl}} [-\eta_{(n+1)}(t) - \eta_{(n-1)}(t) + 2\eta_n(t)] \\ + \beta_{k_{cnl}} \left( [-\eta_{(n+1)}(t) + \eta_n(t)]^3 + [-\eta_{(n-1)}(t) + \eta_n(t)]^3 \right) = -\beta_{f_m} F \cos(\Omega t) \end{aligned} \quad (14)$$

159 where

$$\beta_{k_{cl}} = \int_0^{L_b} \Xi_1(x) k_{cl}^{mag} \Xi_1(x) \delta_d\left(\frac{L_b}{2}\right) dx, \quad \beta_{k_{cnl}} = \int_0^{L_b} \Xi_1(x) k_{cnl}^{mag} \Xi_1^3(x) \delta_d\left(\frac{L_b}{2}\right) dx. \quad (15)$$

160 After incorporating the coupling terms, Equation (14) (apart from disorder effects) retains  
161 the same structure as presented in Equation [34], enabling the formal derivation of the  
162 DDNLS from the proposed architecture.

### 163 **III. EXPERIMENTAL DESCRIPTION AND DDNLS FORMULATION**

164 This section is organized as follows. First, the physical structure and its fabrication  
165 process are described, followed by the experimental methodology used for data acquisition  
166 (Subsection III A). The structure calibration procedure is presented in Subsection III B,  
167 and the response of the oscillator with tuned disorder is analyzed in Subsection III B 1.  
168 Subsection III C addresses the identification of physical parameters and the formulation  
169 of the DDNLS model. Finally, Subsection III C examines the experimentally identified  
170 parameters from the DDNLS perspective.

#### 171 **A. Design, fabrication and experimental methodology**

172 Figure 5 (a) presents the isometric structure's CAD model. The arrangement of the  
173 oscillators resembles a piano, with the keys represented by beams placed side by side, and  
174 the positioning of the magnets indicated in Figure 5(b). It should be emphasized that the  
175 oscillators are manufactured as a single piece with a common base (Figure 5(c)), ensuring  
176 that any imposed base acceleration is uniformly transmitted to all oscillators.

177 Table I summarizes the beam parameters along with the selection of other components.  
178 For pedagogical reasons, the validation of the equations describing the vibration modes (Fig-  
179 ure 28) is performed using the experimental parameters; however, the geometry is selected  
180 only after validation. That is, once the validity of Matrix (A9) and Equation (A7) is estab-  
181 lished, the bandwidth allowed by the experimental setup can be used to guide the geometric  
182 design. The arrangement of the magnets, when their polarity induces an attractive force  
183 between oscillators, results in a positive linear coupling and a negative nonlinear one (in  
184 accordance with Equations (13)). The distance  $\Delta s$  is measured between the cube faces, and  
185 the point where displacement measurements are taken is located at the center of the beam,  
186 at  $\frac{L_b}{2}$ .

187 Figure 5 (d) details the structure manufactured from the design shown in Figure 5 (a) -  
188 (c). To minimize manufacturing errors, the metal plates are cut using water jets, while other  
189 components are acquired from specialized suppliers. The electrical resistances are adjusted  
190 through nine independent potentiometers, each responsible for controlling the electrical volt-  
191 age of the nine MFCs distributed along the middle of the chain. The chosen architecture

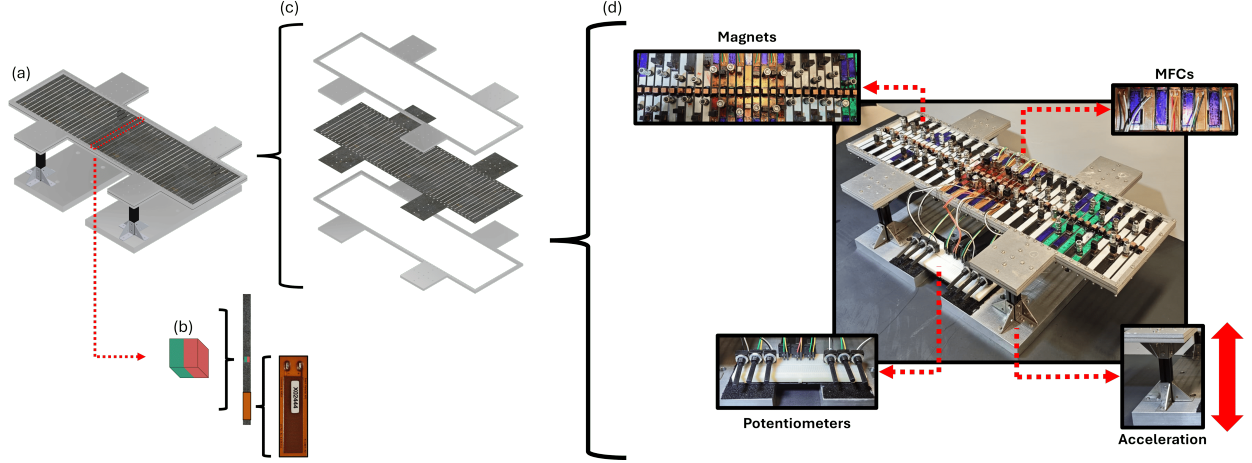


FIG. 5. Experimental structure designed and constructed for the validation of soliton-based energy management. (a) CAD model in isometric view, highlighting the complete assembly. (b) Detail of a single oscillator, showing the placement of the Macro Fiber Composite (MFC) patch, the arrangement of the permanent magnets, and their polarity (north/south faces indicated by green/red). (c) Exploded isometric view illustrating that the structure is manufactured from a single metal sheet, water-cut to ensure geometric uniformity and structural integrity. (d) Photograph of the final assembled structure, with zoomed-in views of specific components: MFC patches, magnet positioning, potentiometers, and the base excitation direction. The coloring scheme adopted is illustrative and was employed during the experimental tuning to facilitate disorder configuration.

TABLE I. Design specifications of the experimental setup.

| Parameter          | Value      | Unit | Parameter  | Value | Unit |
|--------------------|------------|------|------------|-------|------|
| Number of beams    | 45         | —    | Cube side  | 7     | mm   |
| Beam material      | DC01 steel | —    | $L_b$      | 150   | mm   |
| Piezo (MFC)        | M-2807-P2  | —    | $h_b$      | 0.8   | mm   |
| Cube magnetization | N42        | —    | $b$        | 10    | mm   |
| Cube mass          | 2.6        | g    | $\Delta s$ | 5     | mm   |

provides experimental flexibility in controlling the coupling (adjustable through the orientation and positioning of the magnets), the natural frequency (by adding or removing masses and metal plates), and the choice of the measurement points (as the beam motion remains unobstructed by neighboring components).



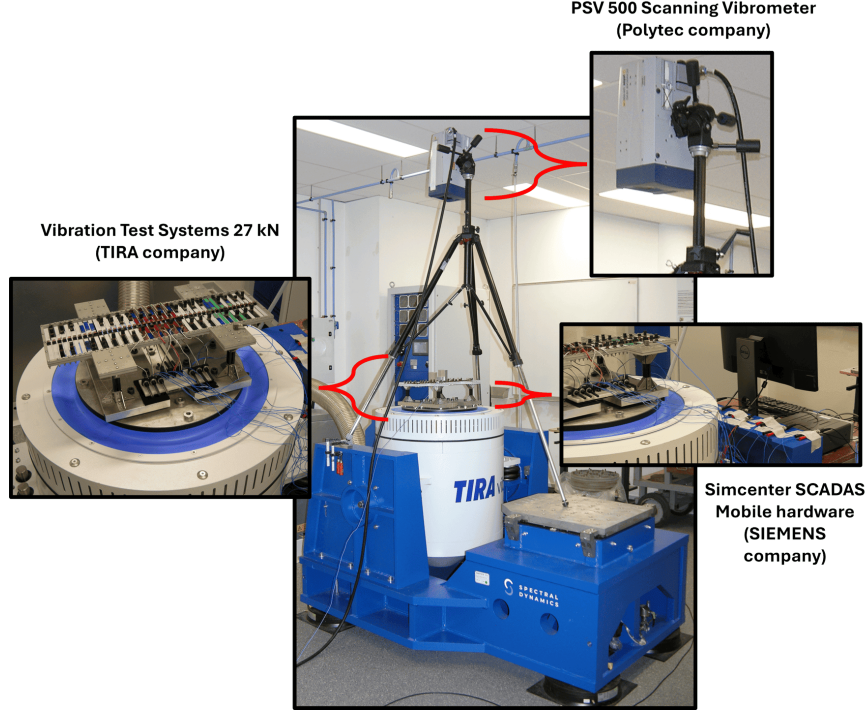


FIG. 6. Experimental setup used for the vibrational characterization and data acquisition. The structure is mounted on an electrodynamic shaker (Vibration Test Systems 27 kN, TIRA), enabling base excitation. A PSV-500 scanning laser Doppler vibrometer (Polytec) is positioned on a tripod and directed toward the oscillator array to capture full-field out-of-plane velocity responses. The electrical and mechanical signals are synchronously acquired using the Simcenter SCADAS Mobile system (SIEMENS). The inset views highlight: (i) the mechanical coupling between the structure and the shaker head, (ii) the laser vibrometer optical head and its alignment, and (iii) the data acquisition interface and wiring layout.

Figure 6 presents the experimental setup used to obtain the Frequency Response (FR)<sup>3</sup>. Due to the weight of the structure (9.4 kg), a high-capacity shaker is necessary (Vibration Test Systems 27 kN - TIRA<sup>®</sup>). By imposing an acceleration at the base of the structure, two variables are recorded and subsequently processed: the electrical voltage at the piezo terminals and the oscillation velocity, measured by a laser positioned above the structure (PSV 500 Scanning Vibrometer - Polytec<sup>®</sup>). The data processing is performed using a commercially available software and hardware (Simcenter SCADAS Mobile hardware -

<sup>3</sup> As in [34], we corroborate the notion that the concept of a frequency response curve is strictly applicable to linear systems. Here, we refer to the term "frequency response" as the response of the model in the frequency domain.

203 SIEMENS®).



FIG. 7. Experimental strategy for studying isolated oscillators. Aluminum masses gradually allowed to oscillate, starting from the central oscillator (index 22).

204 Due to the “Piano” assembled structure, studying the individual dynamics of the oscilla-  
 205 tors presents an experimental challenge. To overcome this difficulty, aluminum masses are  
 206 placed on top of the neodymium cubes adjacent to the oscillator under investigation, as  
 207 shown in Figure 7. This approach not only enables the study of a single oscillator but also  
 208 allows for the gradual increase of  $N$ . The minimum aluminum mass required to isolate the  
 209 FR curve is not self-evident. Using Ansys®, we verified that a mass greater than  $\sim 17\text{ g}$   
 210 would result in an amplitude difference of more than 34 times between the free beam and the  
 211 beam with the added masses (see Figure 8), which defines the minimum mass value used.

## 212 B. Structure calibration

213 The study of individual oscillators enables the calibration of the lattice periodicity. Con-  
 214 sistent with the results reported in [65], we reaffirm that disorders associated with the natural  
 215 frequency have significantly greater impacts than those affecting other physical parameters.  
 216 In this context, a calibration in  $\omega_1$  is conducted prior to the experimental tests.

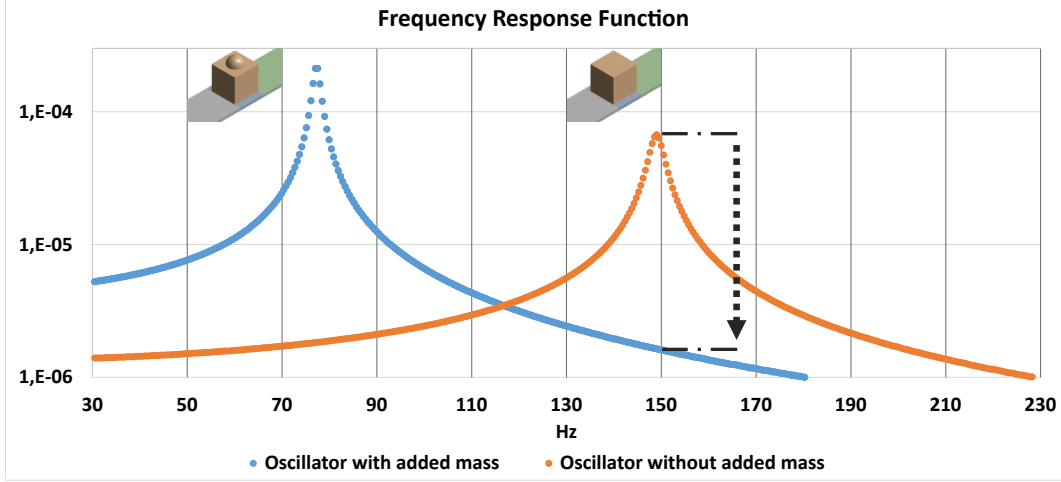


FIG. 8. FRs Functions of the beams with and without the addition of aluminum masses. Damping of  $\sim 1\%$  and aluminum mass of  $\sim 17.4\text{ g}$ . Vertical axis in logarithmic scale (displacement over acceleration). Reference point at the central node of the mesh.

As demonstrated in the first part of the study [34], periodicity disruptions that reduce the natural frequency have the potential for spontaneous soliton nucleation<sup>4</sup>. As a consequence, two types of calibration are performed: one at the center of the chain (where the transducers are coupled,  $n = 18, 19, \dots, 26$ ) and another outside this region.

After determining the maximum natural frequency allowed by the oscillators (eigenvalues of matrix (A9)), the possibility of controlling/tuning the lattice becomes feasible, and the experimental nominal values are imposed. It is important to note that the sources of uncertainty in the structure shown in Figure 5 (d) range from electrical parameters (resistance value, electromechanical coupling, etc.) to mechanical parameters (screw torque, damping, geometry, etc.), but are beyond the scope of the present study<sup>5</sup>.

Figure 9 shows the FRs of the electrical domain for the central oscillators, excluding oscillator 22 (which is analyzed separately in the subsequent section III B 1). The charts are presented for two disorder configurations,  $D_1$  and  $D_2$ , measured using the method described in Section III A. Initially, in the context of soliton stability, it is natural to hypothesize that the configuration in  $D_2$  is more stable than that in  $D_1$ . However, this investigation can only be properly conducted by considering all other physical parameters. For instance,

<sup>4</sup> We limit our statement to this observation, i.e., we do not imply that only impurities that decrease  $\omega_1$  manifest this phenomenon.

<sup>5</sup> A comprehensive analysis of the influence of uncertainties on soliton nucleation has been carried out in [65].

although the periodicity break is more pronounced in  $D_1$ , for sufficiently high coupling, soliton nucleation may occur in frequencies where  $D_2$  does not, thus requiring a global analysis. In this context, both arrangements,  $D_1$  and  $D_2$ , are used and indicated as needed.

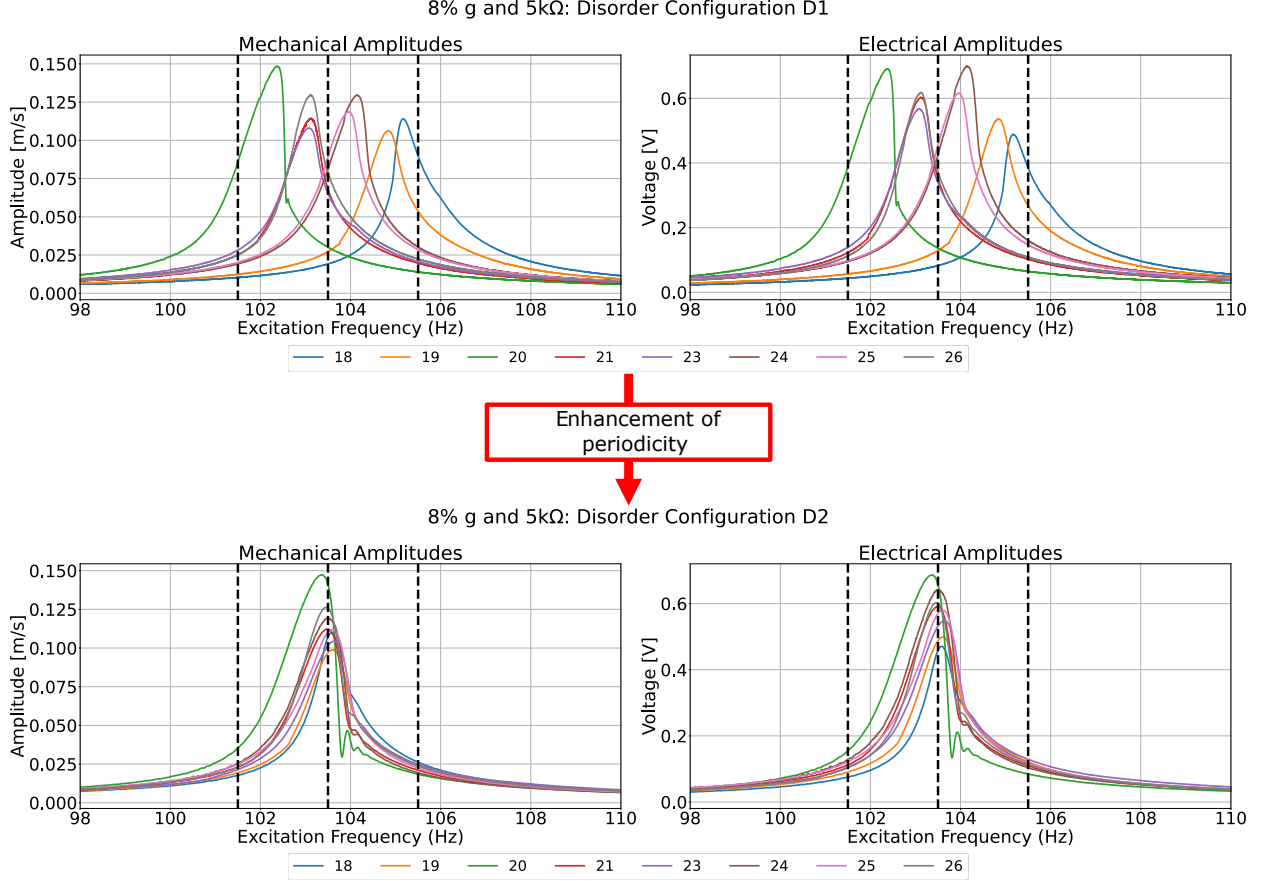


FIG. 9. Steady-state amplitudes versus excitation frequency for oscillators near the central site (oscillator 22), under an acceleration level of 8%  $g$  and potentiometer setting of 5  $k\Omega$ . The top and bottom panels correspond to disorder configurations  $D_1$  and  $D_2$  (reduced periodicity), respectively. Results obtained from sweep-up tests.

After adjusting the central oscillators, the tuning of the remaining lattice is carried out. The nominal frequency for this region is set to  $106 \text{ Hz} \pm 2 \text{ Hz}$ , which is higher than that of the central region, guiding soliton nucleation toward the areas where the piezos are positioned (as predicted in [34]). Figure 10 presents the FRs of the remaining 36 beams, with three vertical lines demarcating the frequencies corresponding to peak amplitude. The nonlinear effect is more pronounced compared to the oscillators with MFCs attached, which is an expected result given the absence of damping caused by the MFC adhesive layer and energy

dissipation in the potentiometers.

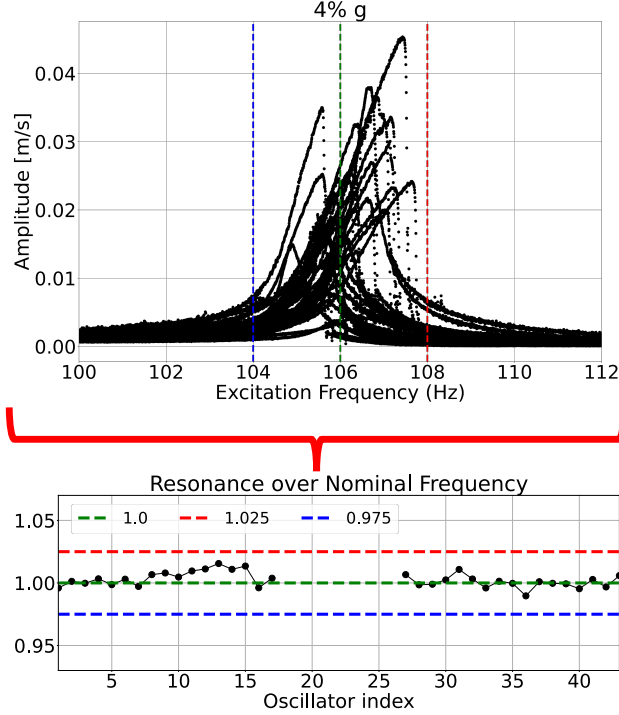


FIG. 10. Steady-state response amplitudes versus excitation frequency measured in the non-central region of the oscillator chain. The excitation corresponds to an acceleration level of 4%  $g$ , and the data are acquired through sweep-up tests.

Throughout the experimental tests, different beam arrangements are explored, including a negative coupling configuration. The mathematical formulation of the magnetic force in this setup follows the same algebraic approach described in Section II B, except for the polarity change in  $\alpha_{mag}$ , which becomes negative. Although useful in various applications involving multistable harvesters [66–68], implementing  $k_{cl}^{mag} < 0$  (Equation (13)) imposes a higher experimental challenge, leading to polarity alternations in the coupling pattern (negative coupling between adjacent oscillators but positive coupling between non-adjacent ones). Reassembling and recalibrating the structure introduces defects that manifest as disorder in the DDNLS, interfering with the identification and control of physical parameters. The periodicity shown in Figures 9 and 10 correspond to the final configurations, featuring positive linear coupling while accounting for the effects of structural deviations.

The curves shown in Figures 9 and 10 are obtained by adjusting the frequencies associated with the maximum oscillation amplitudes so as to match the nominal experimental values. To enable such control, the positioning  $x_A$  of the masses along the beam is adjusted (see Figure 1). Figure 11 details this approach, where pre-measured specimens are used to set the distances of interest.

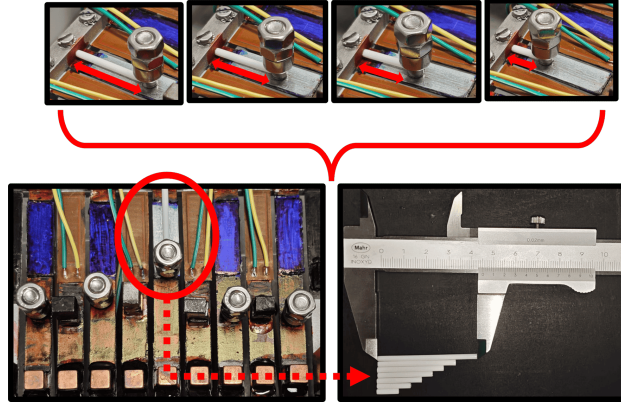


FIG. 11. Mass positioning control along the oscillators.

The design strategy outlined in [34] optimizes the lattice architecture by manipulating the natural frequency of the central oscillator. In order to validate this strategy, the central beam (22) is used to investigate the soliton formation. Figure 12 presents the FRs of beam 22 as a function of the control mass position  $x_A$  (Figure 11). In line with the modal description derived in A, increasing  $x_A$  (which brings the mass closer to the oscillator's center) reduces the resonance frequency by more than 20 Hz, a bandwidth that is employed in the handling of disorder.

### C. Identification of Physical Parameters and DDNLS Derivation

Figures 9, 10, and 12 are obtained using the methodology described in Section III A. Consequently, the coupling between the oscillators is always present in the experiment, meaning they are not isolated from the stiffness responsible for resonance. The linear coupling  $\beta_{k_{cl}}$  (Equation (14)) can be estimated by comparing the peak amplitude frequencies of two configurations under distinct conditions: one with aluminum masses and the other allowing free

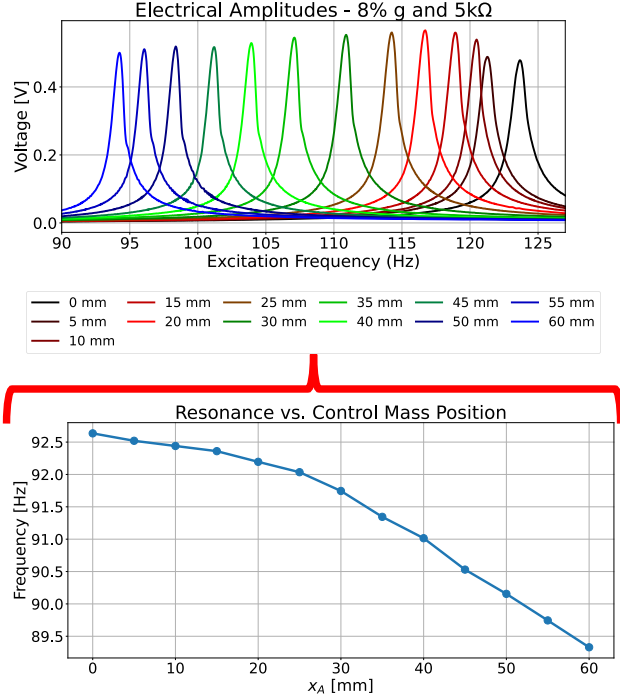


FIG. 12. FRs of beam 22 as a function of  $x_A$ .

oscillation of the beams. Figure 13 shows the shift in overall resonances as the adjacent oscillators are no longer restricted in their movement.

Due to the the magnetic nature of the system and the resulting continuity of the magnetic field along the neodymium cubes, coupling between non-adjacent beams is expected. In [34], the possibility of multi-coupling is modeled using an equivalent parameter, denoted as  $\beta_{k_{cleq}}$ , which (neglecting couplings between oscillators separated by more than four indices) is initially estimated by the following equation:

$$\beta_{k_{cleq}} = \left( \frac{\beta_{k_{1cl}}}{\sum_{n=1}^4 \beta_{k_{ncl}}} \right) \beta_{k_{1cl}} + \sum_{n=2}^4 \left( \frac{\beta_{k_{ncl}}}{\beta_{k_{1cl}}} \right) n^2 \beta_{k_{ncl}}. \quad (16)$$

By assuming that

$$\sum_{n=1}^4 \beta_{k_{ncl}} \approx \frac{(2\pi \cdot 103.3)^2 - (2\pi \cdot 91.8)^2}{2}, \quad (17)$$

the remaining physical parameters can then be estimated. For this purpose, the analytical formulation of the FRs of Duffing oscillators (obtained using the Harmonic Balance Method) is employed, [69, 70]. Figures 14 and 15 compares the predictions of the Duffing model (fitted with experimental parameters) with the measured curves of the central oscillators, considering two disorder configurations,  $D_1$  and  $D_2$ . It is worth noting that the modal



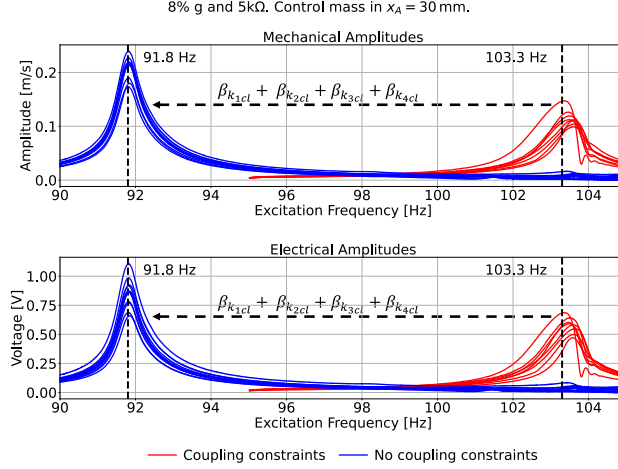


FIG. 13. Change in the FRs of the 9 central oscillators where piezos are attached. Disorder used in the experiment:  $D_2$ .

displacement in Equation (14) is not directly accessible in the experiments, since what is effectively measured is the physical displacement. This limitation does not apply in the electrical domain. Although Equation (6) is formulated in terms of the modal displacement  $\eta(t)$ , the voltage  $v(t)$  is directly acquired from the piezoelectric components.

After verifying the individual FRs, a refinement in the identification of the parameters must be carried out. The objective of the next campaigns is to understand how the couplings are distributed along the chain (Equation (16)) as well as to identify the other physical variables in a more direct manner. For this purpose, two disorder configurations in beam 22 are considered:  $x_A = 45$  mm and  $x_A = 60$  mm.

Several strategies can be found in the literature for parameter identification, such as Bayesian inference [71], stochastic methods [31], and filtering-based identification techniques [72]. One should remark that the magnetic field is sensitive to the spatial distribution of the masses along the chain [73]. Such observation implies that all physical parameters are slightly modified under different disorder configurations. In this scenario, no optimization strategy is employed at a particular disorder, as a potential overfitting of Equations (14) and (6) under a specific condition would reduce the predictive capability of the model in other  $x_A$  arrangements.

The refinement of parameter identification is conducted through numerical sweeps around the parameters previously identified in Figures 14 and 15. The selection criterion is defined by a maximum error of 1 Hz in the frequencies associated with peak amplitudes and a



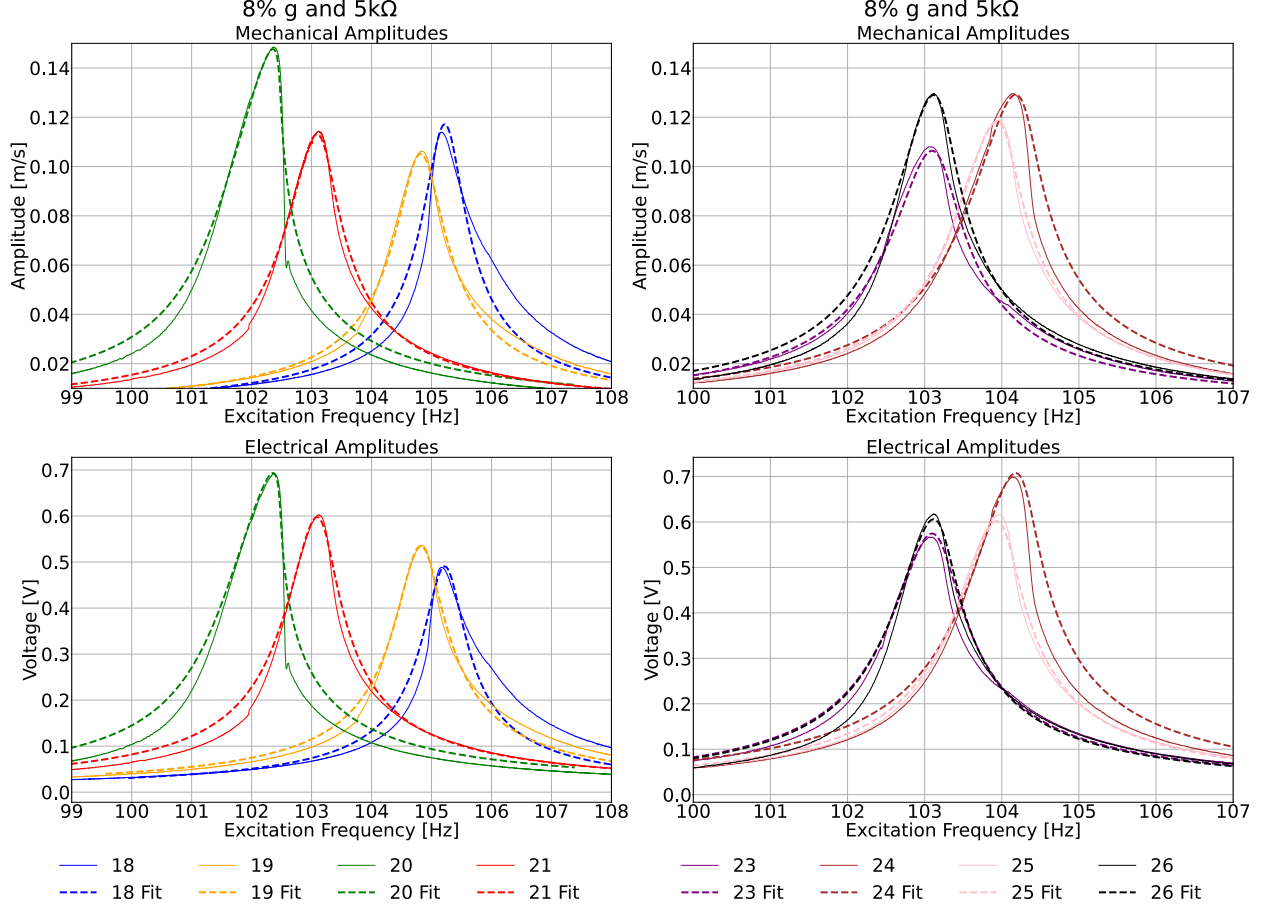


FIG. 14. Duffing model identification for the central oscillators. Disorder configuration D1.

deviation not exceeding 5% between the model-predicted and experimentally measured amplitudes for beam 22. By parametrically varying the proportions that compose  $\sum_{n=1}^4 \beta_{k_{ncl}}$ , a variation in the hierarchy of the resonance peaks across the beams is observed, which gives rise to a second criterion for parameter selection. Figure 16 presents the identified parameters along with their respective variations around the mean values.

Figure 17 presents the experimental amplitudes of the nine central beams along with the predicted response obtained from Equations (14) and (6) under chirp signal excitation [74, 75]. Two distinct dynamic behaviors are observed: one for excitation frequencies below the average natural frequency, and another for frequencies above it. For instance, when  $x_A = 45$  mm, oscillators 1 and 9 exhibit the lowest amplitudes among all oscillators when  $\Omega < \omega_1$ , a pattern that no longer holds when  $\Omega > \omega_1$ .

The assumed model (dashed lines) is derived from the beam equation in the absence of buckling effects (Equation (11)), which excludes the possibility of nonlinearities other

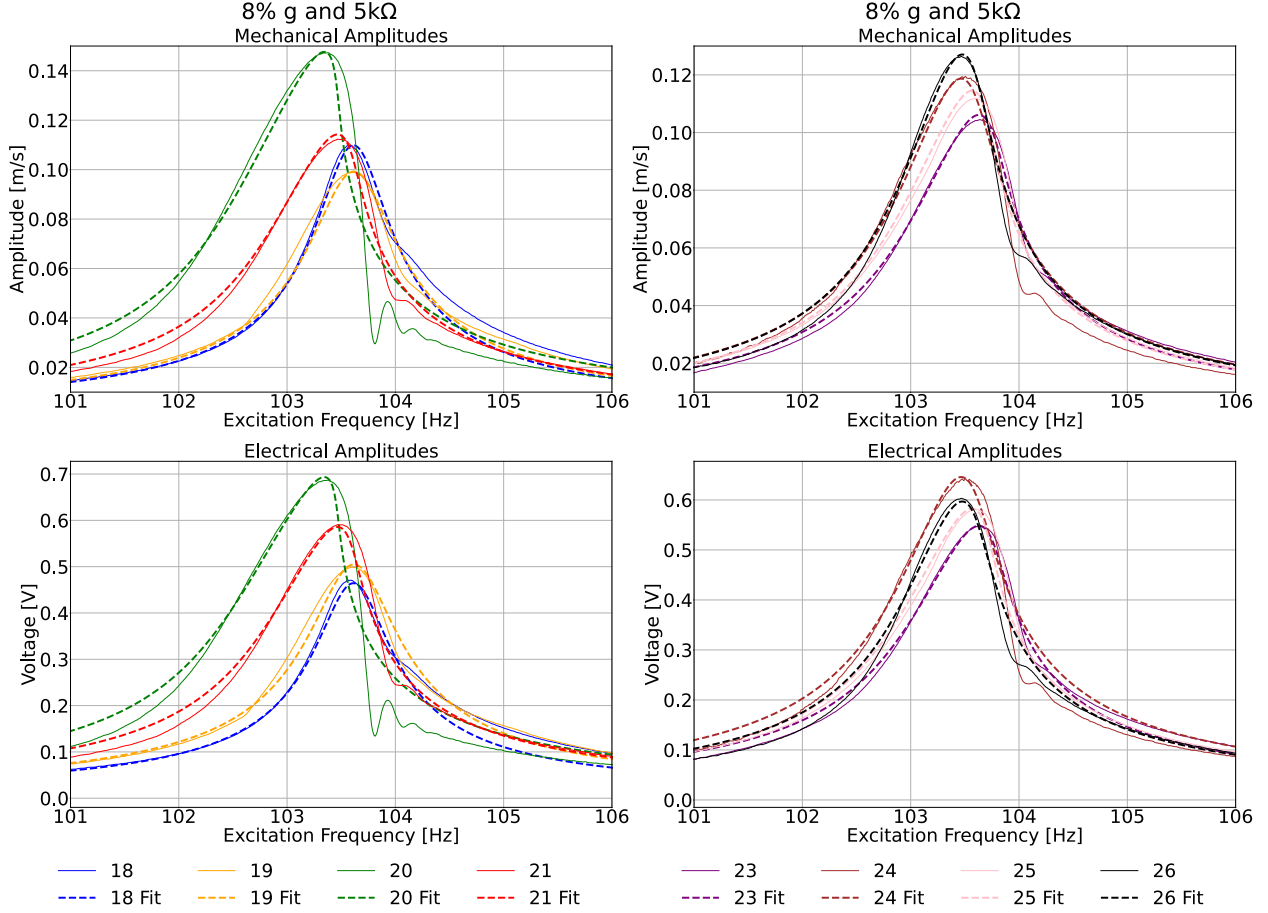


FIG. 15. Duffing model identification for the central oscillators. Disorder configuration D2.

than the cubic type, such as quadratic terms [76, 77]. The consequence of this simplifying assumption is a reduction in the accuracy when predicting hardening and softening effects. As shown in Figure 17, for  $x_A = 60$  cm, the the frequency values between 86 Hz and 90 Hz exhibits stronger hardening behavior than that predicted by the model, whereas for  $x_A = 45$  cm the opposite effect is observed. Also, the correction of the electrical response is achieved through electromechanical coupling, whereas the correction of the mechanical displacement prediction  $D(x, t)$  (which could be addressed via the modal basis  $\Xi(x)$ ) is neglected<sup>6</sup>, since the focus of this work is directed towards the electrical voltage response.

Once the experimental correlation between the lattice differential equation with the measured results is verified (Figure 17), the verification of the soliton dynamics can then be investigated. Recalling the DDNLS formulation presented in the first part of this study [34],

<sup>6</sup> Recall that  $D(x, t) = \Xi_1(x), \eta_1(t)$ .

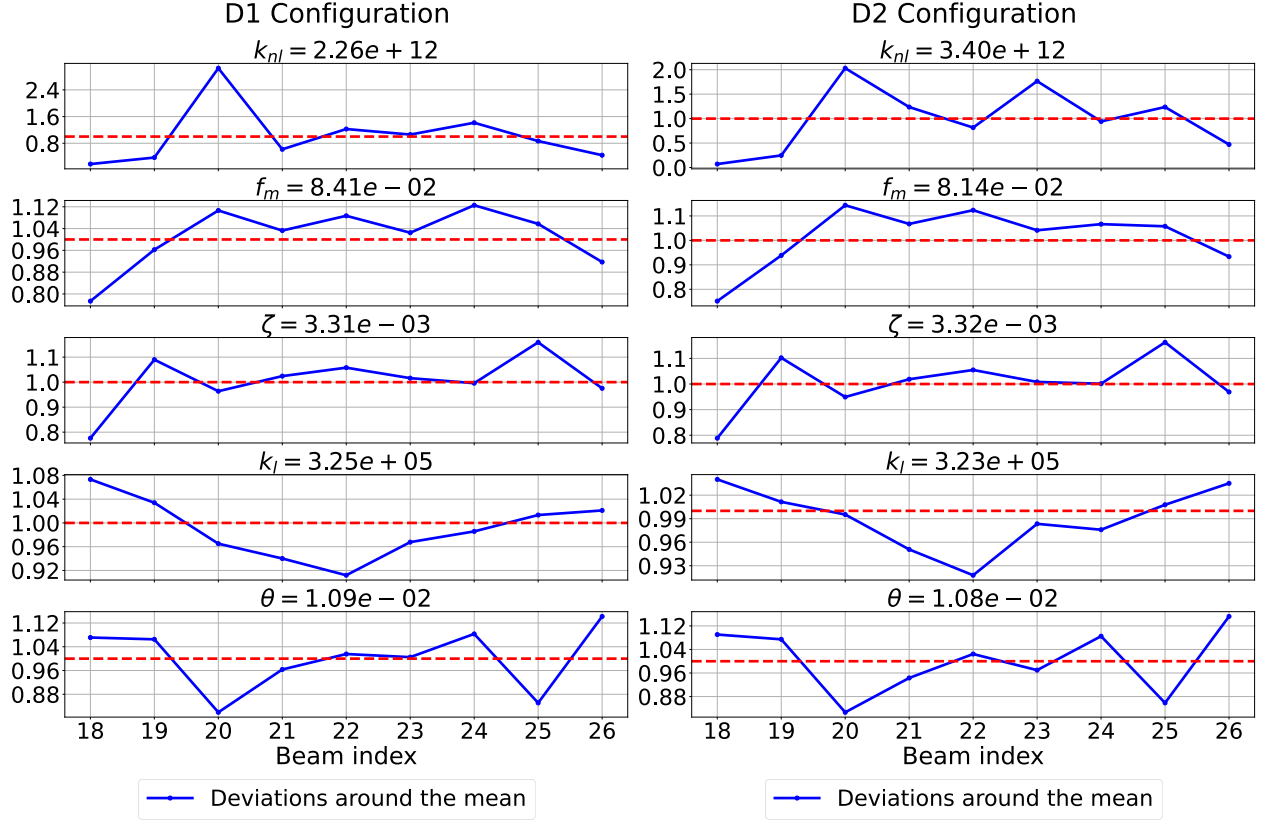


FIG. 16. Physical parameters numerically identified after comparison with the model defined by Equations (14) and (6). Linear coupling:  $\left(\sum_{n=1}^4 \beta_{k_{ncl}}\right) = 41$  [kN/m], with  $\beta_{k_{1cl}} = 0.48 \left(\sum_{n=1}^4 \beta_{k_{ncl}}\right)$ ,  $\beta_{k_{2cl}} = 0.20 \left(\sum_{n=1}^4 \beta_{k_{ncl}}\right)$ ,  $\beta_{k_{3cl}} = 0.18 \left(\sum_{n=1}^4 \beta_{k_{ncl}}\right)$ , and  $\beta_{k_{4cl}} = 0.14 \left(\sum_{n=1}^4 \beta_{k_{ncl}}\right)$ . Nonlinear coupling:  $k_{1cnl} = -9.85 \times 10^{11}$  [N/m<sup>3</sup>],  $k_{2cnl} = -2.05 \times 10^{11}$  [N/m<sup>3</sup>],  $k_{3cnl} = -9.23 \times 10^{10}$  [N/m<sup>3</sup>], and  $k_{4cnl} = -3.23 \times 10^{10}$  [N/m<sup>3</sup>]. Dashed red line indicating the mean value of the variable.

the steady-state behavior of the metastructure can be described by:

$$\begin{aligned}
 & H(k_{nl}, k_l, \Omega) - 2F(\chi, \tau)|F(\chi, \tau)|^2 + [-\Gamma_1(\chi, R, k_l, \Omega) + i[\zeta_\varepsilon(\Omega) + \Gamma_2(\chi, R, k_l, \Omega)]]F(\chi, \tau) \\
 & + [\alpha_\Omega + q(k_l, \phi, \Omega)\delta_d(\chi)]F(\chi, \tau) + i\frac{\partial}{\partial \tau}F(\chi, \tau) + \frac{\partial^2}{\partial \chi^2}F(\chi, \tau) = 0,
 \end{aligned} \tag{18}$$

where  $\alpha_\Omega = -1$  for  $\Omega < \omega_1$  and  $\alpha_\Omega = 1$  for  $\Omega > \omega_1$  (variables  $H, F, \chi, \tau, \Gamma_1, \zeta_\varepsilon, \Gamma_2, q, \phi$ , and  $\delta_d$  are defined in [34]).

It is important to highlight that, despite the good accuracy of the oscillator chain model, possible inaccuracies in the coupling-related variables may lead to significant discrepancies

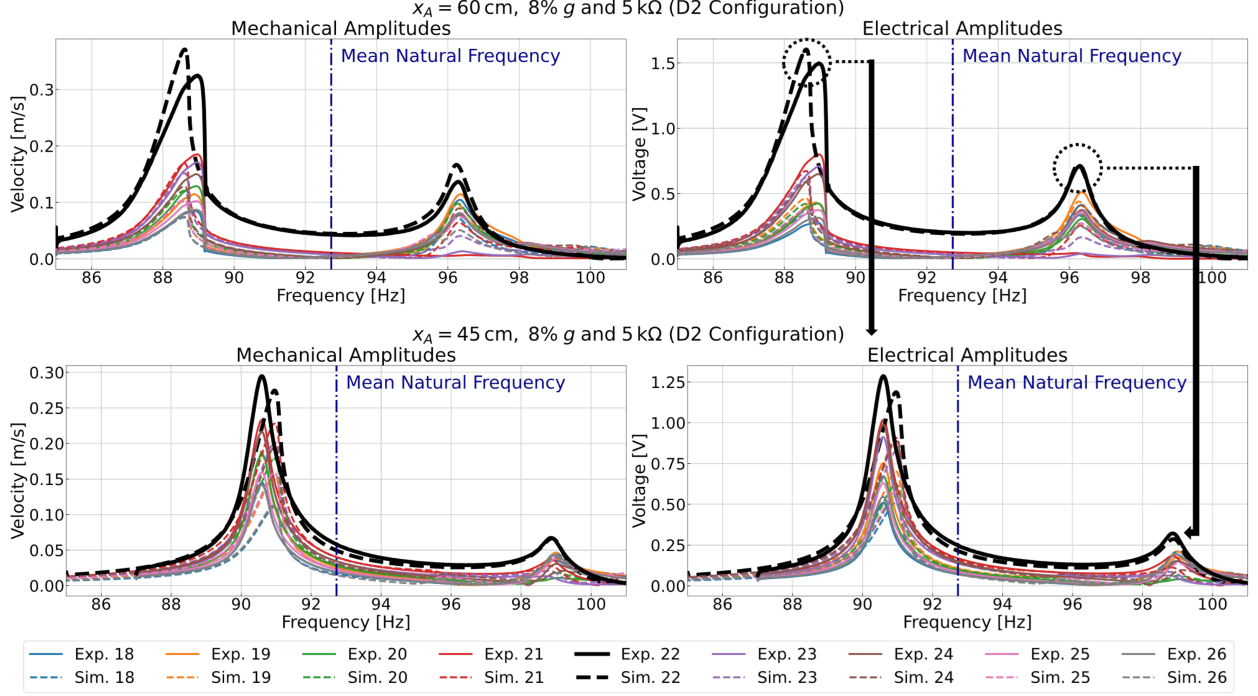


FIG. 17. Steady-state amplitude prediction for the nonlinear oscillator chain. Experimental curves are shown as solid lines, while simulation results are represented by dashed lines.  $D_2$  configuration.

in the prediction of the equivalent coupling, as defined in Equation (16). For instance, a 2% prediction error in  $\beta_{k_{3cl}}$  results in an 18% error in the same variable within the equivalent coupling expression  $\beta_{k_{cleq}}$ , which constitutes a drawback of employing multi-coupled oscillator configurations.

In the context of VEH, the description of  $F$  must necessarily be addressed in the frequency domain<sup>7</sup>. To the best of the authors' knowledge, this is the first study on stationary solitons in mechanical systems that treats the Nonlinear Schrödinger Equation from this perspective [24]. Figure 18 presents the numerical sweep of Equation (18) under the physical parameters extracted experimentally. It is relevant to point out that, under steady-state and stable conditions,  $F$  becomes solely a function of the spatial distribution. Since the dimensionless spatial domain  $\chi$  varies with the excitation frequency,  $F$  is plotted as a function of the physical coordinate  $s$ .

Unlike the simulations employed for validating the lattice differential equation, Figure 18 is built based on initially motionless oscillators. As a consequence of this assumption,

<sup>7</sup> Although the spatiotemporal dynamics of  $F$  are useful for visualizing the soliton properties, such analysis can only be performed at a single frequency.

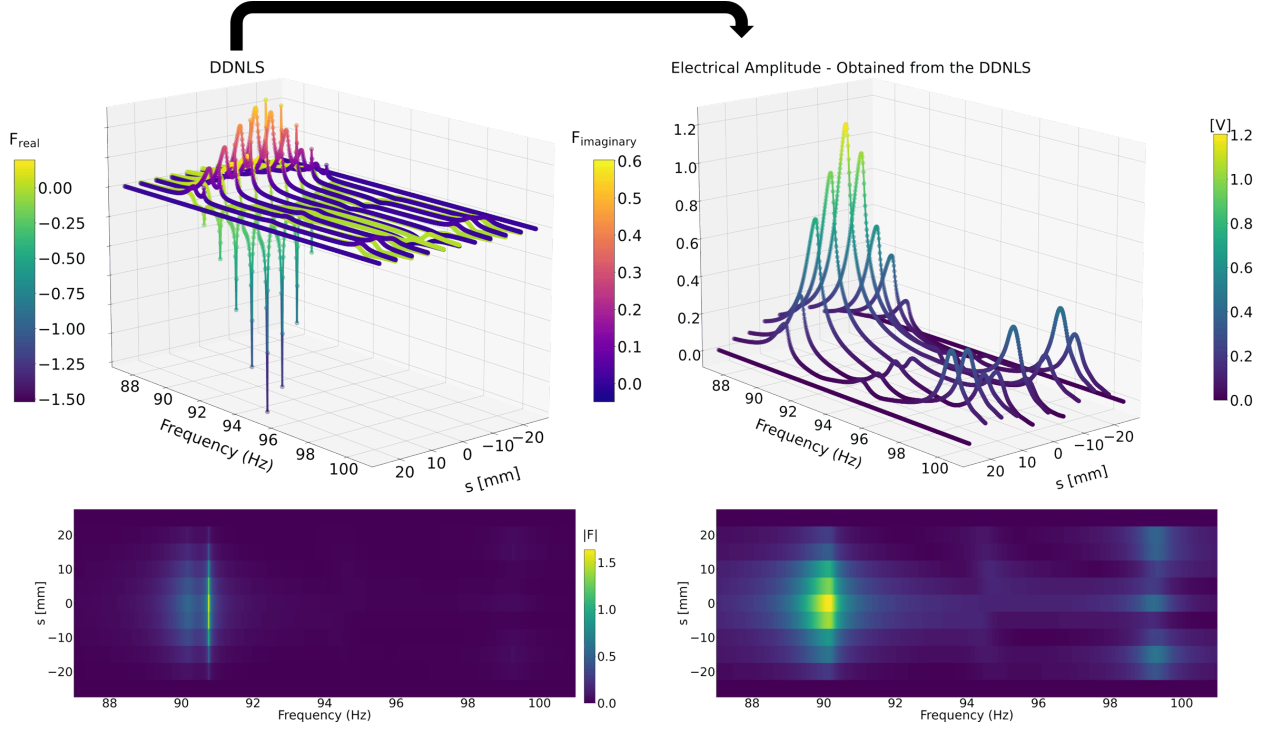


FIG. 18. DDNLS diagram built from Equation (18).  $D_2$  condition and  $x_A = 45$  cm. On the right, the predicted electrical amplitudes extracted from  $F$ .

the soliton nucleations observed are spontaneous, in agreement with the theoretical frame-  
work reported in [78]. It should also be noted that Figure 18 provides information on the  
steady-state behavior over the entire range of excitation frequencies, indirectly capturing  
the spatiotemporal evolution of  $F$ .

The diagram presented by the DDNLS may, at first glance, appear not to capture the nu-  
cleations shown in Figure 17. In the mechanical domain, one could note only that the region  
of maximum amplitude, between 90 Hz and 92 Hz, is in agreement with the experimental  
results. Although prominent amplitudes are not observed in the DDNLS between 98 Hz and  
100 Hz, its electrical prediction does reveal such behavior, in accordance with the experimen-  
tal findings. This apparent contradiction arises due to the normalization applied to  $F(\chi, \tau)$ ,  
which, being frequency-dependent, prevents direct correlation with physical behavior. Such  
correspondence can only be established after reverting the DDNLS normalization as defined  
in Equation (A15) of the Appendix in the first part of this study [34].

#### IV. PREDICTIVE CAPABILITY OF THE DDNLS: EXPERIMENTAL VALIDATION

In [34], the use of the solitons in the design of smart structures is based on the hypothesis that the DDNLS encapsulates a feasible dynamic regime of the system without requiring direct resolution of the full lattice differential equation. Stating that solitonic dynamics are possible in Equation (14) does not imply that such dynamics necessarily emerge, a fact that underscores the importance of incorporating the DDNLS into the design process.

The framework proposed in Section 5.1 of [34] (Part I) is tailored to engineering-oriented applications. Its implementation example in Section 5.2 assumes known quantities such as the excitation amplitude and the bandwidth within which the soliton is expected to operate. The Proof of Concept validation in the current work does not aim to validate the framework for a specific application (as in Section 5.2 of [34]), but to verify the theoretical findings and the predictive capability of the DDNLS within a physical structure (Figure 5).

The parameter optimization in [34] targeted (heuristically) five design parameters simultaneously. Although this approach reflects realistic design conditions, an analysis of the individual effects must be conducted in experimental settings. In other words, only isolated investigations enable the verification of the properties discussed in Section 2.1 of [34] (Physical interpretations underlying the DDNLS). For our structure in particular, the advantage of the ease disorder manipulation (controlled by  $x_A$ ) is counterbalanced by the difficulty in isolating individual parameters, since they are derived based on the vibration modes, which are, in turn, affected by disorder (Equations (5)).

Following the guidelines presented in [34], the first step for implementing the proposed design methodology is the identification of the parameters that can be potentially tuned within the structure. The nonlinear stiffness originates from the geometry of the beams and cannot be tuned throughout the tests. In contrast, the coupling (being a function of the position of the magnets along the beams and their mutual distance) although possible, it is not easily adjustable. The energy dissipation, represented by the variable  $\zeta$ , is not controlled by the design but rather measured a posteriori. The acquisition hardware is limited to a maximum of 10 V, making the use of high resistance values in the MFCs impractical. Within this context, once the experimental setup is assembled, the controllable physical variables are: oscillator disorder, acceleration level, excitation frequency, and the number of terms in

the lattice (by means of adding or removing aluminum masses).

### A. Influence of disorder

The disorder evaluation is centered around the variable  $q$  (Equation (18)). It should be noted that this variable is not only a function of the variation in natural frequency (position of  $x_A$ ), but also depends on the excitation frequency, thus requiring its analysis over the entire domain of  $\Omega$ . As a practical hypothesis, the parametrization/variation of  $x_A$  is exclusively associated with changes in the linear stiffness of beam 22, denoted by  $\phi$  (Equation (18)).

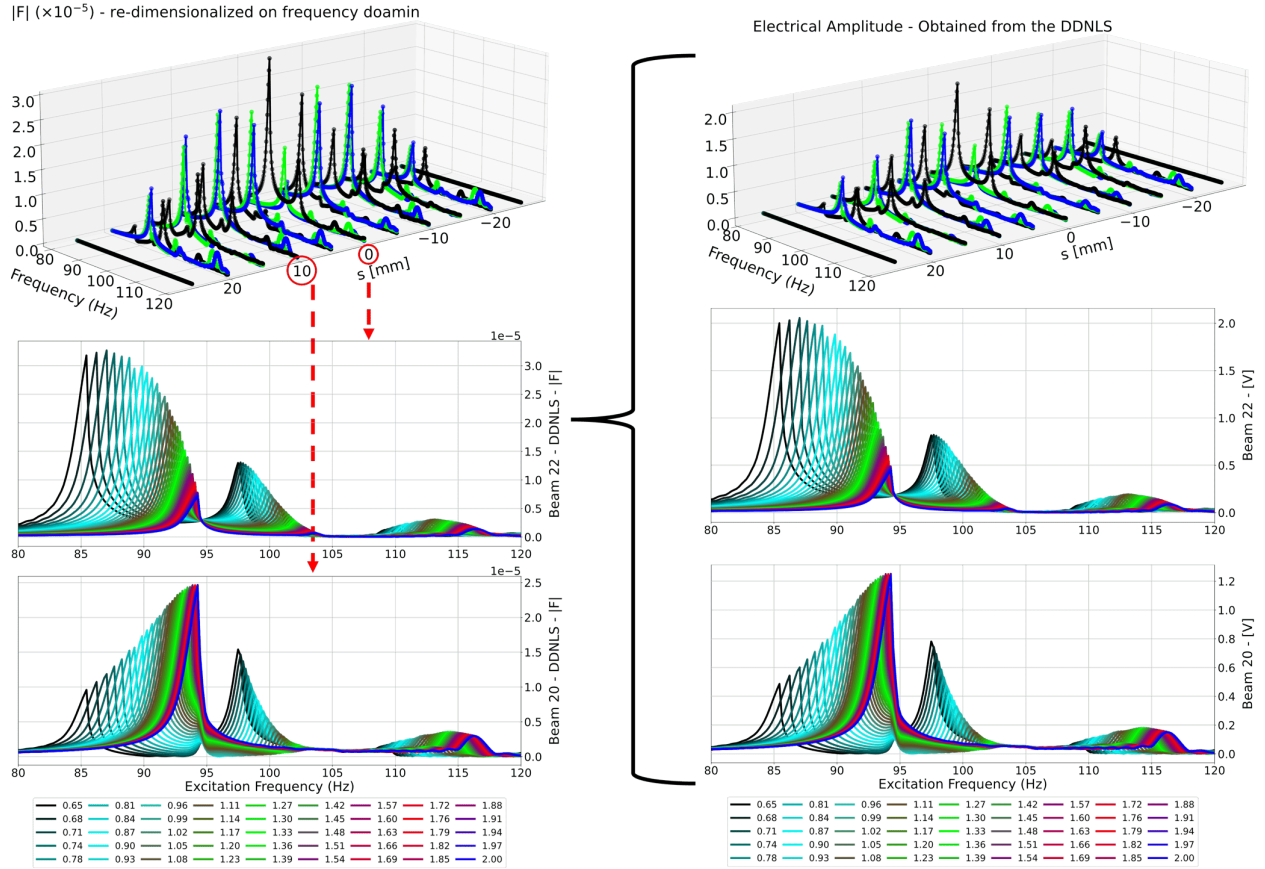


FIG. 19. Soliton nucleation and prediction of the electrical domain. Figure legends indicate the parametric variation: the numerical values (color-coded) represent the modulation of the equivalent linear stiffness of beam 22. In blue, where the linear stiffness is twice the average stiffness of the lattice, no nucleation is observed. In black, the equivalent linear stiffness is set to 0.65. Simulation parameters: 12%, g, 5 k $\Omega$ , and D1 configuration.

Figure 19 shows the behavior of the DDNLS as the linear stiffness of beam 22 ( $s = 0$  [mm]) is varied. According to the DDNLS, energy from the remaining beams migrates toward the beam where the soliton is nucleated. To better visualize this phenomenon, we analyze the amplitude response of beam 20 ( $s = 10$  [mm]). For values of  $q$  that do not lead to soliton nucleation, its peak voltage reaches approximately 1.2 V. As  $q$  increases, this voltage peak bifurcates into two. Such bifurcation is associated with the variable  $\alpha_\Omega$ , which, as demonstrated in [34], delimits the DDNLS into a domain preceding and following the mean nominal natural frequency of the lattice. Both peaks exhibit lower amplitude compared to the peak observed when no soliton is nucleated (in blue)<sup>8</sup>. Concurrently with the energy loss in beam 20, a power gain is observed in beam 22. Let us recall that the physical correlation of  $F$  must be obtained after reverting the DDNLS normalization, given that the normalization is dependent on the excitation frequency.

Figure 20 presents the experimental results obtained for the lattice subjected to different values of  $x_A$ . Although  $x_A$  influences all physical parameters and not only the variable  $\phi$ , the qualitative behavior predicted by the DDNLS is recovered. The bifurcation of the voltage peaks in beam 20 and the consequent energy transfer to beam 22 follow the same patterns outlined by the DDNLS, where the bifurcation point is indicated by the dashed orange line. The analyses conducted for the remaining beams lead to the same conclusions as those observed between beams 20 and 22.

## B. Influence of the number of oscillators

The mathematical development presented in [34] approximates the second derivative of  $F(\chi, \tau)$  using the finite difference method along  $\chi$ . Recall that the new spatial domain (once the other lattice parameters are fixed) is built based on the number of oscillators  $N$ . Such correlation allows us to investigate the dynamics predicted by  $F(\chi, \tau)$  as a function of variations in  $N$ , thereby enabling an understanding of the lattice saturation level, i.e., how the increase in the number of oscillators affects the amount of harvested energy.

Section IV A demonstrates, both numerically and experimentally, that the nature of the periodicity breaking is a key factor soliton formation. In our specific system, a reduction in

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<sup>8</sup> Although energy transfer is being evaluated through the electrical voltage, recall that power is proportional to the square of the voltage, which further highlights the transfer between peaks.



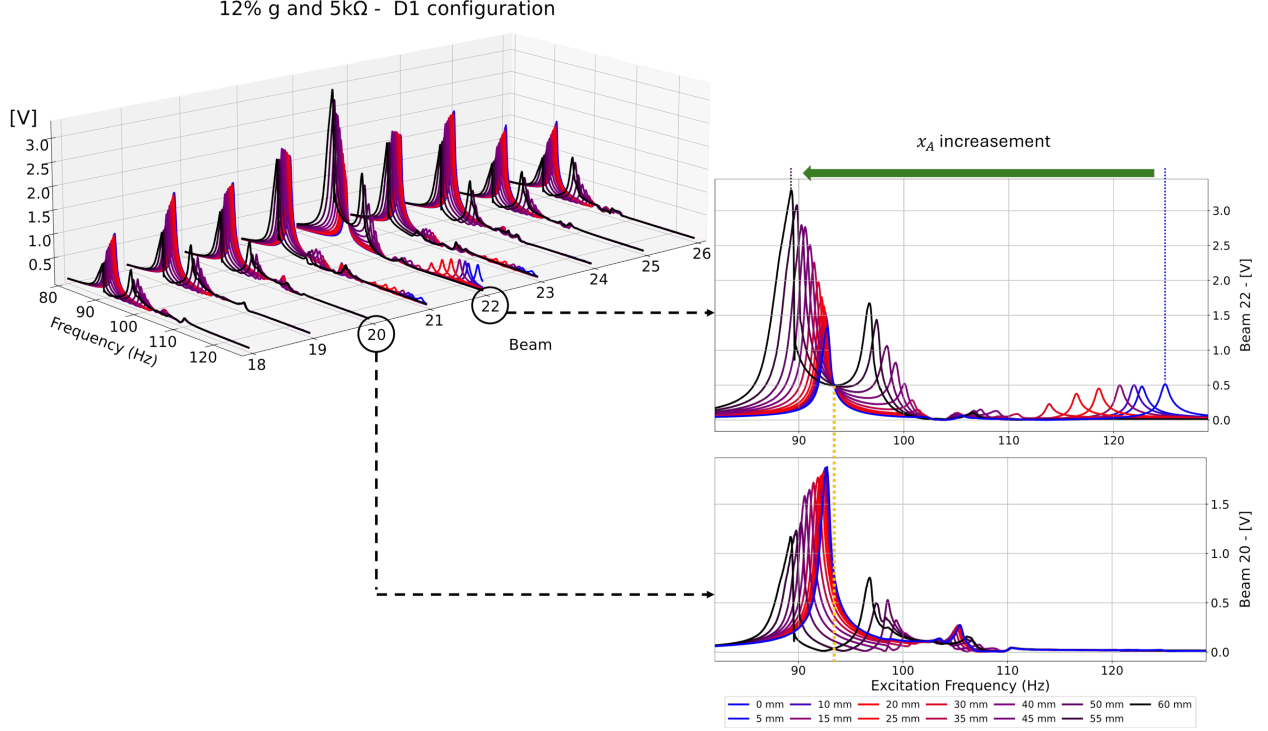


FIG. 20. Experimental data related to the voltage amplitude of the nine central beams as a function of the control mass position on beam 22. The blue curves correspond to the absence of control mass ( $x_A = 0$ , mm, Figure 1), while the black curves represent the configuration with the mass placed near the center ( $x_A \rightarrow \frac{L_b}{2}$ ). The bifurcation point is highlighted by the dashed orange line.

the natural frequency of beam 22 induces soliton nucleation at its location<sup>9</sup>. Motivated by this fact, the oscillators located outside the central region (where piezoelectric transducers are not placed) are tuned to increase their natural frequencies (see Figure 10). This strategy ensures that, at the soliton nucleation frequencies, even in the presence of disorders, any increase in the number of oscillators necessarily transfers energy from the boundaries toward the center, thereby enabling the investigation of an optimal value for  $N$ .

Recalling the methodology presented in Section III A, Figure 21 presents the experimental results and the DDNLS predictions for the FR of beam 22 as more beams are allowed to oscillate. Note that the variation in the number of lattice elements does not aim to increase the oscillator density along the spatial domain, but rather to extend its length  $L$  (Figure

<sup>9</sup> A reduction in the natural frequency corresponds to a negative  $\phi$ , which results in a positive linear potential field in the DDNLS (i.e.,  $q > 0$  in Equation (18)).

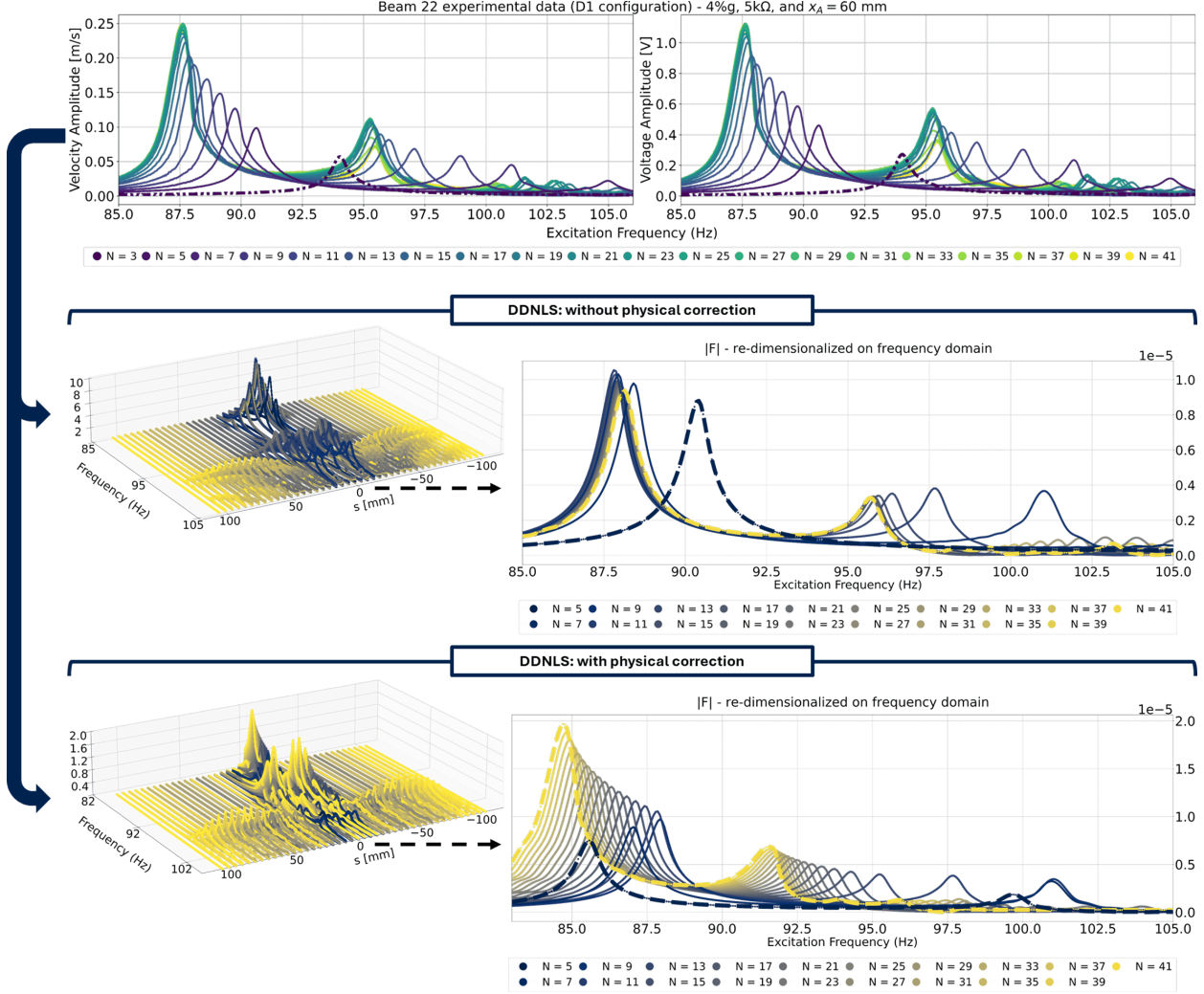


FIG. 21. Experimental and DDNLS-based FRs of beam 22 for different values of  $N$ . Numerical results shown with and without parameter adjustments related to equivalent stiffness and modal force, as a function of  $N$ .

7). Following the findings reported by Shim *et al.* [79] and Xie *et al.* [80], the effects of coupling, nonlinearity, and intentional periodicity breaking significantly enhance both the maximum extracted power and the bandwidth of the harvesters. It is also observed that the voltage obtained for  $N = 33$  does not represent a substantial improvement when compared to  $N = 43$ , whereas a notable increase is found as  $N$  increases from 5 to 9, revealing a saturation level in energy nucleation.

Assuming average parameters for the oscillators located outside the central region (Figure 10), the increase in  $N$  can be analyzed from the DDNLS perspective. It should be emphasized

that incorporating additional oscillators into the system dynamics does not merely stretch or compress  $L_\chi$  due to changes in  $N$ , but also alters  $\beta_{k_{cleq}}$ , since the magnetic stiffnesses that previously acted as local oscillator stiffness now contribute to the coupling stiffness between indices not accounted for in Equation (16). In fact, all physical parameters defined by Equations (5) are modified with the reconfiguration of the magnetic field, particularly the modal force  $\beta_{f_m}$ . As a consequence of this fact, a physical adjustment to the values of  $H$  and  $\beta_{k_{cleq}}$  must be taken into account. Figure 21 presents the numerical integration of Equation (18) with and without incorporating such adjustments. In the absence of this refinement, the DDNLS fails to reproduce the saturation trend observed in the experimental results. However, by assuming an asymptotic decrease in the equivalent stiffness and an asymptotic increase in  $\beta_{f_m}$ , a more accurate qualitative representation is achieved.

### C. Energy localization

Energy localization in mechanical lattices may result from either intentional design or intrinsic emergent dynamics that constrain vibratory energy within limited spatial regions. In linear quasi-periodic systems, localization can arise from minor deviations in periodicity (whether due to manufacturing imperfections or deliberate detuning) which disrupt the dynamic equilibrium of normal modes, thereby inducing modal confinement near structural irregularities [81–83]. In contrast, periodic nonlinear lattices enable intrinsic localization mechanisms, allowing for the emergence of spatially confined excitations referred to as ILMs, breathers [84], or lattice solitons [85–88]. These phenomena could manifest in defect-free structures, exhibiting robustness to external perturbations [89]. The nucleation and stabilization of such states are governed by several physical parameters, including nonlinearity, inter-element coupling stiffness, damping, and external forcing [90–92]. In the context of energy harvesting, the capacity to manipulate the spatial distribution of vibratory energy is particularly advantageous, as it permits the design of systems in which energy is intentionally concentrated near the transduction elements, thereby facilitating the power extraction for several purposes [93].

Figure 22 shows the energy localization predicted by the DDNLS as a function of the disorder variation in beam 22. For values of  $\phi$  that enable soliton nucleation, a significant energy concentration is observed, supporting the use of this metric in potential optimization

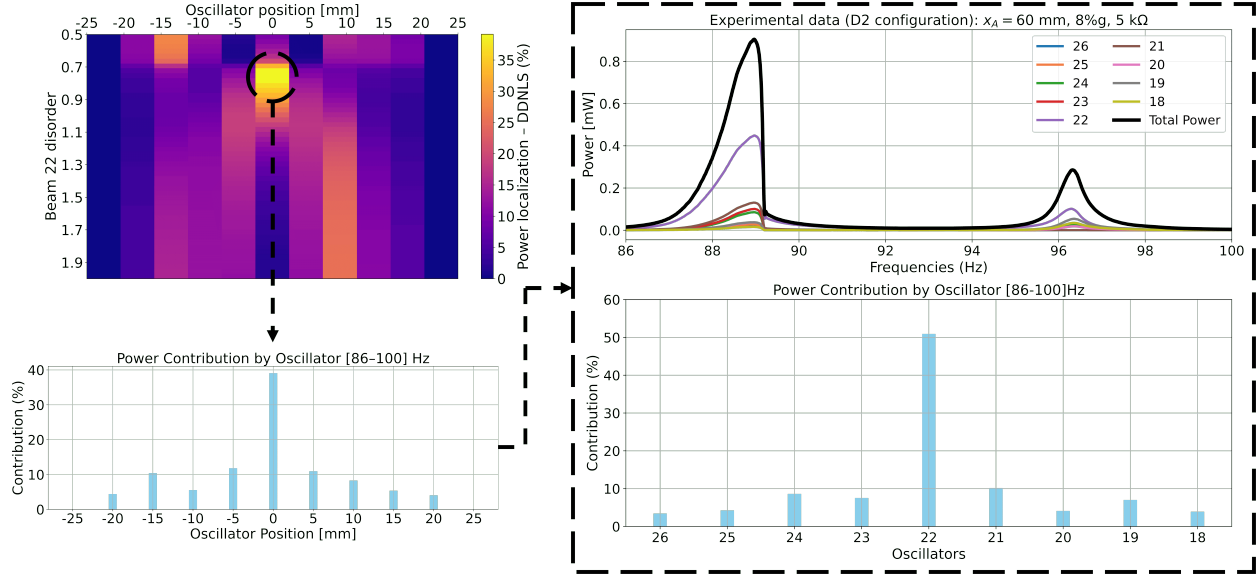


FIG. 22. Numerical analysis of electrical energy localization in the DDNLS as a function of the disorder parameter  $\phi$ . In the colormap, the  $y$ -axis represents the ratio between  $\phi$  and the average linear stiffness of the chain, revealing the SFED for the given disorder. The configuration that yields the energy localization is imposed on the piano, and the corresponding experimental results are plotted with respect to the oscillator index. Eleven oscillators without motion constraint ( $N = 11$ ).

procedures for harvester architectures. When the structure is subjected to nucleation conditions, an energy distribution consistent with that predicted by the DDNLS is obtained, within the expected limitations associated with the assumed disorder functions. The experimental data shown in Figure 22 indicate that the three central beams (21, 22, and 23) trap 77.44% of the electrical energy within the high-amplitude frequency range (86 Hz to 100 Hz), and 33.97% within the low-amplitude range (95 Hz to 99 Hz), under the imposed disorder level. The pronounced frequency-dependent energy localization reveals promising avenues for the development of nonlinear sensing strategies [94–99], as soliton-induced robustness against dispersion is preserved even in the presence of structural disorders when compared to the linear localization (as shown in [34]).

The beam lattice employed in the experiments is not subjected to design optimizations concerning its volume, bandwidth, or key VEH parameters such as the electrical load. The main objective is to provide an experimental validation of the theory developed in [34], where the superiority of the solitonic phenomenon over its linear counterpart was demonstrated in terms of robustness against uncertainties (experimentally verified, Figure 22). Still within

the context of stability, but now in comparison with other nonlinear systems, the enforcement of solitonic regimes also ensures stable dynamics despite the possibility of strong nonlinearities. By analyzing the maximum Lyapunov exponent [65], it was verified that spontaneous nucleation are non sensitive to initial conditions and, therefore, non-chaotic. Building upon the theoretical findings reported in [78], it can also be stated that the tunability of disorder expands the stability diagram, thus, expanding the design possibilities. In order To illustrate these aspects, Figure 23 is presented. In this experimental test, the polarity of the magnets is inverted (resulting in negative linear couplings between adjacent magnets) and random disorders are introduced along the chain. By disregarding the stability conditions dictated by the DDNLS, the guarantee of stability is lost, leading to energy distributions that cannot be controlled.

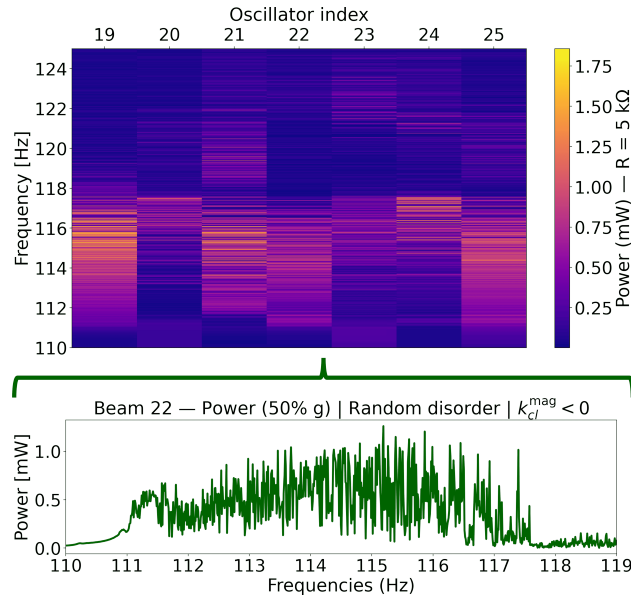


FIG. 23. Experimental result illustrating an unstable energy distribution regime resulting from random spatial disorder and repulsive magnetic coupling. Top: SFED across central oscillators in the chain, showing irregular and broadband energy spreading. Bottom: power spectral density of beam 22, revealing incoherent energy amplification with multiple broadband peaks. In this configuration, the polarity of the magnets is inverted, inducing negative linear coupling between adjacent oscillators. A constant resistance of  $5\text{ k}\Omega$  is applied, and the system is excited by a base acceleration of  $0.5g$ .

## D. Energy storage

An important aspect with respect to the use of the electrical output provided by piezo electric energy harvesters concerns the feasibility of storing the harvested energy. Several studies reporting self-charging power mechanisms highlight this necessity [100, 101]. It must be pointed out that, since the output signals from VEHs are alternating (AC), they must be rectified/regulated into direct current (DC) prior to storage. To this end, an energy management circuit composed of a rectifier, a filter, and a voltage regulator is employed, as illustrated in Figure 24. The circuit for energy storage management is the E-821 module, a commercially available device provided by PI Ceramic<sup>®</sup>. The primary focus of this study is the experimental investigation of the DDNLS, which naturally raises the question of how to store the generated energy. However, a more in-depth analysis of the associated electronics is still required, and is outside the scope of this work. Several studies have reported the use of supercapacitors [102, 103], adaptive inductance [104], lithium-ion rechargeable batteries [105], and multiple coupled circuits [106], providing examples of mechanisms that can enhance the electronic architecture used for energy harvesting.

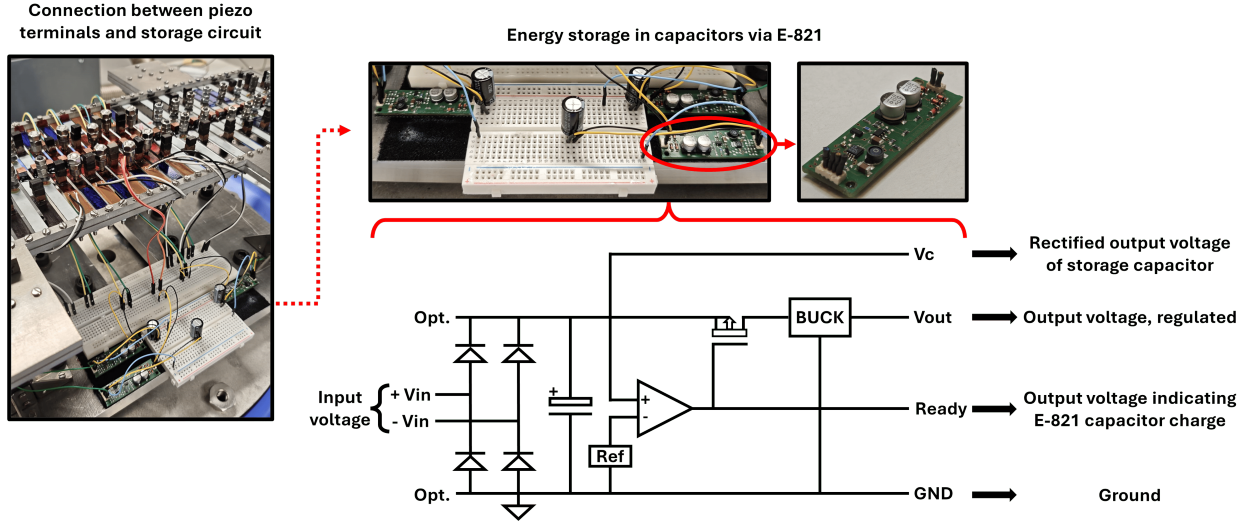


FIG. 24. Description of the signal conditioning circuit for the AC output provided by the MFC terminals. The piezoelectric component terminals are connected to the input pins of the E-821 module, which enables two usage strategies: the rectified signal  $V_c$  and the regulated signal  $V_{out}$ .

The first experiment, presented in Figure 25, aims to analyze the behavior of the electrical outputs  $V_c$  and  $V_{out}$ . For this purpose, the three central oscillators are connected to the

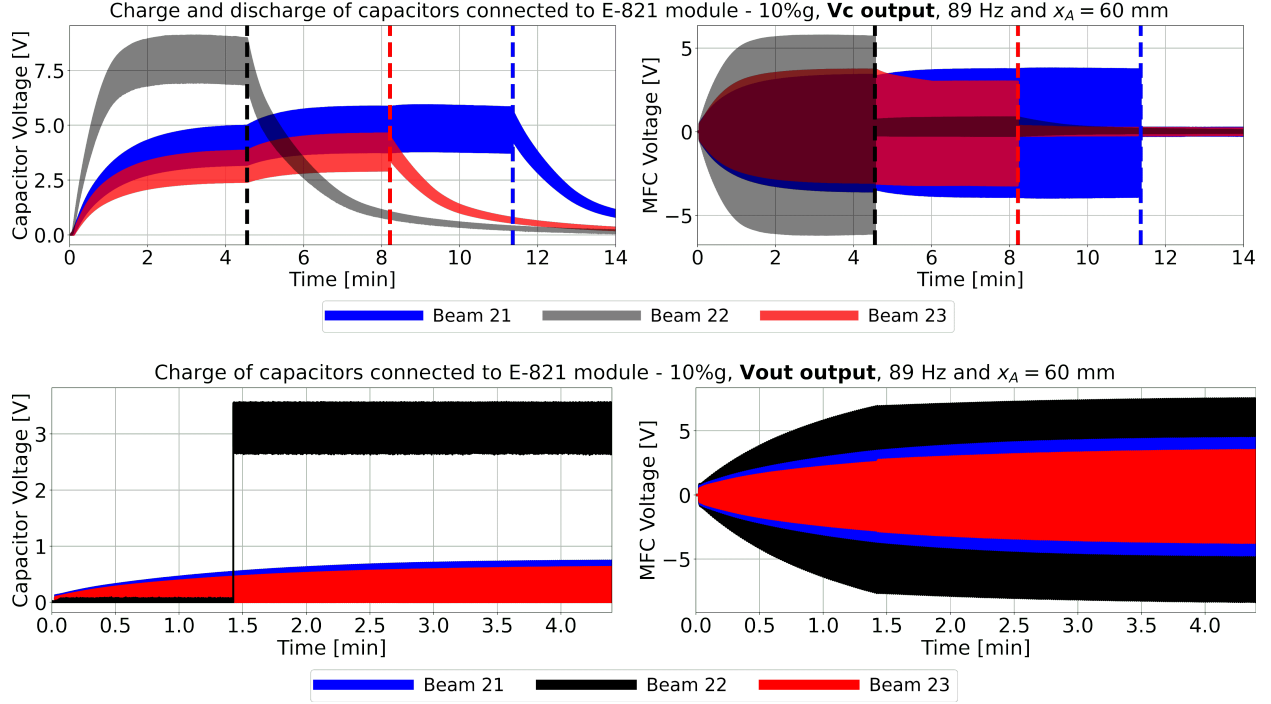


FIG. 25. Electrical response comparison between rectified  $V_c$ ) and regulated ( $V_{out}$ ) outputs during capacitor charging. In the  $V_c$  test, dashed vertical lines indicate the moment when the power supply from the piezo elements is interrupted.

input terminals of E-821 modules, while the remaining harvesters are short-circuited. The rectified signal  $V_c$ , which does not pass through the BUCK stage (DC-DC voltage step-down converter), exhibits a higher voltage level, charging the capacitors<sup>10</sup> in approximately 1.5 minutes. As observed, more electrical energy is stored by the harvester of beam 22, which is predicted by the DDNLS due to energy localization at the frequency of 89 Hz. Throughout the test, the connections between the piezo elements and the modules are intentionally interrupted to observe the energy discharge from the capacitors into the acquisition system, confirming that the capacitors were properly charged. Once the system's ability to charge the capacitor is validated, a subsequent test is conducted using the regulated voltage output. By connecting the capacitor terminals to the  $V_{out}$  output, the charging cycle is again completed in approximately 1.5 minutes. As expected, the BUCK circuit imposes a reduction in the electrical voltage. It is important to observe that the choice between a regulated or rectified voltage depends on the specific requirements of the intended application.

<sup>10</sup> Capacitor of  $4.7 \mu\text{F}$  with maximum voltage of 400 V.



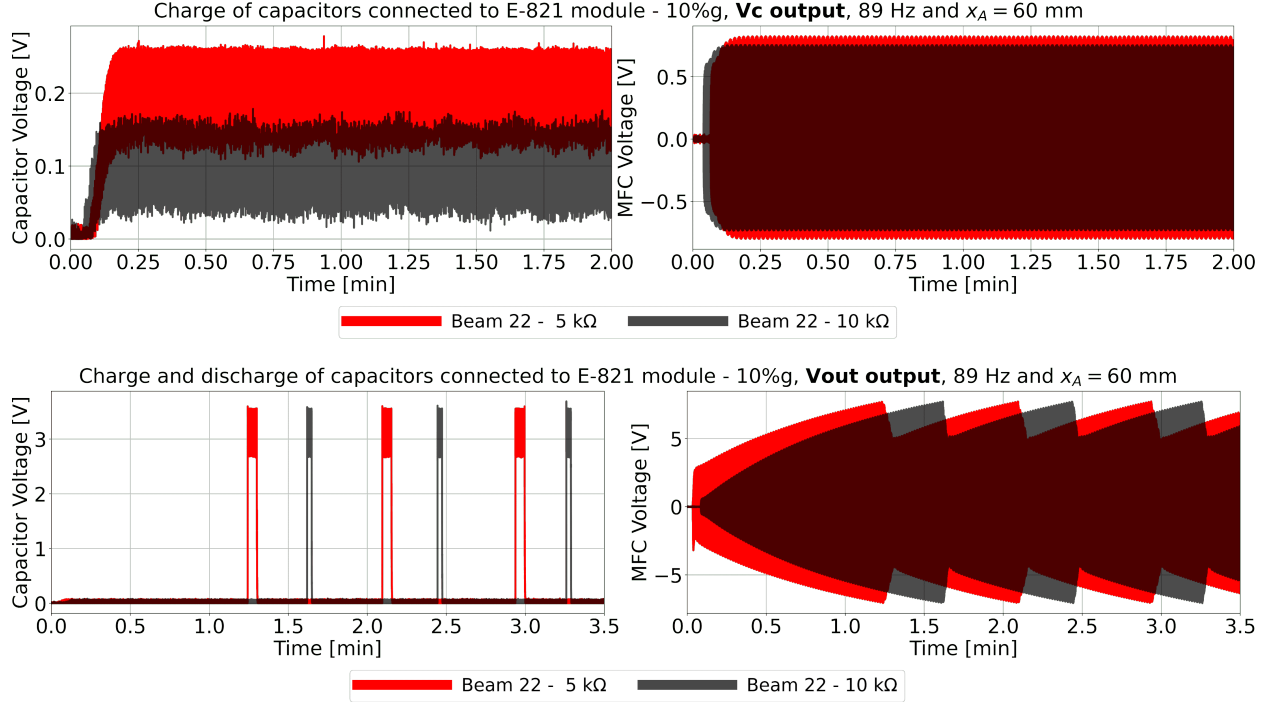


FIG. 26. Output electrical energy under resistive loading conditions: comparison between rectified and regulated cases with parallel capacitors.

534 The second experiment, presented in Figure 26, aims to analyze the electric energy flow  
 535 within the circuit when subjected to electrical loads. By connecting resistors in parallel to  
 536 the capacitor terminals, the charging cycle does not complete or stabilize in the same manner.  
 537 It is observed that when the voltage  $V_c$  is used to power the system, a significant amount  
 538 of energy is dissipated in the resistors, yet a stable behavior is achieved. In contrast, when  
 539 the voltage  $V_{out}$  is employed, the capacitor charging cycle completes; however, the stored  
 540 energy is rapidly transferred to the resistive load, and a new charging cycle is immediately  
 541 initiated.

## 542 E. Performance metrics

543 Given the diversity of works related to VEH, performance metrics for system evaluation  
 544 become mandatory. As an example, we highlight the metric proposed by Liu *et al.* [107],  
 545 where the authors employed power density and frequency bandwidth to create a Systematic



Figure of Merit SFoM<sub>BW</sub>, given by:

$$\text{SFoM}_{\text{BW}} = \frac{16\pi P_{\text{av}} \mathcal{B}}{F^2 V_{\text{tot}}}, \quad (19)$$

where  $P_{\text{av}}$  is the average power,  $\mathcal{B}$  is the total frequency bandwidth,  $F$  is the acceleration amplitude, and  $V_{\text{tot}}$  is the volume of the harvester. Following this approach, several studies have assessed how competitive nonlinear oscillator chains perform when compared to other architectures [108–110]. A noteworthy example is the work of Dowlati *et al.* [111], in which the proposed VEH, simultaneously activating two internal resonances, significantly expanded the operational bandwidth (even without any optimization related to electromechanical coupling), revealing a remarkable advantage of nonlinear metastructures.

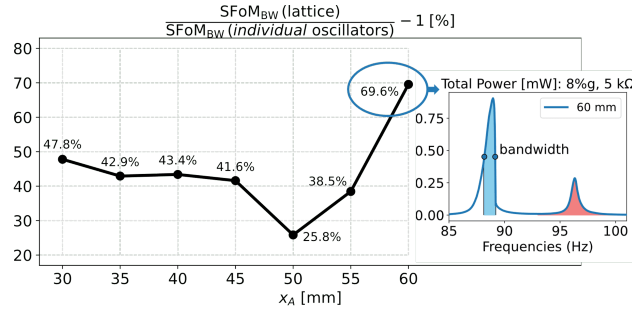


FIG. 27. Comparison between the performance of the collective dynamics governed by the DDNLS equation and the simple superposition of the individual oscillator contributions. The frequency bandwidth  $\mathcal{B}$  is defined as the range within which the power remains above half of the maximum value (half-power bandwidth), illustrated in the inset plot in blue. Tests are performed under  $D_2$ -type disorder, with excitation level of  $8\%g$  and electrical resistance of  $5\text{ k}\Omega$ . By construction, the metric does not account for secondary response regions below the half-power threshold, despite their non-negligible contribution to the total harvested energy (further reinforcing the advantage of the collective nonlinear dynamics). The percentage values shown correspond to the relative improvement in SFoM<sub>BW</sub> achieved by the metastructure.

A conclusion drawn from the theoretical development presented in [34] is that the performance of the collective dynamics surpasses the simple sum of the individual oscillator contributions. To support this statement, let us consider SFoM<sub>BW</sub> in two distinct cases: one in the oscillatory regime described by the DDNLS and another as the sum of the average power contributions of the individual components (Figure 9). When comparing the two sce-

559 narios (Figure 27), we observe that the performance of the metastructure is superior across  
 560 all investigated disorder levels, experimentally validating the conclusion proposed in [34].

## 561 V. CONCLUDING REMARKS

562 This study presents the experimental validation of a energy harvesting strategy based on  
 563 stationary solitons in nonlinear multiphysics metastructures. Building upon the theoretical  
 564 formulation of the Disordered Damped Nonlinear Schrödinger Equation (DDNLS), previ-  
 565 ously derived in [34], the present work confirms its predictive capability through a combi-  
 566 nation of analytical modeling, experimental measurements, and numerical simulations. The  
 567 derivation of an equivalent Duffing oscillator model from a piezoelectric beam array enables  
 568 the incorporation of electromechanical coupling and nonlinear magnetic interactions into the  
 569 DDNLS, which accurately describes the steady-state dynamics of localized oscillations and  
 570 their correlation with spatially confined energy regions.

571 Experimental investigations corroborate the theoretical prediction that intentional disor-  
 572 ders (introduced through localized modifications in the equivalent stiffness of oscillators) act  
 573 as nucleation sites for stationary solitons. The correspondence between DDNLS solutions  
 574 and measured electrical responses validates the hypothesis that soliton-induced energy local-  
 575 ization is not only predictable but also controllable. Furthermore, the influence of physical  
 576 parameters (including nonlinearity, coupling stiffness, electrical resistance, and excitation  
 577 frequency) on soliton stability and spatial distribution aligns with the qualitative dynamics  
 578 predicted by the Schrödinger equation.

579 Following the design guidelines presented in [34], and once the DDNLS has been vali-  
 580 dated, a design criterion can be established, enabling the optimization of energy harvesting  
 581 performance across a specified frequency range. The proposed framework accommodates  
 582 both sub-resonant ( $\Omega < \omega_1$ ) and super-resonant ( $\Omega > \omega_1$ ) excitation regimes and accounts  
 583 for practical limitations associated with experimental uncertainties. By identifying the dis-  
 584 order functions responsible for breaking the chain's periodicity, the DDNLS is shown to be  
 585 an effective tool for optimizing energy localization, thereby offering a robust approach to  
 586 energy management in metastructures.

587 To the best of our knowledge, this work represents the first experimental validation of  
 588 stationary soliton dynamics in a mechanical lattice formulated in the frequency domain.

Such approach introduces perspectives for the design of metastructures by endowing the lattice with robustness against waveform dispersion (a defining property of nonlinear solitary waves). The experimental findings, grounded in the theoretical framework developed in [34], demonstrate the superior performance of coupled oscillators compared to the mere sum of their individual responses, particularly in terms of maximum extracted power, operational bandwidth, and energy localization capacity.

In conclusion, when considered alongside the first part of this study, the present article confirms the feasibility of a design tool for energy management in multiphysical metastructures, resilient to uncertainty, system dispersion and chaotic behavior. The experiments also confirm the feasibility of energy storage using rectified and regulated outputs from soliton-induced localized vibrations. Future research will focus on refining parameter identification techniques, integrating adaptive control strategies for real-time soliton tracking, and designing optimized architectures for applications subjected to multi-frequency excitations.

## ACKNOWLEDGMENTS

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## DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Appendix A: Derivation of the modal basis

The formulation of the modal basis is derived from the linear, undamped and unforced form of Equation 1:

$$m(x)\ddot{D}(x, t) + E_{eq}I_{eq}D''''(x, t) = 0, \tag{A1}$$

611 while, in our specific problem,

$$m(x) = m_b + M_M \delta_d \left( x - \frac{L_b}{2} \right), \quad (\text{A2})$$

612 where  $m_b$  is the mass per unit of length of the beam in the absence of  $M_M$ . By assuming  
 613 an oscillatory response for the time-dependent variable in the form  $\eta_r(t) = \Upsilon e^{i\omega_r t}$ , Equation  
 614 (A1) simplifies to a purely spatial equation:

$$-\omega_r^2 \left( m_b + M_M \delta_d \left( x - \frac{L_b}{2} \right) \right) \Xi_r(x) + E_{eq} I_{eq} \frac{\partial^4}{\partial x^4} \Xi_r(x) = 0. \quad (\text{A3})$$

615 The solution to Equation A3 can be obtained using the Laplace transform  $\mathcal{L}_x$  in all terms,  
 616 leading to the following expression:

$$\mathcal{L}_x [\Xi_r(x)](s) = \frac{\mu k^4 \Xi_r(L_b/2) e^{-sL/2}}{s^4 - k^4} + \frac{s \frac{d^2 \Xi_r}{dx^2}(0)}{s^4 - k^4} + \frac{\frac{d^3 \Xi_r}{dx^3}(0)}{s^4 - k^4}, \quad (\text{A4})$$

617 in which

$$k^4 = \frac{\omega_r^2 m_b}{E_{eq} I_{eq}}, \quad \mu = \frac{M_{conc}}{m_b}, \quad (\text{A5})$$

618 and the following boundary conditions are imposed:

$$\Xi_r(0) = 0, \quad \left. \frac{d\Xi_r(x)}{dx} \right|_{x=0} = 0, \quad \Xi_r(L_b) = 0, \quad \left. \frac{d\Xi_r(x)}{dx} \right|_{x=L_b} = 0. \quad (\text{A6})$$

619 Once the inverse transform is applied to Equation (A4), bringing back the spatial variable  
 620  $x$ , we derive the equation that governs the modal basis of the system having the format:

$$\Xi_r(x) = C_1 H_e \left( x - \frac{L_b}{2} \right) \left[ \sinh \left( k \left( x - \frac{L_b}{2} \right) \right) - \sin \left( k \left( x - \frac{L_b}{2} \right) \right) \right] \quad (\text{A7})$$

$$+ C_2 (\cosh(kx) - \cos(kx)) + C_3 (\sinh(kx) - \sin(kx)),$$

621 where

$$\begin{aligned}
C_1 &= \frac{Mk}{2m} \Xi_r \left( \frac{L_b}{2} \right), \\
C_2 &= \frac{1}{2k^2} \frac{d^2 \Xi_r(0)}{dx^2}, \\
C_3 &= \frac{1}{2k^3} \frac{d^3 \Xi_r(0)}{dx^3},
\end{aligned} \tag{A8}$$

622 and  $H_e \left( x - \frac{L_b}{2} \right)$  represents the Heaviside function.

623 Let us note that Equation (A7) is undetermined due to the unknown constants  $\Xi_r \left( \frac{L_b}{2} \right)$ ,  
624  $\frac{d^2 \Xi_r(0)}{dx^2}$ , and  $\frac{d^3 \Xi_r(0)}{dx^3}$ . The first step toward their determination is the formulation of a  
625 system of equations based on the boundary conditions presented in Equations (A6), leading  
626 to a matrix  $\Lambda$  whose determinant imposes the natural frequencies  $\omega_r$  associated with the  
627 modal basis. In our specific case,  $\Lambda$  is defined by

$$\Lambda = \frac{1}{2k^2} \begin{bmatrix} 2k^2 & f' \left( \frac{L_b}{2} \right) & f \left( \frac{L_b}{2} \right) \\ \mu k^4 f \left( \frac{L_b}{2} \right) & f' (L_b) & f (L_b) \\ \mu k^4 f' \left( \frac{L_b}{2} \right) & f'' (L_b) & f' (L_b) \end{bmatrix}, \tag{A9}$$

628 being associated with the system of equations:

$$[\Lambda] \begin{bmatrix} \Xi_r \left( \frac{L_b}{2} \right) \\ \frac{d^2 \Xi_r(0)}{dx^2} \\ \frac{d^3 \Xi_r(0)}{dx^3} \end{bmatrix} = 0. \tag{A10}$$

629 The auxiliary function  $f(x)$  is introduced for notational convenience and is defined as follows:

$$f(x) = \frac{\sin(kx)}{k} - \frac{\sinh(kx)}{k}. \quad (\text{A11})$$

630 The system (A10) allows for the derivation of relationships between  $\Xi_r \left( \frac{L_b}{2} \right)$ ,  $\frac{d^2 \Xi_r(0)}{dx^2}$ ,  
631 and  $\frac{d^3 \Xi_r(0)}{dx^3}$ , but does not yet provide their complete determination. This can only be  
632 achieved through the modal orthogonality property imposed by the following relation:

$$\int_0^{L_b} \Xi_r(x) m(x) \Xi_j(x) dx = \delta_{kj}, \quad (\text{A12})$$

633 where  $\delta_k$  is the Kronecker delta, which equals 1 when  $r = j$  and 0 otherwise. A detailed  
634 discussion on the algebraic process involved in the derivation of the modes can be found in  
635 [112, 113].

636 The modal basis verification and the prediction of the first undamped natural frequency  
637  $\omega_1$  are conducted using Ansys<sup>®</sup> through a finite element (FE) simulation. Employing the  
638 geometric parameters used in the experiment (Table I, Section III A) and elements with a  
639 maximum dimension of 0.5 mm in the mesh discretization, two simulations are examined:  
640 one considering all dimensions of the magnet in the central position and another employing  
641 the point mass assumption.

642 Figure 28 presents the simulation results, showing that the accuracy of Equation (A7)  
643 and matrix  $\Lambda$  is verified. According to the simulations, the numerical prediction of both the  
644 mode shape and the natural frequency agrees with the finite element results within an error  
645 of  $\sim 0.8\%$ , with numerical discrepancies attributed to mesh refinement<sup>11</sup>.

646 The mathematical formulations describing the isolated beam provide insight into the  
647 operational range of the experiment. In other words, the obtained frequencies serve as an  
648 upper bound for experimental tuning, implying that frequency adjustment is feasible through  
649 mass addition, which consequently decreases the  $\omega_1$  values. Another advantage arising from  
650 the analytical formulation of the beam is the understanding of the order of magnitude of  
651 the stiffness parameters involved. As an example, the dimensionless variable  $H$ , which  
652 accounts for the external excitation in the DDNLS, is a function of the square root of the  
653 nonlinear stiffness. This relationship enables us to estimate the required acceleration level

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<sup>11</sup> The clamped-clamped boundary configuration is adopted, which also may represent a potential source of discrepancies [114].

needed to achieve a specific value of  $H$ . Although the exclusion of the tuning mechanisms (Figure 1) from the model prevents accurate quantitative predictions, the development of an experimental setup still demands a feasible operational range, which is provided by Equation (1).

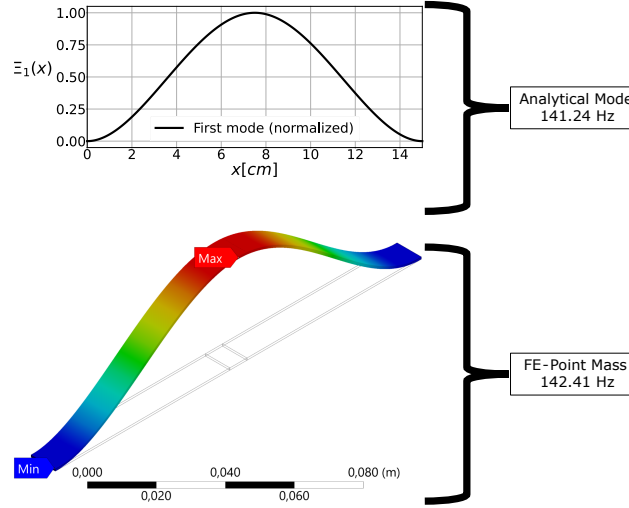


FIG. 28. Comparison between the predictions for the mode shapes and the first natural frequency of the clamped-clamped beam. In order: analytical prediction and finite element model considering a point mass.

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