



Energy localization in damped nonlinear disordered metastructures under superharmonic resonance

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Abstract

This paper proposes a framework for energy localization in nonlinear oscillator chains operating in nonfundamental resonance, with emphasis on superharmonic regimes. The system is modeled as a metastructure composed of Duffing oscillators with both linear and nonlinear coupling, incorporating disorder-induced periodicity breaking. Under the assumption of strong excitation, the method of multiple scales is employed to derive the governing equations for soliton dynamics. At first-order perturbation, the classical form of the Nonlinear Schrödinger Equation emerges, whereas second-order analysis yields a previously unreported equation arising from the restitution of time scales. Analytical and numerical results demonstrate the nucleation of solitons in both hardening and softening regimes, based on two approaches: direct time-domain simulations from initially motionless states and numerical continuation in the frequency domain. A key finding is the distinct role of phase in superharmonic resonances compared to the primary resonance; specifically, the coexistence of multiple frequency components in the steady-state response precludes interpreting the soliton directly as a displacement envelope. Instead, the resulting secular terms capture the soliton associated with the resonant contribution, while transient components remain present under superharmonic excitation. Furthermore, robustness against disorder uncertainty is assessed by determining the tolerance levels that preserve the phenomenon. These results support the development of vibration control strategies aimed at mitigating the increase in resonant frequencies associated with the geometric downscaling of mechanical systems.

Keywords Nonlinear metastructures · Energy localization · Superharmonic resonance · Standing solitons · Disordered Nonlinear Schrödinger Equation

1 Introduction

Increasing attention has been devoted to novel strategies for energy management in mechanical systems [1]. In fact, the development of metamaterials and metastructures has emerged as a promising approach for addressing a broad range of challenges, including wave propagation [2], vibration control [3], impact absorption/mitigation [4], sensing [5], acoustic protection [6], and energy harvesting [7]. Such mechanisms have garnered prominence due to their versatile applications [8] and tunability [9], allowing dual functionalities and multiphysics couplings [10]. Notably, metastructured lattices enable enhanced energy control in multiphysical systems through collective dynamic effects. For instance, energy localization has been demonstrated in coupled-beam systems [11], phononic crystals and topological beams [12], as well as in defect-induced modes [13], acoustic black holes [14], architected geometries with

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instability-driven localization [15], and mistuned bladed structures [16].

With respect to the design of metastructures for vibration-related applications, the literature reports a broad spectrum of methodological approaches, encompassing post-buckling instability [17], energy absorption [18], topology optimization [19], voxel-based modeling [20], bio-inspired architectures [21], wave-based finite element formulations [22], stochastic modeling frameworks [23], and inverse design strategies [24], among others. The introduction of nonlinear effects into these systems opens up a new variety of physical phenomena [25]. In a recent contribution, Bai et al. [26] reviewed the major advances in nonlinear vibration metamaterials. Building upon the classification proposed therein, these advances may be broadly grouped according to the prevailing nonlinear mechanisms, including granular media-induced nonlinearity [27], nonlinear stiffness-based architectures (encompassing quadratic, cubic, negative, and quasi-zero stiffness) [28], multistable and bistable systems exploiting snap-through and hysteresis [29], as well as designs that explicitly exploit nonlinear dissipation [30]. Such strategies have resulted in, among other effects, amplitude-dependent bandgaps [31], broadband low-frequency attenuation [32], and nonreciprocal wave propagation [33].

Within the same scenario, coupled nonlinear oscillators have been widely investigated owing to their rich dynamical behavior [34–38]. Unlike linear chains, nonlinear chains can support Intrinsic Localized Modes (ILMs) in perfectly periodic configurations, without the need for disorder or defects [39–42], although the induction of energy localization is significantly enhanced by deliberate periodicity breaking [43]. When addressing studies related to ILMs, a complementary research direction emerges, providing the mathematical framework of energy localization in nonlinear oscillator chains, namely, the investigation of stationary solitons. Following the early observation of solitary waves by Russell [44] and their subsequent theoretical formalization [45], solitons have become fundamental features across a broad range of physical systems. Their dynamics have been extensively described from the standpoint of the Nonlinear Schrödinger Equation (NLS), which has been employed to model phenomena in diverse fields [46–49], and, more recently, in mechanical and elastic metamaterials [50]. From the perspective of mechanical metastructure architecture, ILMs/stationary solitons have been reported in a broad class of systems, including micromechanical arrays [51], coupled pendula [52], rotors [53], and linear oscillator networks [54]. Their properties and practical applications have been investigated in the literature, particularly in sensing [55], sensitivity enhancement [56], and nonlinear energy sink technologies [57].

Recently, Gao et al. [58] published a comprehensive review classifying the state of the art in solitary waves within

nonlinear metamaterials and metastructures. The reported examples encompass pulse-type solitons [59], kink-type solitons [60], envelope-type solitons [61], breathers [62], vector solitons [63], rarefaction solitons [64], and topological solitons [65], among others. The reviewed contributions highlight the dynamical advantages of nonlinear wave phenomena over linear ones, particularly regarding robustness against waveform dispersion [66]. Consistent with such an advantage, the pursuit of self-balanced states (in which energy remains predominantly confined to a region of space) can be induced by nonpropagating solitary waves, an idea that has been established in the literature for several years [67]. Studies have derived closed-form analytical solutions and the corresponding stability maps for standing/stationary solitons under parametric [68, 69] and external [41, 70] excitation. Extending beyond periodic configurations, quasiperiodic lattices with localized impurities have been shown to promote multi-soliton nucleation and to expand the existence domain of damped solutions [43, 71]. Furthermore, the nature of the excitation plays a critical role in solitonic dynamics, and analytical solutions for stationary damped solitons subjected to external forcing remain unavailable [70, 72]. Compared with other physical domains, the systems considered here involve a further layer of complexity, since damping is generally non-negligible in mechanical systems. In [73, 74], Barbosa et al. [73] proposed a design methodology grounded in the Disordered Damped Nonlinear Schrödinger Equation, demonstrating the feasibility of using standing solitons as design guidelines for energy harvesting applications. The authors further highlight the challenges posed by damping, particularly with respect to the generation of solitons in the frequency domain.

Despite the extensive literature on the topic, to the best of the authors' knowledge, several nonlinear phenomena involving nonfundamental harmonic resonances and metastructures have attracted limited research effort. Indeed, resonances not associated with the fundamental frequency have been documented for decades [75, 76] and continue to attract attention in recent studies [77–80]. Applications such as pedestrian-induced excitation of vibrating structures [81], energy harvesting exploiting superharmonic behavior [82], MEMS dynamics [83], multi-staged clutch damper systems [84], time-delayed feedback control [85], ultraharmonic resonance [86], and damage regulation [87] further motivate the investigation of secondary resonances in nonlinear oscillators. From the perspective of metastructure development under secondary resonance regimes, the study of coupling among unit cells becomes of particular interest [88, 89], raising questions related to energy management under such conditions. For instance, in [90], Wu et al. investigated the generation of superharmonic waves in a chain-type multicell aperiodic structure with nonlinear stiffness. In a later development, Lepidi et al. [91] presented a study on superharmonic

wave propagation in two-dimensional mechanical metamaterials with inertia amplification. Beyond superharmonic responses, subharmonic resonances have been reported in locally resonant metamaterials induced by autoparametric resonance [92], in two-dimensional wings incorporating bistable metamaterials [93], and in piezoelectric metamaterial beams subjected to viscous flow [94]. Nevertheless, these two directions remain disconnected: energy-localization frameworks based on stationary solitons/ILMs, most recently extended to a disorder-tuned design methodology for energy harvesting [73, 74], have so far addressed only excitation at the primary resonance. In contrast, studies on superharmonic metastructures [88–91] focus on dispersion, internal resonance, and harmonic generation without treating the associated envelope/soliton dynamics or disorder-induced localization. To the best of the authors' knowledge, this work is the first to combine both frameworks, demonstrating soliton-based energy localization under superharmonic excitation.

Given the current state of the art, this paper demonstrates the existence of stationary solitons in secondary resonance regimes, particularly superharmonic ones. To this end, a theoretical mechanical metastructure is investigated, and, based on the method of multiple scales under the strong excitation assumption [95, 96], a discrete soliton equation is derived. Motivated by a recent study [73, 74], in which the behavior of multiphysics solitons was assessed in the frequency domain, the present work establishes parallels between the mechanisms associated with primary and superharmonic resonances, highlighting their implications for energy localization. The main contributions of this article consist of (i) demonstrating the role of phase dynamics in lattice energy localization under superharmonic excitation, (ii) describing the wave nucleation process from initially motionless oscillators and (iii) deriving a novel methodology for energy localization in nonlinear lattices under disorder uncertainty; contributions that have so far been addressed in the literature only around the fundamental frequency. Given the general character of the theoretical formulation, which is not restricted to a specific metastructure architecture or to a particular type of nonlinearity, the analyses are conducted for both hardening and softening behavior, thereby clarifying the numerical differences between the two scenarios from the perspective of the frequency domain.¹ This work contributes to the field of metastructures by proposing a new design paradigm that leverages the robustness of solitons against waveform dispersion. By establishing the viability of solitary waves operating under superharmonic resonance, the realization of systems with reduced geometric dimen-

sions becomes a more attainable prospect, as the increase in natural frequency associated with geometric miniaturization can be offset by the availability of harmonic frequencies, which may act as submultiples of the fundamental one and accordingly broaden the design space.

The remainder of the paper is structured as follows. Section 2 presents the derivation of the lattice soliton equation in the superharmonic regime, based on the method of multiple scales. The subsection on first-order secular terms establishes the connection with the Disordered Damped Nonlinear Schrödinger equation, while the subsection on second-order secular terms introduces a novel restitution equation that accounts for higher-order effects. Section 3 investigates the frequency response of an isolated Duffing oscillator under superharmonic excitation, providing a reference for interpreting the collective behavior of the lattice. Section 4 analyzes the nucleation and properties of stationary solitons, including the influence of disorder, the saturation with respect to the number of oscillators, and the uncertainty levels tolerated by the metastructure. Finally, Sect. 5 summarizes the main findings and discusses perspectives for future work.

2 Derivation of the lattice soliton equation in the superharmonic regime

The goal of this section is to present the mathematical description of the steady-state regime of the nonlinear metastructure shown in Fig. 1. Based on the resulting equation, the differences relative to primary resonance regimes are detailed, highlighting theoretical features that arise exclusively in secondary resonances, such as the emergence of multiple frequency components in the composition of the response. Additionally, the NLS is derived at the first slow temporal scale, thus demonstrating the possibility of superharmonic soliton generation. To extend the validity of the model, a higher temporal order is incorporated in the derivation of the secular terms, resulting in an equation not yet reported in the literature.

By enforcing force equilibrium among the N components illustrated in Fig. 1, the governing equation of the lattice dynamics under base acceleration can be written as follows:

$$\begin{aligned} & \frac{d^2 U_n(t_f)}{dt_f^2} + 2\zeta_n \omega_0 \frac{dU_n(t_f)}{dt_f} + \omega_0^2 U_n(t_f) \\ & + f(n) U_n(t_f) \pm \beta U_n^3(t_f) \\ & + \sum_{k=1}^{\alpha} C_l^{(k)} (2U_n(t_f) - U_{n+k}(t_f) - U_{n-k}(t_f)) \\ & + \sum_{k=1}^{\alpha} C_{nl}^{(k)} [U_n(t_f) - U_{n+k}(t_f)]^3 \end{aligned}$$

¹ For didactic consistency, the same color schemes are adopted throughout the study for the hardening (“viridis”) and softening (“plasma”) regimes.

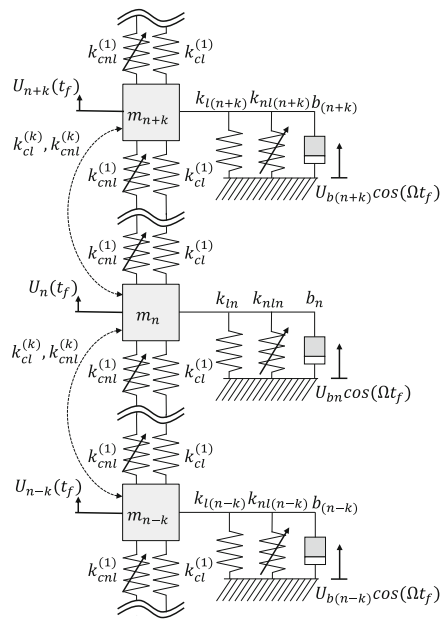


Fig. 1 Equivalent lumped-parameter model of the metastructure. The system consists of a chain of oscillators coupled through linear stiffnesses k_{cl} and nonlinear stiffnesses k_{nl} , with coupling interactions extending up to the α -th neighbor and the superscript (k) denotes the coupling order. Each unit cell is defined by the mass m , linear stiffness k_l , nonlinear stiffness k_{nl} , damping coefficient b , and base excitation of amplitude U_b , indexed by n to identify each oscillator. The excitation is tuned around the third-order superharmonic resonance of the primary mode

$$+ (U_n(t_f) - U_{n-k}(t_f))^3] = -F_n \cos(\Omega t_f), \quad (1)$$

where the natural frequency ω_0 , the damping ratio ζ_n , the spatial variation of the linear stiffness $f(n)$, the nonlinear stiffness β , the linear couplings $C_l^{(k)}$, and the nonlinear couplings $C_{nl}^{(k)}$ are defined in Table 2. It is important to emphasize that the system incorporates couplings not only among neighboring oscillators but also among oscillators separated by up to order α . Additionally, periodicity breaking is allowed in the damping ratio, linear stiffness, and oscillator excitation, while the couplings and nonlinear effects remain periodic.

In order to address Eq. (1), the method of multiple scales is applied [97]. The lattice physical parameters are therefore rescaled by a factor ε according to the following orders²:

$$\begin{aligned} & \frac{d^2 U_n(t_f)}{dt_f^2} + \omega_0^2 U_n(t_f) \\ & + \varepsilon \left[2\zeta_n \omega_0 \frac{dU_n(t_f)}{dt_f} + U_n(t_f) f_\varepsilon(n) \pm \beta_\varepsilon U_n^3(t_f) \right. \\ & \left. + \sum_{k=1}^{\alpha} C_{l\varepsilon}^{(k)} (2U_n(t_f) - U_{n+k}(t_f) - U_{n-k}(t_f)) \right] \end{aligned}$$

² For notational convenience, the newly introduced variables are defined using the same notation as in Eq. (1), indexed by ε .

$$\begin{aligned} & + \varepsilon^2 \sum_{k=1}^{\alpha} C_{nl\varepsilon}^{(k)} \left[(U_n(t_f) - U_{n+k}(t_f))^3 \right. \\ & \left. + (U_n(t_f) - U_{n-k}(t_f))^3 \right] = -F_n \cos(t_f \Omega). \quad (2) \end{aligned}$$

As indicated by Eq. (2), the high-excitation hypothesis is adopted. Under this assumption, the external forcing is of the same order of magnitude as the free dynamics, so that F_n appears at leading order in the expansion and is therefore not perturbed [97, 98]. Within the same perturbation framework, damping and nonlinear terms are assumed to be weak and are consequently scaled one order of magnitude below the linear stiffness terms [97]. The linear and nonlinear coupling stiffnesses are likewise assumed to be one order of magnitude lower than the corresponding linear and nonlinear stiffnesses. Such scaling is supported by experimental systems, in which the coupling elements are weak relative to the local restoring terms: Barbosa et al. [74] identifies linear and nonlinear couplings smaller than the corresponding linear and nonlinear stiffnesses, while Mahmoudi et al. [99] obtains a curve fit of a magnetic interaction force whose identified coefficients are of the same weak order. Given the extensive algebraic manipulations associated with the method of multiple scales, Eq. (2) is nondimensionalized according to the relations presented in Table 2, from which the following governing equation is derived:

$$\begin{aligned} & \frac{d^2 u_n(t)}{dt^2} + u_n(t) + \varepsilon \left[2\zeta_n \varepsilon \frac{du_n(t)}{dt} \pm u_n^3(t) \right. \\ & + \sum_{k=1}^{\alpha} \chi_l^{(k)} (2u_n(t) - u_{n+k}(t) - u_{n-k}(t)) \\ & + u_n(t) \mu(n) \left. + \varepsilon^2 \sum_{k=1}^{\alpha} \chi_{nl}^{(k)} \left[(u_n(t) - u_{n+k}(t))^3 \right. \right. \\ & \left. \left. + (u_n(t) - u_{n-k}(t))^3 \right] \right] = -K_n \cos(t\Theta). \quad (3) \end{aligned}$$

In Eq. (1), the nonlinear coefficient β is defined as positive, while hardening and softening behaviors are represented by the sign $\pm$. Such a convention is used consistently in the algebraic development.

Hassan [100] demonstrated that, in the third superharmonic regime of the Duffing oscillator, at least a second-order perturbation expansion is required to properly capture nonlinear harmonic interactions and yield quantitatively reliable steady-state predictions. This requirement arises because, when only a first-order perturbation is considered, the fundamental harmonic acts as an ideal energy source for the third-order superharmonic, such that the amplitude of the superharmonic does not influence the fundamental component of the response, leading to inaccurate predictions when this assumption is no longer valid. In light of this result, the

solution of Eq. (3) is expanded up to the second order as follows:

$$u_n(t) = u_n^0(T_0, T_1, T_2) + \varepsilon u_n^1(T_0, T_1, T_2) + \varepsilon^2 u_n^2(T_0, T_1, T_2) + \mathcal{O}(\varepsilon^3), \quad (4)$$

where $T_j = \varepsilon^j t$, for $j = 0, 1, 2$, with secular terms up to ε^2 . Owing to the mathematical ansatz in Eq. (4), a hierarchy of linear equations in powers of ε is obtained, as follows³:

$$\mathcal{O}(\varepsilon^0): \frac{\partial^2 u_n^0}{\partial T_0^2} + u_n^0 = -K_n \cos(T_0 \Theta), \quad (5)$$

$$\begin{aligned} \mathcal{O}(\varepsilon^1): \frac{\partial^2 u_n^1}{\partial T_0^2} + u_n^1 = & - \left(\pm \left(u_n^0 \right)^3 + 2\zeta_{n\varepsilon} \frac{\partial u_n^0}{\partial T_0} + 2 \frac{\partial^2 u_n^0}{\partial T_1 \partial T_0} + u_n^0 \mu(n) \right. \\ & \left. + \sum_{k=1}^{\alpha} \chi_l^{(k)} \left(2u_n^0 - u_{n+k}^0 - u_{n-k}^0 \right) \right), \end{aligned} \quad (6)$$

and

$$\begin{aligned} \mathcal{O}(\varepsilon^2): \frac{\partial^2 u_n^2}{\partial T_0^2} + u_n^2 = & - \left(\pm 3 \left(u_n^0 \right)^2 u_n^1 + 2\zeta_{n\varepsilon} \frac{\partial u_n^0}{\partial T_1} + 2\zeta_{n\varepsilon} \frac{\partial u_n^1}{\partial T_0} \right. \\ & + u_n^1 \mu(n) + \frac{\partial^2 u_n^0}{\partial T_1^2} + 2 \frac{\partial^2 u_n^0}{\partial T_2 \partial T_0} + 2 \frac{\partial^2 u_n^1}{\partial T_1 \partial T_0} \\ & + \sum_{k=1}^{\alpha} \chi_l^{(k)} \left(2u_n^1 - u_{n+k}^1 - u_{n-k}^1 \right) \\ & \left. + \sum_{k=1}^{\alpha} \chi_{nl}^{(k)} \left[\left(u_n^0 - u_{n+k}^0 \right)^3 + \left(u_n^0 - u_{n-k}^0 \right)^3 \right] \right). \end{aligned} \quad (7)$$

A distinguishing feature from the primary resonance case is that, unlike the situation in which the variable K_n is perturbed, Eq. (5) admits a solution composed of two terms, formulated as:

$$u_n^0(T_0, T_1, T_2) = \Lambda_n e^{iT_0 \Theta} + A_n(T_1, T_2) e^{iT_0} + \text{c.c.}, \quad (8)$$

³ For notational compactness, $u_n^0(T_0, T_1, T_2) = u_n^0$ and $u_n^1(T_0, T_1, T_2) = u_n^1$.

where c.c. denotes the complex conjugate and Λ_n is defined as

$$\Lambda_n = \frac{K_n}{2(\Theta^2 - 1)}. \quad (9)$$

Given the focus on superharmonic resonance, the nondimensional excitation frequency Θ is defined in terms of a tuning parameter σ , according to:

$$\Theta = \frac{1}{3} (1 + \varepsilon \sigma). \quad (10)$$

Equation (10) constitutes the starting point for the subsequent steps. It specifies the harmonic of interest, from which the analysis can be extended to higher-order subharmonic or superharmonic cases. Although the present study focuses on the formulation given in Eq. (10), the mathematical treatment can be readily generalized to other particular configurations.

2.1 First-Order Secular Terms

Upon substitution of Eqs. (8), (9), and (10) into the first-order Eq. (6), the first-order resonant terms, proportional to e^{iT_0} , arise and must be canceled. Accordingly, the following Equation must be satisfied⁴:

$$\begin{aligned} & \pm \Lambda_n^3 e^{iT_1 \sigma} \pm 3 \Lambda_n |A_n|^2 \\ & + \left(\pm 6 \Lambda_n^2 + 2i \zeta_{n\varepsilon} + \mu(n) \right) A_n + 2i \frac{\partial A_n}{\partial T_1} \\ & + \sum_{l=1}^{\alpha} \chi_l^{(l)} (2A_n - A_{n+l} - A_{n-l}) = 0. \end{aligned} \quad (11)$$

Equation (11) corresponds to the Discrete NLS, widely reported in the literature [101–104] and still the subject of ongoing research [105]. The resulting mathematical framework already enables the use of superharmonic ILMs as mechanisms for controlling energy localization in the structure shown in Fig. 1. However, unlike solitons derived under primary resonance conditions, higher-order powers of ε should be considered for a more accurate amplitude quantification [100].

Developing the derivation further and disregarding the continuity hypothesis, one verifies that, once the first-order secular terms are removed, Equation (6) yields the following solution:

$$\begin{aligned} u_n^1(T_0, T_1, T_2) = & \frac{i(5T_0 - T_1 \sigma)}{3} \frac{i(5T_0 + 2T_1 \sigma)}{3} \\ & \pm \frac{3\Lambda_n A_n^2 e}{(\Theta - 1)(\Theta - 3)} \pm \frac{3\Lambda_n^2 A_n e}{4\Theta(\Theta + 1)} \end{aligned}$$

⁴ For notational compactness, the time dependence is omitted.

$$\begin{aligned}
& \pm \frac{3\Lambda_n^2 A_n e}{4\Theta(\Theta-1)} \pm \frac{3\Lambda_n A_n^2 e}{(\Theta+1)(\Theta+3)} \\
& \pm \frac{A_n^3 e^{3iT_0}}{8} + \left(\frac{2i\Theta\Lambda_n\zeta_{n\varepsilon}}{\Theta^2-1} + \frac{\Lambda_n\mu(n)}{\Theta^2-1} \right) \\
& \pm \frac{3\Lambda_n^3}{\Theta^2-1} \pm \frac{6\Lambda_n|A_n|^2}{\Theta^2-1} \Big) e^{\frac{i(T_0+T_1)\sigma}{3}} + \text{c.c.}, \quad (12)
\end{aligned}$$

which, together with Eq. (8), constitutes the solution proposed in Eq. (4). A further parallel with primary resonance concerns the amplitudes involved in the construction of the solution. As shown, both A_n and Λ_n must be evaluated in steady state, although the soliton is associated with the localization of A_n .

2.2 Second-order secular terms and restitution equation

In analogy with the derivation of the ε -order secular terms in Eq. (11), the substitution of (12) into the ε^2 -order problem in (7) leads to additional resonant terms, expressed as:

$$\begin{aligned}
& \mathcal{L}_1 A_n^2 e^{-iT_1\sigma} + \left(\mathcal{L}_2(n) + \mathcal{L}_3 |A_n|^2 \right) e^{iT_1\sigma} \\
& + \left(\mathcal{L}_4(n) + \mathcal{L}_5(n) |A_n|^2 - \frac{15|A_n|^4}{8} \right) A_n \\
& + 2i \frac{\partial A_n}{\partial T_2} + \mathcal{F}_{\chi_l^{(k)}} + \mathcal{F}_{\chi_l^{(k)}\chi_l^{(j)}} + \mathcal{F}_{\chi_{nl}^{(k)}} = 0, \quad (13)
\end{aligned}$$

where the definitions of the variables $\mathcal{L}_i$ ($i = 1, 2, 3, 4$ and 5) and the coupling functions $\mathcal{F}_{\chi_l^{(k)}}$, $\mathcal{F}_{\chi_l^{(k)}\chi_l^{(j)}}$, and $\mathcal{F}_{\chi_{nl}^{(k)}}$ are detailed in Table 3.

In contrast to the equation presented in (11), the resonant terms of Eq. (13) do not correspond to a previously reported formulation in the literature and therefore lack established stability diagrams. Precisely, due to the presence of contributions from distinct temporal scales in $u_n(t)$, the emergence of secular terms across these scales requires the formulation of a restitution equation in the nondimensional time t , as expressed in the following equation:

$$\frac{dA_n}{dt} = \frac{\partial A_n}{\partial T_0} + \varepsilon \frac{\partial A_n}{\partial T_1} + \varepsilon^2 \frac{\partial A_n}{\partial T_2}. \quad (14)$$

Through the transformation $A_n(t)e^{i\varepsilon\sigma} \rightarrow A_n(t)$, which removes the oscillatory character of Eq. (14), one obtains the steady-state governing Equation as:

$$\begin{aligned}
& \varepsilon^2 \mathcal{L}_1 A_n^2 + \varepsilon^2 \mathcal{L}_2 \pm \varepsilon \Lambda_n^3 + \varepsilon^2 \mathcal{L}_3 |A_n|^2 \\
& + \left(\pm 6\varepsilon \Lambda_n^2 + 2i\varepsilon \zeta_{n\varepsilon} + \varepsilon \mu(n) - 2\varepsilon\sigma + \varepsilon^2 \mathcal{L}_4 \right) A_n \\
& + \left(\pm 3\varepsilon + \varepsilon^2 \mathcal{L}_5 \right) |A_n|^2 A_n - \frac{15\varepsilon^2 |A_n|^4 A_n}{8}
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \sum_{k=1}^{\alpha} \chi_l^{(k)} (2A_n - A_{n+k} - A_{n-k}) \\
& + \varepsilon^2 \mathcal{F}_{\chi_l^{(k)}} + \varepsilon^2 \mathcal{F}_{\chi_l^{(k)}\chi_l^{(j)}} + \varepsilon^2 \mathcal{F}_{\chi_{nl}^{(k)}} + 2i \frac{d}{dt} A_n = 0, \quad (15)
\end{aligned}$$

where the previously defined coupling functions remain unchanged. Additionally, A_n in Eq. (13) depends on the time scales T_1 and T_2 , while in Eq. (15) it depends only on the reconstituted time t .

Not being inherently bound to the system depicted in Fig. 1, Eq. (1) retains a general, architecture-independent character (e.g., pendulums [52], linearly coupled masses [54], or circular structures [53]). This character is likewise inherited by Eqs. (11) and (15), with their respective limitations. For a one-degree-of-freedom system, Hassan [100] shows that in the first-order approximation, Eq. (11), the excitation acts as an ideal energy source for the superharmonic component, which restricts its validity to single-valued superharmonic responses. The second-order approximation, Eq. (15), remains valid for multi-valued responses, but underestimates the response as the excitation amplitude increases. Hence, to preserve physical consistency, the theoretical soliton predictions associated with Eq. (15) are numerically validated using Eq. (3) (see Fig. 14 in Appendix B).

From a mathematical standpoint, nontrivial aspects can be identified. The first deals with the type of nonlinearity present in Eq. (15). Although it originates from a Duffing oscillator with cubic nonlinearity, Eq. (15) also exhibits quadratic and quintic nonlinearities, which emerge from the secular terms described by Eq. (13). Another observation pertains to the influence of Λ_n . Qualitatively, since it depends on the external excitation K_n in Eq. (9), it is natural to interpret it as an external forcing term in Eq. (11) [70]. The nontrivial aspect lies in the role played by Λ in conjunction with the periodicity breaking $\mu(n)$. Such a mathematical consequence does not arise under primary excitation, where the periodicity breaking in the natural frequency does not couple to the applied excitation.

A major difference between the energy localization around ω_0 when compared to the superharmonic energy localization concerns the importance of the phase associated with A_n . Even when considering the approximation $u_n(t) \approx u_n^0(T_0, T_1, T_2)$ [97], Eq. (8) presents terms whose direct algebraic summation cannot be performed. The amplitudes Λ_n and A_n must be considered jointly in the composition of the amplitude of $u_n(t)$, although the soliton Eq. (11) addresses only A_n .

2.3 Continuity hypothesis

The formulation presented in Eq. (15) is established within a discrete spatial framework. This consideration provides a more general modeling approach than strategies based on the continuity hypothesis. At the same time, by imposing approximations associated with the oscillator spacing Δs ,

theoretical results previously reported in the literature can be recovered. Let us begin by assuming the following approximation along the space $s = n\Delta s$:

$$A_n \rightarrow A(s, t), \text{ and } A_{n\pm 1} \rightarrow A(s \pm \Delta s, t). \quad (16)$$

The adopted approximation allows the use of the following expansion:

$$A_{n\pm 1} \approx A \pm \Delta s \frac{\partial A}{\partial s} + \frac{\Delta s^2}{2} \frac{\partial^2 A}{\partial s^2} + \mathcal{O}(\Delta s^4), \quad (17)$$

which results in the transformation of the operator $\Delta_1\{X\}$ (see Table 3) into an operation that maps a discrete function to its corresponding second-order spatial derivative:

$$\Delta_1\{X\} = -\frac{\Delta s^2}{2} \frac{\partial^2 X}{\partial s^2} + \mathcal{O}(\Delta s^4), \quad (18)$$

where X denotes a general function, such that:

$$(2A_n - A_{n+1} - A_{n-1}) \rightarrow -\frac{\Delta s^2}{2} \frac{\partial^2 A(s, T_1)}{\partial s^2}. \quad (19)$$

For the sake of theoretical simplification, the couplings between nonadjacent oscillators ($\alpha = 1$, $\chi_l^{(k)} = \chi_l$), nonlinear couplings ($\chi_{nl}^{(k)} = 0$), and periodicity-breaking effects ($\mu(n) = 0$, with indices omitted) are neglected. Under these conditions, upon applying the relations in Eqs. (16)–(18) to Eq. (15), while neglecting terms of order $\mathcal{O}(\Delta s^4)$, the resulting continuous-domain soliton is expressed as:

$|\Phi|^2 \frac{\partial^2 \Phi}{\partial s^2}$ and $\Phi^2 \frac{\partial^2 \bar{\Phi}}{\partial s^2}$ arise in high-order envelope equations for gravity waves [107–109], in ultrashort-pulse propagation in optical fibers [110, 111], in the continuum limit of the discrete Heisenberg spin chain [112], and in electromagnetic metamaterials and nonlinear transmission lines [113]. As here, these terms emerge from the asymptotic reconstitution, not as a constitutive property of the medium. Sedletsky and Gandzha obtained them for the nonlinear Klein-Gordon model by three independent routes: multiple scales, an extended averaged-Lagrangian method, and Routhian reduction [114–116]. In line with Sedletsky and Gandzha, once the expansion is carried one order beyond the cubic NLSE, the dispersion relation acquires an intensity-dependent correction that generates $|\Phi|^4\Phi$, while the even powers of the potential excite second and zeroth harmonics that both produce quadratic terms in the envelope equation and renormalize the cubic and derivative coefficients.

One may note that, even when Eq. (20) is approximated at first order in ε , the parameters composing the equation remain coupled. The external excitation, described by the variable Λ , multiplies the function Φ , and its effects are therefore balanced by the variable σ . Although physical parameter combinations capable of stabilizing the first-order equation in ε theoretically exist, large values of Λ compromise the predictive capability of the model, thus requiring the inclusion of the additional terms.

$$\begin{aligned} & 2i \frac{\partial \Phi}{\partial t} - \varepsilon \left[\underbrace{\pm \Lambda^3 \pm 6\Lambda^2\Phi - 2\sigma\Phi - 2i\zeta_\varepsilon\Phi \pm 3|\Phi|^2\Phi - \chi_l \frac{\Delta s^2}{2} \frac{\partial^2 \Phi}{\partial s^2}}_{\text{Classical Damped Externally Driven Nonlinear Schrödinger Equation}} \right] \\ & - \varepsilon^2 \left[\underbrace{\bar{\mathcal{L}}_1\Phi^2 + \bar{\mathcal{L}}_2 + \bar{\mathcal{L}}_3|\Phi|^2 + \bar{\mathcal{L}}_4\Phi + \bar{\mathcal{L}}_5|\Phi|^2\Phi - \frac{15|\Phi|^4\Phi}{8}}_{\text{Newly added terms}} \right] \\ & + \underbrace{\chi_l^2 \frac{3\Delta s^2}{4} \frac{\partial^2 \Phi}{\partial s^2} \pm 3\chi_l \Delta s^2 \left[\Lambda^2 \frac{\partial^2 \Phi}{\partial s^2} - \frac{1}{4}\Phi^2 \frac{\partial^2 \bar{\Phi}}{\partial s^2} + \frac{1}{4} \frac{\partial^2 [\Phi|\Phi|^2]}{\partial s^2} + \frac{1}{2} |\Phi|^2 \frac{\partial^2 \Phi}{\partial s^2} \right]}_{\text{Newly added terms}} \Bigg] = 0, \end{aligned} \quad (20)$$

where $\Phi = \overline{A(s, t)}$.

Equation (20) is composed of terms classified according to the order of the perturbation factor ε . At first order, a classical formulation of the NLS is obtained, whose stability has been established for decades, both for bounded spatial domains [106] and infinite spatial domains [70]. However, when attention is directed to terms proportional to ε^2 , the stability of the phenomenon becomes unreported, particularly due to the emergence of fifth-order nonlinearities of the form $|\Phi|^4\Phi$.

Terms of this kind are not exclusive to the present system. The quintic term $|\Phi|^4\Phi$ and the nonlinear dispersive terms

To investigate this premise, Figs. 2 and 3 are presented for the hardening and softening cases, respectively. The comparison is performed using the Root Mean Squared Error ($RMSE$) metric, computed for both the first-order perturbation approximation ($RMSE_\varepsilon$) and the second-order perturbation approximation ($RMSE_{\varepsilon^2}$). The results show that both $RMSE_{\varepsilon^2}$ and $RMSE_\varepsilon$ increase with excitation intensity for the hardening and softening regimes, indicating that the accuracy of the estimates improves under weaker excitation conditions. Moreover, $RMSE_\varepsilon$ exceeds $RMSE_{\varepsilon^2}$, demonstrating the greater sensitivity of this metric to changes in F .

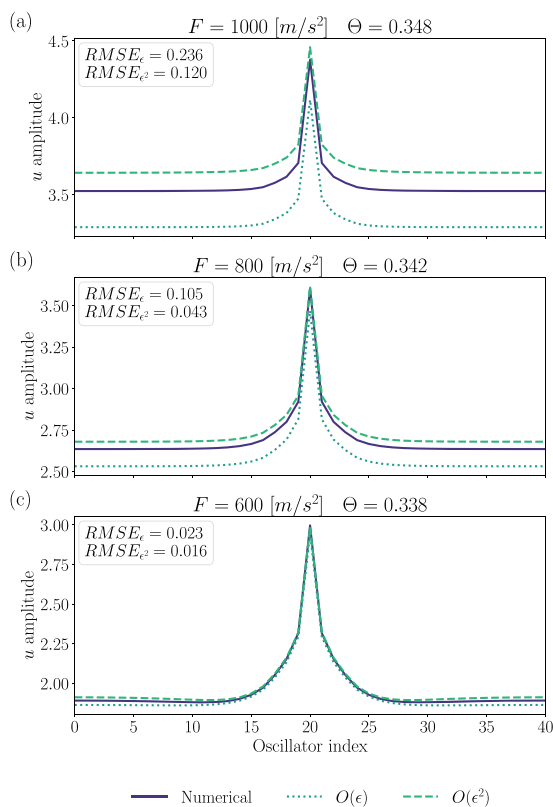


Fig. 2 Comparison between the first- and second-order analytical responses and the numerical solution of Eq. (3), for the hardening case ($\pm = 1$). Panels (a–c) present the comparison for different excitation amplitudes and frequencies. Disorder $f = -0.05 \omega_0^2$

It should be emphasized that the solitons shown in Figs. 2 and 3 are obtained for fixed values of the excitation frequency Θ . Since the $RMSE$ varies across the frequency domain, the accuracy of the first-order approximation in ϵ depends on the excitation condition. Consequently, although the approximation may accurately describe the lattice response at specific frequencies, its validity cannot be generalized to the entire excitation range. In this context, the numerical sweeps presented hereafter incorporate all terms of Eq. (15), rather than being restricted to those associated with the classical NLS.

This section develops the theoretical features inherent to discrete solitons associated with superharmonic resonance. Through the method of multiple scales, the role of phase in the composition of the response is demonstrated, as the steady-state response involves multiple interacting frequencies. Although the NLS is associated with the first order of perturbation ϵ , the use of higher-order perturbations leads to relations whose stability has not been reported in the literature. Given the absence of closed-form solutions for Eq. (20), the results of this study are obtained numerically through integration of Eq. (15). Furthermore, owing to the difficulty of stabilizing the periodic case, disorder is introduced as a mechanism for soliton nucleation, in agreement with previ-

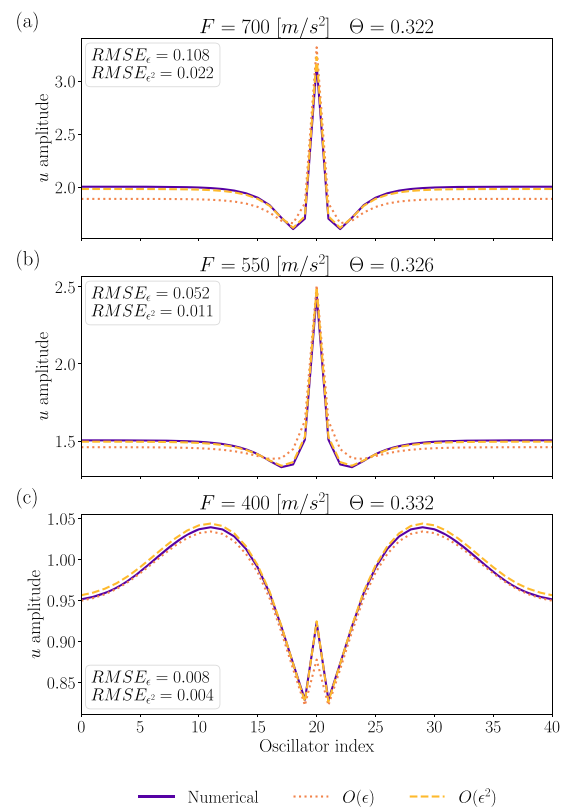


Fig. 3 Comparison between the first- and second-order analytical responses and the numerical solution of Eq. (3), for the softening case ($\pm = -1$). Panels (a–c) present the comparison for different excitation amplitudes and frequencies. Disorder $f = -0.05 \omega_0^2$

ous studies reporting an expansion of the stability diagram induced by impurities in nonlinear lattices [43, 71].

3 Duffing oscillator frequency response in superharmonic regime

This section analyzes the dynamic behavior of the Duffing oscillator in the superharmonic regime, considered individually. The purpose of this study is to establish a basis for comparison between individual and collective dynamics, thereby enabling the identification of energy flow along the oscillators. The numerical sweeps are performed in both the hardening and softening regimes, under the assumptions of base acceleration and linear damping. It should be noted that alternative strategies for extracting the steady-state response of the single-degree of freedom (1-DOF) system may be employed [36]. However, the method of multiple scales is adopted to maintain consistency with the soliton derivation.

The mathematical description of the 1-DOF system is obtained by suppressing the coupling and disorder terms in Eq. (3), consequently yielding the decoupled form of the system. This formulation is equivalent to the response of an

isolated oscillator, for which the following polar representation is adopted:

$$A_n(t) = \frac{a_n(t) e^{-i\gamma_n(t)}}{2}, \quad (21)$$

with $a_n(t)$ and $\gamma_n(t)$ being the real variables associated with the amplitude and phase of $A_n(t)$, respectively. By introducing the polar representation into the restitution equation (15), while separating the real and imaginary components and imposing the steady-state condition, two equations are obtained:

$$\sin(\gamma_n) = \frac{ZX - VW - VY + Xa_n \varepsilon \sigma}{W^2 + X^2 - Y^2}, \quad (22)$$

and

$$\cos(\gamma_n) = \frac{-VX - (Z + a_n \varepsilon \sigma)(W - Y)}{W^2 + X^2 - Y^2}, \quad (23)$$

where the variables Z , X , V , W , and Y are defined in Appendix A, Table 4. From the algebraic manipulation of Eqs. (22) and (23),⁵ an algebraic equation for the amplitude a_n of the 1-DOF system is obtained and can be solved. This amplitude does not correspond to the oscillator displacement, but rather to the components associated with 3Θ . The analytical solution for $u_n(t)$ follows from substituting Eqs. (8) and (12) into (4), together with the polar representation of Eq. (21), yielding⁶:

$$\begin{aligned} u_n(t) = & a_n \cos(3t\Theta - \gamma_n) + 2\Lambda_n \cos(t\Theta) \\ & + \varepsilon \left(\left(\frac{2\Lambda_n \mu(n)}{\Theta^2 - 1} \pm \frac{3\Lambda_n a_n^2}{\Theta^2 - 1} \pm \frac{3\Lambda_n^2 a_n \cos(\gamma_n)}{4\Theta(\Theta - 1)} \right. \right. \\ & \left. \left. \pm \frac{6\Lambda_n^3}{\Theta^2 - 1} \right) \cos(t\Theta) + \left(\pm \frac{3\Lambda_n^2 a_n \sin(\gamma_n)}{4\Theta(\Theta - 1)} \right. \right. \\ & \left. \left. - \frac{4\Theta \Lambda_n \zeta_n \varepsilon}{\Theta^2 - 1} \right) \sin(t\Theta) \right) + \text{NRT}, \end{aligned} \quad (24)$$

where NRT refers to the nonresonant terms, which are neglected in the numerical simulations.

A direct consequence of the superposition of cosine and sine terms with different frequencies is the requirement to evaluate the steady-state amplitude based on the individual contributions of its components. In other words, the amplitude associated with u_n does not coincide with that obtained from the secular terms presented in Eq. (15), since these describe the variable A_n , whose amplitude is defined by a_n in Eq. (21). To illustrate this issue, the simulation presented

in Fig. 4 is examined. Panels (a, b, and c) display the hardening behavior, whereas panels (d, e, and f) correspond to the softening condition. The figure is organized into three columns: the first presents the amplitude of u_n , the second the amplitude of A_n (a_n), and the third the phase of A_n (γ). The results highlight the distinct shapes of the amplitude curves while preserving a comparable order of magnitude, thereby reinforcing the inadequacy of an isolated analysis. As the analytical formulation is derived using the method of multiple scales, a numerical validation is superimposed on the amplitude curves of u_n (black square markers), confirming the validity of the solution in Eq. (24).

To further examine the validity of the analytical solution, Eq. (3) is simplified by the following transformation:

$$\sqrt{\varepsilon} u(t) \rightarrow v(t), \quad (25)$$

where the 1-DOF case is considered and ε is reincorporated into the parameters, so that:

$$\frac{d^2 v(t)}{dt^2} + 2\zeta \frac{dv(t)}{dt} + v(t) + v^3(t) = -\hat{K} \cos(t\Theta), \quad (26)$$

where $\hat{K} = F\sqrt{\beta}/\omega_0^3$. For numerical simplification, only the hardening case is considered ($\pm = 1$). Figure 5 shows a parametric analysis in the damping ζ and the normalized excitation force $\hat{K}$, the latter a function of the excitation force F and the nonlinear coefficient β . Analytical frequency responses are obtained for the first-order model (Eq. (11)) and the second-order reconstituted model (Eq. (15)) and compared with numerical sweeps (up and down).

The superimposed frequency response curves on the left (Fig. 5a, b, and c) show agreement between the analytical models at all nonresonant frequencies. In fact, when the solution is single-valued, the first-order approximation accurately captures the system response, whereas once the solution becomes multi-valued, the second-order approximation becomes necessary. It is also worth noting that, for a substantial increase in the excitation acceleration magnitude, the second-order approximation underestimates both the resonance peak amplitude and frequency. The color map on the right, shown in Fig. 5d, illustrates the validity region of each model for an error threshold of 1%, computed according to the mean relative error. Such threshold is adopted because a 1% error, although typically regarded as negligible, is diluted over the entire frequency range, so that the mean error underestimates the local error at resonance. For completeness, the diagrams composing the combined map are individually analyzed in Fig. 5e and f for the first- and second-order approximations, respectively.

In conclusion, this section establishes a reference for analyzing energy flow in oscillators subjected to collective dynamics. With the 1-DOF behavior characterized, abrupt

⁵ $\sin^2(\gamma_n) + \cos^2(\gamma_n) = 1$

⁶ This equation holds for both the 1-DOF and N-DOF cases, with $\mu(n) = 0$ in the 1-DOF case.

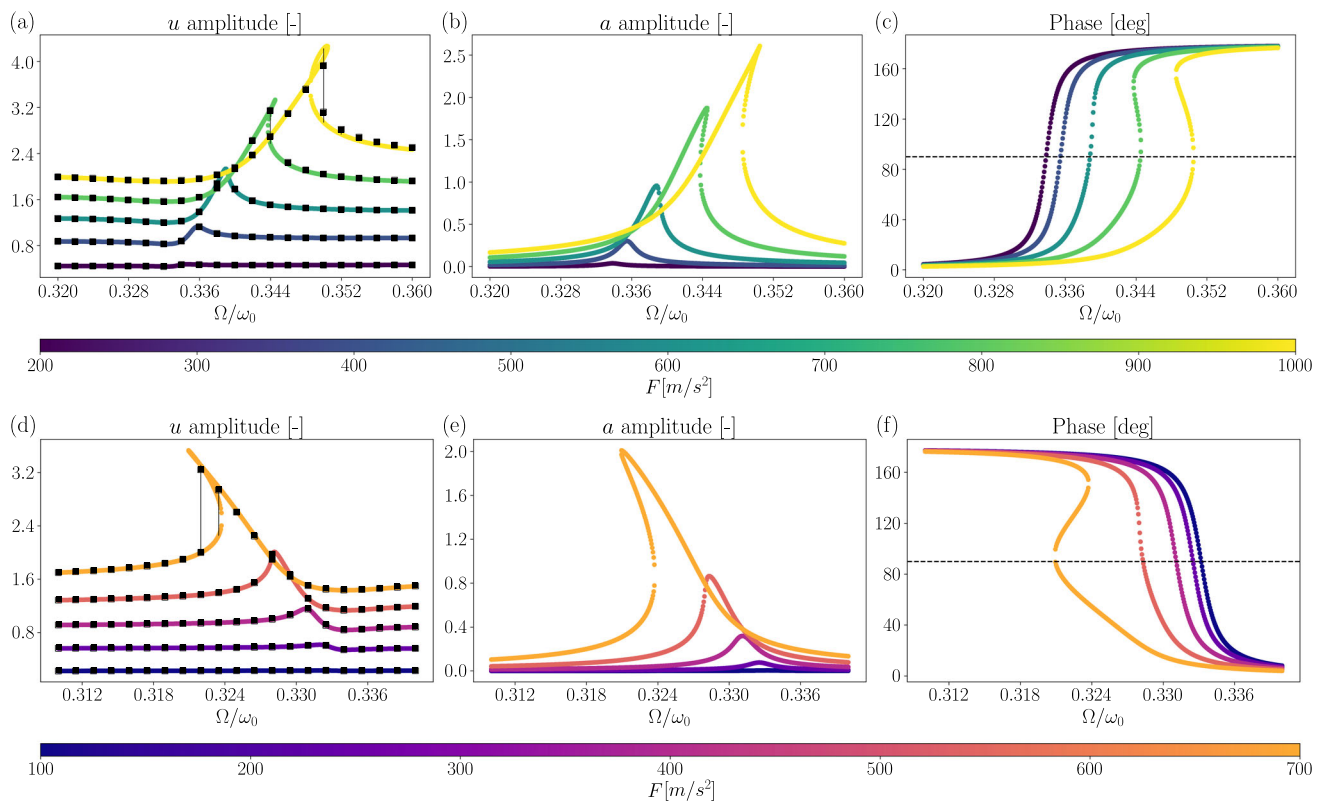


Fig. 4 1-DOF simulation illustrating the limitations of an isolated amplitude analysis, for the hardening (**a–c**) and softening (**d–f**) cases. The three columns show the amplitude of u_n , the amplitude a_n , and the phase γ_n , respectively. Numerical validation is performed by selecting

points on the u_n analytical curve as initial conditions for the numerical integration of Eq. (3), producing the black squares. Excitation values for the softening regime are reduced to achieve numerical convergence

increases in oscillator amplitude can be interpreted in light of the discrete soliton equation. The validity regions shown in Fig. 5 are consistent with Hassan [100]: the first-order model is valid for low excitation amplitudes, where no multi-valued solution exists, and the second-order reconstituted model remains valid up to a point at which it starts to underestimate the response. In contrast to the commonly reported behavior under primary excitation, the oscillator phase plays a critical role in determining the amplitude of u_n , a result supported by numerical validation.

4 Energy nucleation and stationary soliton analysis

This section demonstrates the nucleation of solitons from a set of oscillators. The primary goal is to establish the phenomenon numerically, focusing on three aspects: the influence of disorder, the number of oscillators, and the robustness of the phenomenon against uncertainties in the physical parameters. The analyses are conducted in terms of the amplitudes of u_n (or U_n when considering lattice

dimensions), since the solution for A_n does not fully characterize the steady-state amplitude of the metastructure. It should be noted that Eq. (14) reflects a stationary soliton only when the temporal variation of A_n vanishes. Accordingly, a convergence metric is defined in the present work as follows:

$$\max_n \left| \frac{dA_n}{dt} \right| < 10^{-7} \quad \text{and} \quad \max_n \left| \frac{d^2A_n}{dt^2} \right| < 10^{-8}. \quad (27)$$

It is also emphasized that the simulations may be conducted from motionless oscillators, treated independently, i.e., without numerical continuation, conditions explicitly stated in the figure captions.

The frequency-domain simulations are derived from time-domain integrations. Specifically, to obtain an amplitude profile at a prescribed value of Θ , the governing equations are integrated until the system satisfies Eq. (27); otherwise, the solution is classified as nonconvergent. Figure 18 in Appendix B illustrates this procedure by presenting spatio-temporal simulations of Eq. (15).

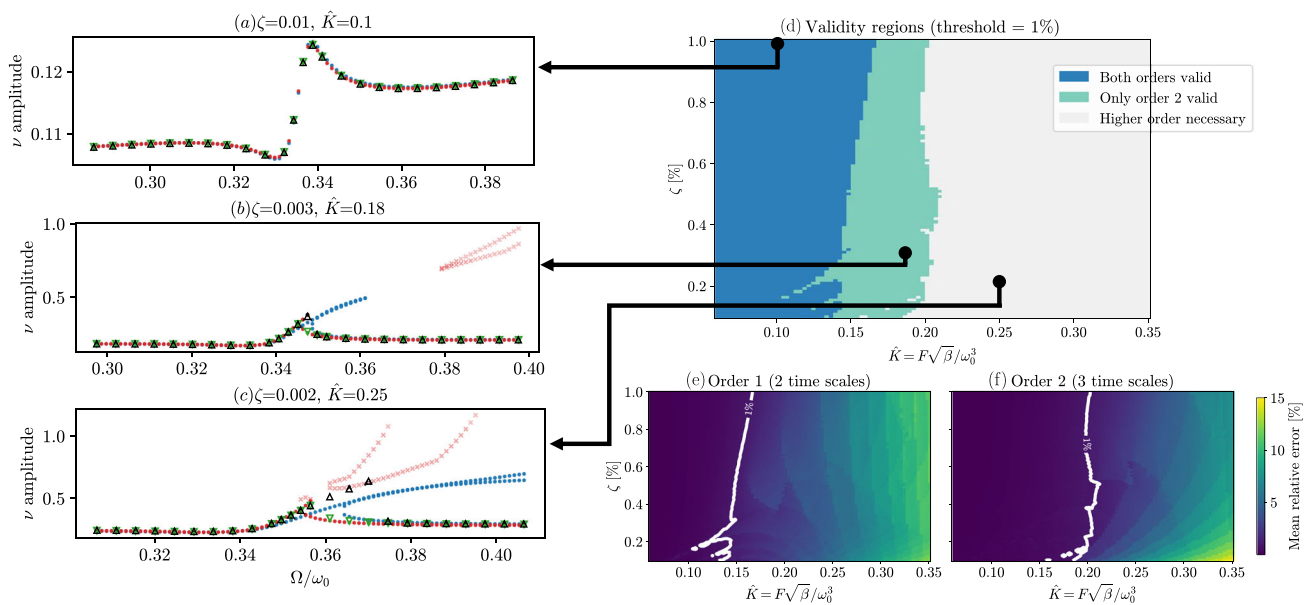


Fig. 5 Parametric analysis of the damped Duffing oscillator in the damping ζ and normalized excitation force $\hat{K} = F\sqrt{\beta}/\omega_0^3$. The validity regions for an error threshold of 1% are shown, distinguishing where both orders are valid, where only the second-order model is valid, and where a higher-order approximation is required, with the mean relative error heatmaps of the first-order (Eq. (11)) and second-order (Eq. (15))

4.1 On the influence of the disorder

It is established in the literature that energy localization in chains of Duffing oscillators can occur in periodic structures with spatially uniform physical properties. Concurrently, disorder is known to enhance the stability of the phenomenon and promote spontaneous nucleation, particularly through the breaking of periodicity toward decreasing natural frequencies of the oscillators. In this context, the verification of such a strategy in superharmonic regimes constitutes a plausible hypothesis, which is investigated in this subsection. Figure 6 shows four numerical sweeps that elucidate such idea: two corresponding to the hardening case (a) and (b), and two to the softening case (c) and (d). Figures (a) and (c) maintain a constant excitation frequency while assessing the effect of impurity on soliton nucleation, whereas Figures (b) and (d) perform the complementary analysis by fixing the disorder and varying the excitation frequency.

An immediate conclusion can be drawn from Fig. 6: energy nucleation can be tuned through an appropriate selection of disorder, which depends on the operating frequency (or bandwidth) of the metastructure. A second conclusion pertains to the nature of the nonlinearity: not only does the optimal level of disorder vary depending on whether a hardening or softening system is considered, but the mechanisms underlying abrupt nucleation also differ in each case. For simulations involving negative nonlinearity, red and cyan regions

models shown below. Three cases are highlighted, exemplifying the validity of each model through the superimposed frequency response curves, comparing the first-order model ($\bullet$), the second-order model ($\bullet$), and numerical sweeps up ($\blacktriangle$) and down ($\blacktriangledown$). Light red dots show spurious solutions, not considered in the error metric

are delimited. These regions do not satisfy the criterion in Eq. (27), indicating that A_n does not converge to a stable profile. To characterize the dynamical behavior of such regions, the Largest Lyapunov Exponent (LLE) is computed following the approach proposed in by Dabrowski [117], and applied in [42]. A positive LLE is a necessary, but not sufficient, condition for chaos. Accordingly, non-convergent orbits with an LLE close to zero are classified as periodic or quasi-periodic (cyan; Fig. 18 d–f in Appendix B), whereas those with a positive LLE are classified as non-periodic (red; Fig. 18 g–i in Appendix B).

By revisiting the secular terms associated with the first power of ε in Eq. (11), the existence of localized nonlinear modes in periodic chains is readily inferred, as the NLS emerges at this order. However, in contrast to the primary resonance case, the soliton parameters are intrinsically correlated. The term equivalent to the external excitation ($(\pm)\Lambda^3 n e^{iT1\sigma}$) depends on Λ , which also multiplies A_n . This observation becomes more pronounced in higher-order secular terms in Eq. (13), where the physical parameters of the lattice exhibit nonlinear coupling, hindering the identification of their isolated effects, as performed in [41] for the fundamental resonance.

From such standpoint, Fig. 7 shows two sets of simulations, (a) corresponding to the hardening regime and (b) to the softening regime, in which system disorder is selectively activated and deactivated over the course of the simulation. In

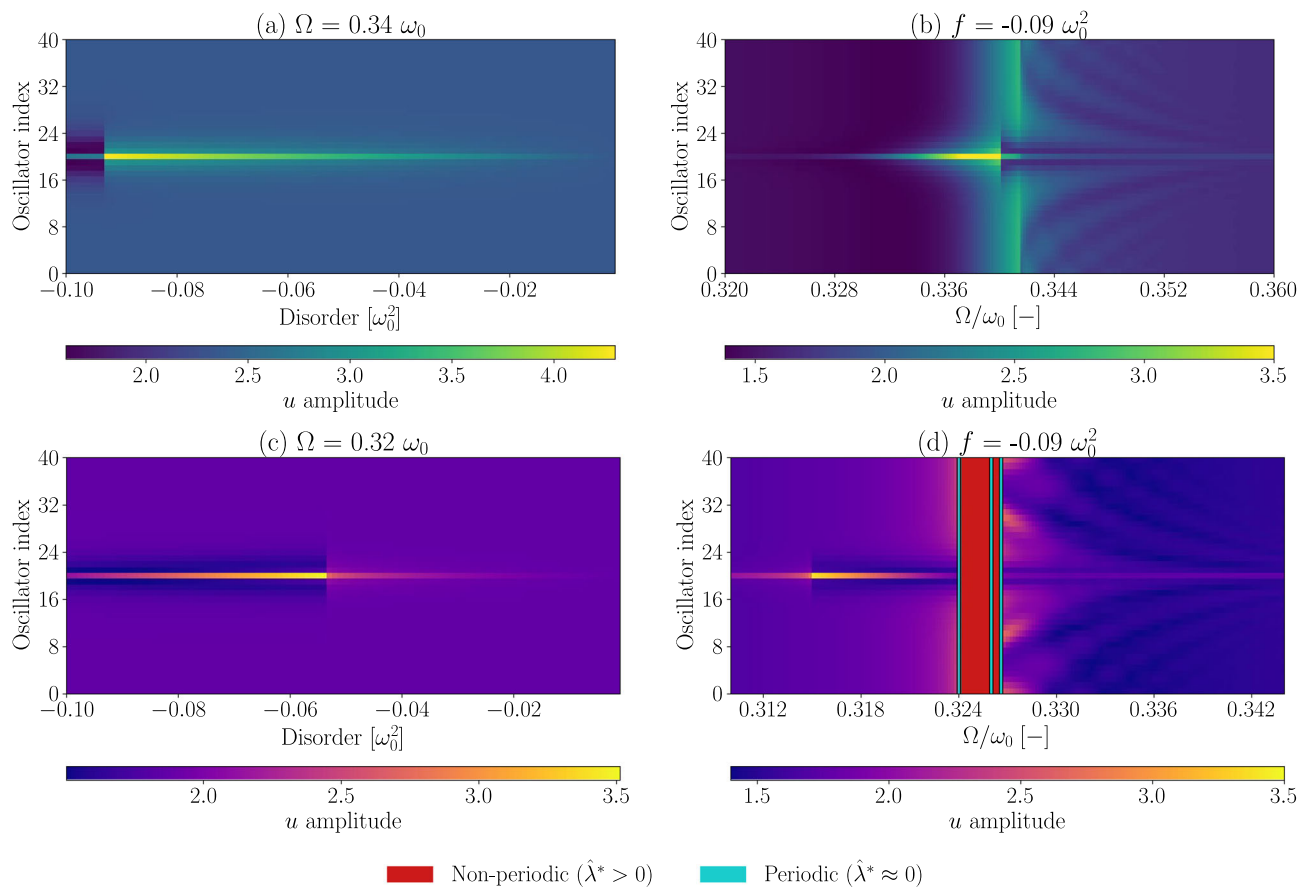


Fig. 6 Numerical sweeps of soliton nucleation for hardening (**a** and **b**) and softening (**c** and **d**) regimes. Panels (**a**, **c**): fixed excitation frequency with varying disorder, (**b**, **d**): fixed disorder with varying frequency.

Numerical continuation is imposed on the frequency domain. Red and cyan regions (**c** and **d**): Nonperiodic and periodic steady-state responses

both cases, the introduction of disorder into the system leads to energy localization, whereas its removal results in delocalization. For clarity, the temporal response of the oscillator at which disorder is introduced is displayed below the heat map. As observed, the resulting solution is not sustained over time. No set of physical parameters yielding localized solutions in the absence of disorder was identified, although this does not imply that such solutions are unattainable. Another relevant design consideration is that, analogously to the primary case, only disorder that reduces the natural frequency was numerically observed to induce soliton nucleation, implying that positive values of $f(n)$ do not promote energy localization (see Fig. 15 in Appendix B).

From a numerical perspective, the nature of the nucleation disorder lies in the fact that reducing the natural frequency enlarges the basin of attraction of high-amplitude solutions at the disordered oscillator, whereas increasing it enlarges that of low-amplitude solutions. From a physical perspective, reducing the natural frequency of selected regions of the chain, while keeping all other physical parameters

unchanged, offers the possibility of higher-energy dynamical states, thereby introducing regions of the chain exhibiting higher-energy/amplitude dynamical states relative to their surroundings. Although nonlinear chains inherently exhibit a similar asymmetry, with higher- and lower-amplitude branches in the frequency domain, it is known that structural disorder further promotes soliton nucleation by enhancing such localized asymmetry state [43, 71]. It is important to distinguish the proposed mechanism from Anderson localization. Under primary resonance, localization may arise from structural disorder, as in linear systems. By contrast, because superharmonic resonance is intrinsically nonlinear, no numerical bridge exists between Anderson localization and superharmonic soliton nucleation.

4.2 On the saturation of the number of oscillators

From the metastructure design perspective, an appropriate selection of the number of DOF is critical, due to the trade-off between the minimum number of oscillators required for the

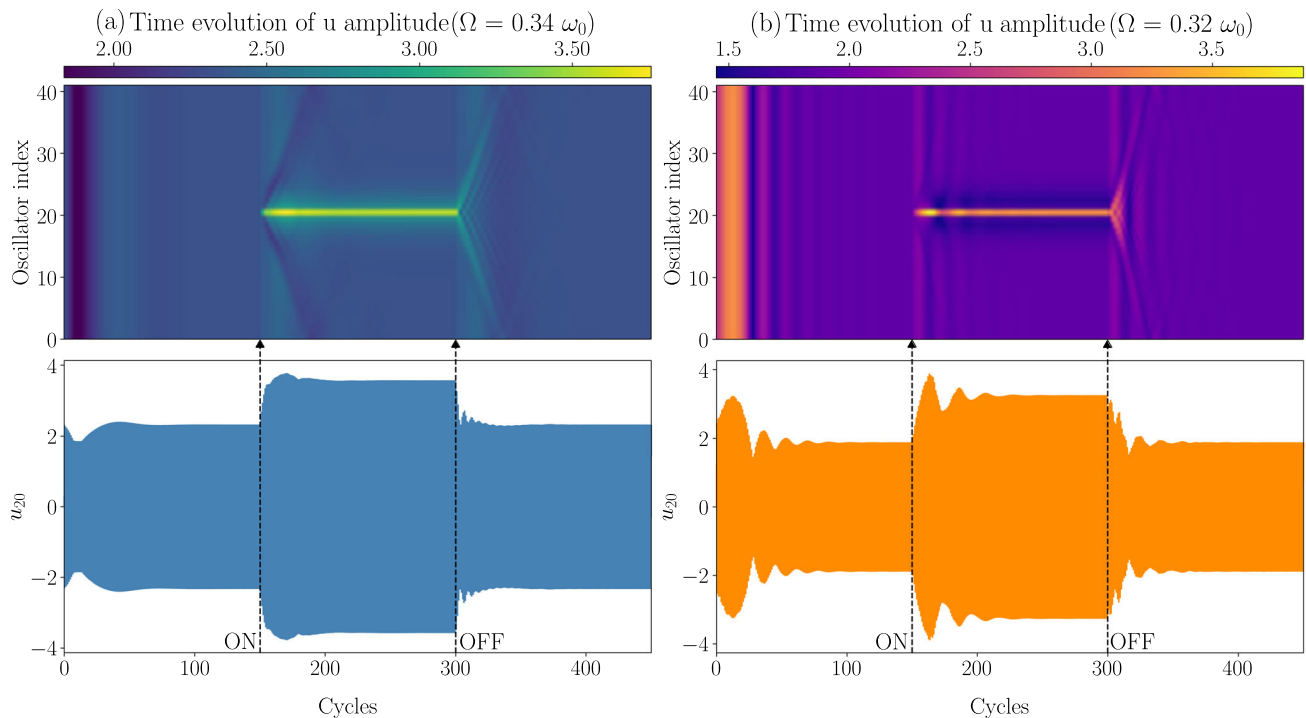


Fig. 7 Simulations with disorder toggled on/off for hardening **(a)** and softening **(b)** regimes, showing induced localization and subsequent delocalization upon removal. The temporal response of the perturbed

oscillator is shown below each heat map, indicating that the localization is not sustained. Disorder $f = -0.06 \omega_0^2$

phenomenon and the maximization of the number of oscillators whose energy migrates toward the soliton core. It should be noted that the admissible range of N is established after fixing the physical parameters of the lattice. As an illustration of this issue, the selection of the number of oscillators can be addressed from the continuum hypothesis standpoint (Limit (16)). Consistent with the spatial nondimensionalization adopted in [41, 43, 73, 74], the allocation of oscillators within the spatial domain governed by the Schrödinger equation depends on the coupling stiffness. In other words, the number of oscillators required for the manifestation of the phenomenon is determined by k_c , a relationship that extends to the perturbation parameter, for instance.

The current subsection investigates the influence of a gradual increase in N as the coupling strength varies. In order to facilitate the analysis, a set of simplifying assumptions is introduced, slightly restricting the generality of Eq. (1). In particular, nonlinear couplings are neglected (as discussed in Appendix C and justified with the parametric analysis in Fig. 20) and the coupling order is limited to $\alpha = 3$, with $C_l^{(2)} = C_l^{(1)}/4$ and $C_l^{(3)} = C_l^{(1)}/9$. In Appendix B (Fig. 16 for the hardening case and Fig. 17 for the softening one), specific cases incorporating nonlinear couplings are presented; however, a detailed analysis of their effects lies beyond the scope of this study.

To quantify the system response, we evaluate the kinetic energy E_c of the oscillator where the disorder is introduced over a full period of motion:

$$E_c = \int_0^{2\pi} \frac{1}{\Theta} \left(\frac{d}{dt} u_n(t) \right)^2 dt = \pi \Theta \left(\Pi_1^2 \varepsilon^2 + \Pi_2^2 \varepsilon^2 + 4\Pi_2 \Lambda_n \varepsilon + 9a_n^2 + 4\Lambda_n^2 \right), \quad (28)$$

where Π_1 and Π_2 are defined in Table 5. Figures 8 (hardening regime) and 9 (softening regime) present simulation panels evaluating the amplitude of $\log(E_c)$ as a function of excitation frequency and the number of oscillators for different coupling strengths. In both regimes, the saturation of N depends on the excitation frequency and on the selected coupling strength. Furthermore, an unbounded increase in the number of oscillators leads to a reduction in the kinetic energy for specific frequency ranges, consistent with the physical expectation that localized modes exhibit preferential formation frequencies. The nonconvergence issue (Relations (27)) for negative nonlinearity values is also governed by the number of oscillators, becoming more pronounced as N increases. Additionally, increasing $C_l^{(1)}$ raises the number of oscillators required to achieve energy saturation, with this effect being more readily observed in the hardening regime. Another relevant aspect concerns the simulation con-

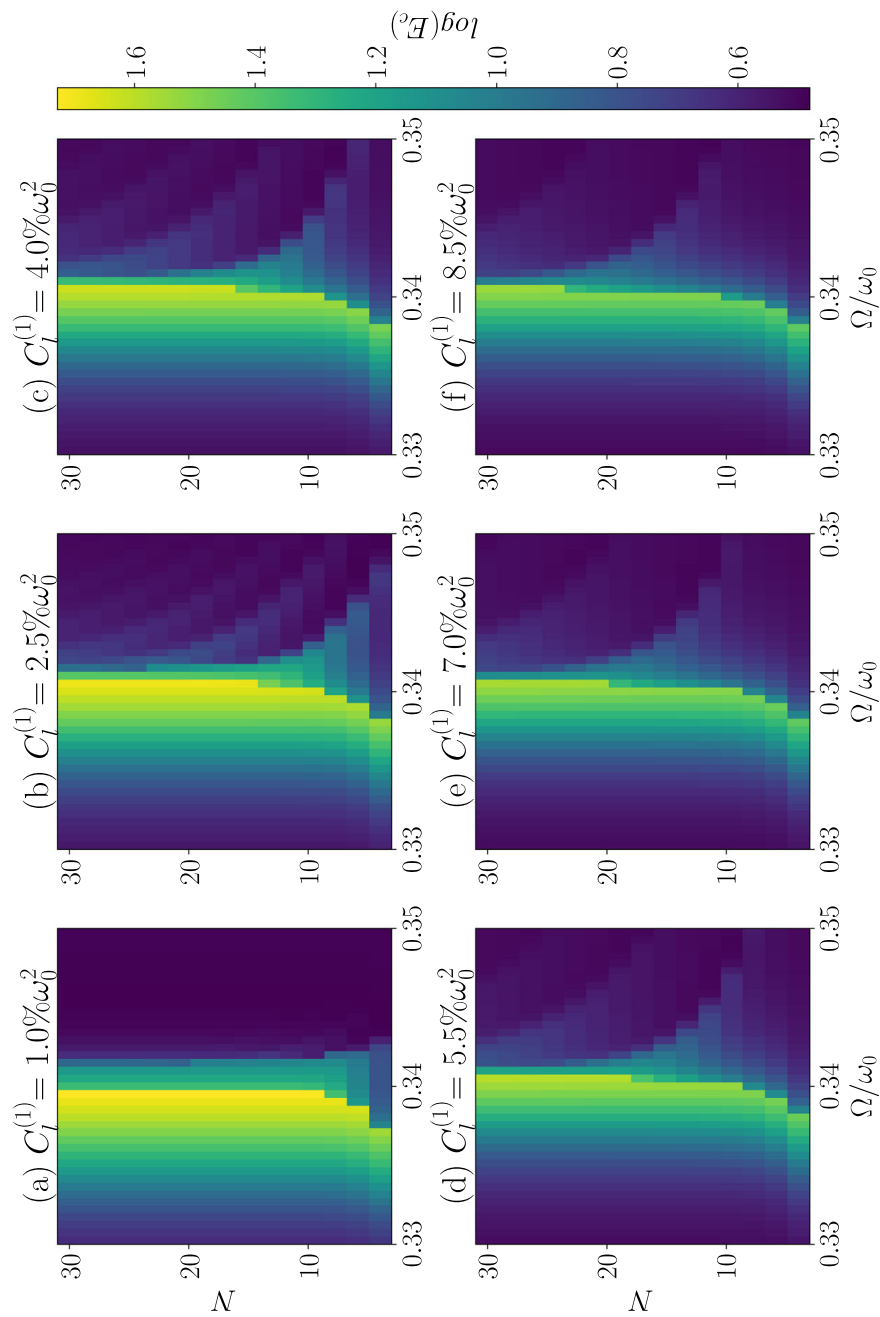


Fig. 8 Parametric sweep conducted to determine the saturation values of the number of oscillators in the hardening regime ($\pm = 1$). Disorder $f = -0.05\omega_0^2$

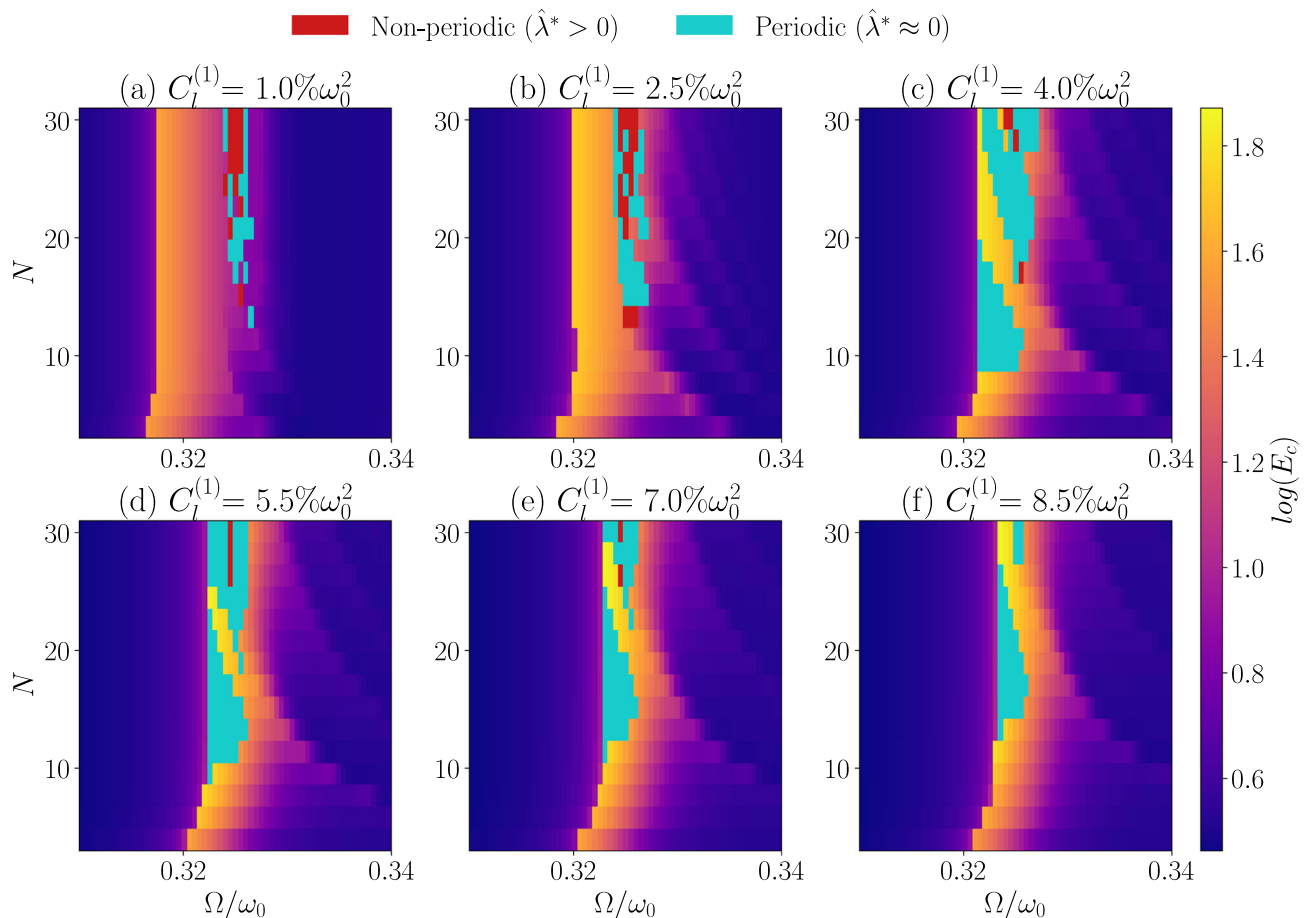


Fig. 9 Parametric sweep conducted to determine the saturation values of the number of oscillators in the softening regime ($\pm = -1$). Disorder $f = -0.05 \omega_0^2$. Red and cyan regions: Nonperiodic and periodic steady-state responses

dition, namely the absence of numerical continuation. In Figs. 6 and 7, the solutions of Eq. (15) are initialized from sufficiently close prior states (either in frequency or time), ensuring convergence within neighboring basins of attraction. In contrast, Figs. 8 and 9 are obtained from initially motionless oscillators, implying spontaneous nucleation; a similar mechanism observed for solitons emerging from primary excitation frequencies. It should be emphasized that, despite the conclusions drawn from simulations represented in Fig. 6 and Fig. 7, alternative coupling configurations may yield different outcomes, particularly under the assumption of numerical continuation.

4.3 On the robustness of the phenomenon to uncertainties in physical parameters

One of the defining characteristics of solitons concerns their stability against waveform dispersion. In fact, when compared with linear defect-induced localizations (Anderson localization, for instance), disorder-nucleated solitons exhibit broader tolerance ranges to uncertainties in the phys-

ical parameters [42], thereby enabling an easier experimental implementation of the phenomenon [74]. This naturally raises the problem of superharmonic nucleation in lattices for which the determinism hypothesis no longer holds.

To address this problem, let us consider that, for an arbitrary variable $\rho(n)$, possible variations around its nominal value $\tilde{\rho}$ are described by:

$$\rho(n) = \tilde{\rho} + \delta_\rho \mathcal{U}(n), \quad (29)$$

where δ_ρ denotes the maximum admissible percentage variation of $\tilde{\rho}$, and $\mathcal{U}(n)$ represents the probability function associated with the spatial occurrence. For the sake of generality, it is assumed to follow a uniform distribution over the interval $[-1, 1]$. In accordance with the lattice description shown in Fig. 1, the parameters subjected to (29) correspond to the damping, natural frequency (by the disorder function), and oscillator excitation, which are evaluated simultaneously.

Figures 10 and 11 present simulations under nondeterministic conditions imposed on Eq. (15) for the hardening and softening cases, respectively. The simulations are conducted

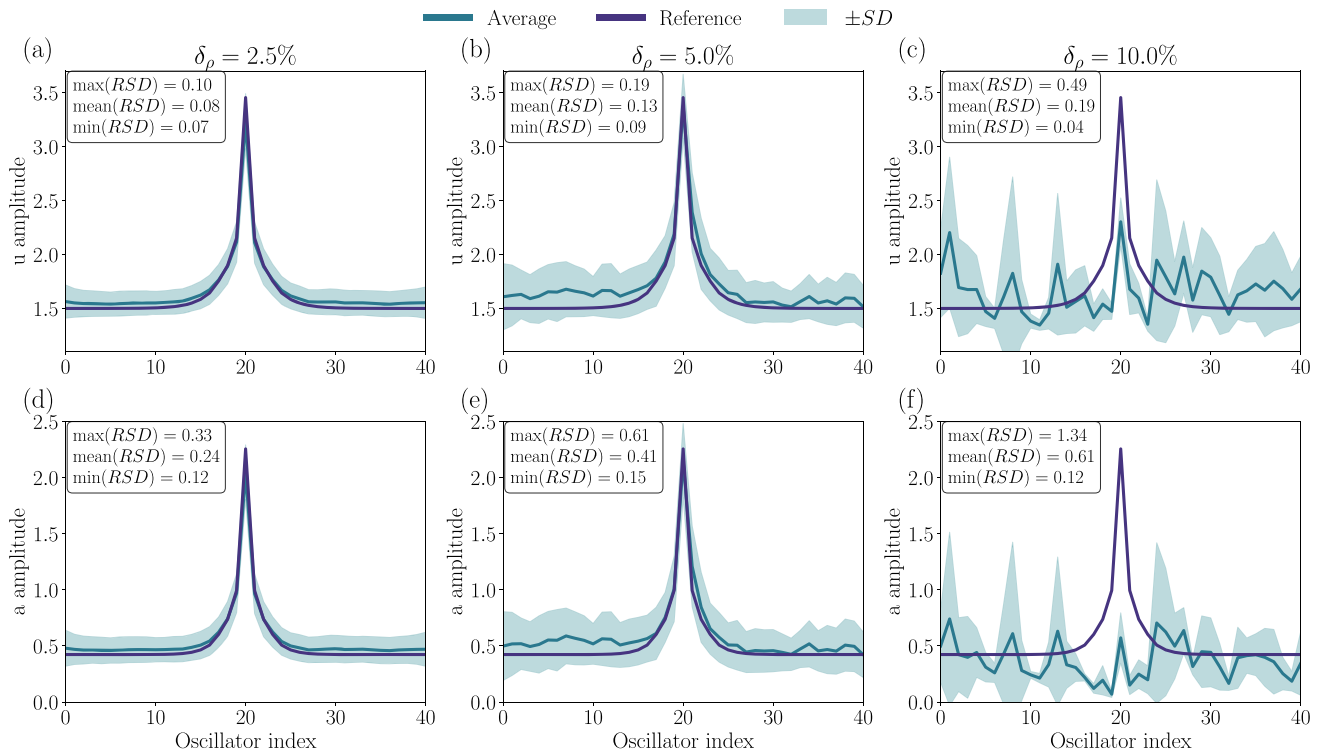


Fig. 10 Simulations under nondeterministic regimes for the hardening case ($\pm = 1$) with $\Theta = 0.336$. The damping, natural frequency, and oscillator excitation are subjected to arbitrary variations described by Eq. (29), whereas the remaining parameters are kept periodic. Vari-

ations in the natural frequency are associated with the introduced disorder, with an intended disorder imposed on the central oscillator ($\tilde{f}(20) = -0.1\omega_0^2$, and $\tilde{f}(n) = 0$ for $n \neq 20$)

under three distinct uncertainty levels, imposed according to the probabilistic law defined by $\mathcal{U}(n)$. For each value of δ_ρ , 1000 samples are generated (see Fig. 19 for convergence), and the corresponding numerical integrations are classified as solitons if they satisfy the convergence criterion defined in (27), while preserving the energy localization. Since Eq. (15) is simulated in the absence of impurities, a reference pattern for the spatial energy distribution can be established. More specifically, to assess the persistence of the phenomenon, the criteria employed in [42] are adopted. To determine whether the solution remains bounded, the oscillation at $n = n_i$ is considered localized when its amplitude satisfies:

$$0.25 \mathcal{REF} \leq |A(n_i, t)| \leq 4 \mathcal{REF}, \quad (30)$$

where $\mathcal{REF}$ denotes the maximum amplitude obtained in the absence of uncertainties. In addition to the amplitude criterion, a spatial criterion is employed. After identifying the peaks of $|A(n, t)|$ (local maxima of the profile), the solution is classified as a soliton when the ratio between the second-highest peak and the highest peak does not exceed 0.5. Despite the practicality of this approach, some considerations are necessary. The verification procedure is based on $A(n, t)$ rather than $u(n, t)$, unlike in [42], since the problem

addressed therein concerned nucleation around the primary harmonic. A further distinction lies in the nature of the non-linearity, which in the present study includes both hardening and softening regimes.

For comparison, Figs. 10 and 11 present the reference curve (without uncertainties), the mean response obtained from the simulations, and a shaded envelope bounded by the standard deviation (SD). As a measure of robustness, the Relative Standard Deviation (RSD) is adopted, defined as the ratio between the standard deviation of the amplitudes and their corresponding mean values. Given the system's multiple degrees of freedom, the maximum, minimum, and mean RSD values are reported in the figures.

Fig. 12 shows a log-logistic fit applied to the nondeterministic samples, describing the evolution of the soliton nucleation probability with the uncertainty level δ_ρ (the corresponding values are summarized in Table 1). Figure 13 shows superimposed violin and box plots of the peak ratio criterion between the oscillator with the highest amplitude and that with the second one, describing the distribution of the sample localizations with respect to the criterion for each δ_ρ .

In line with the simulations presented in this subsection, some conclusions can be drawn. First, regarding the persis-

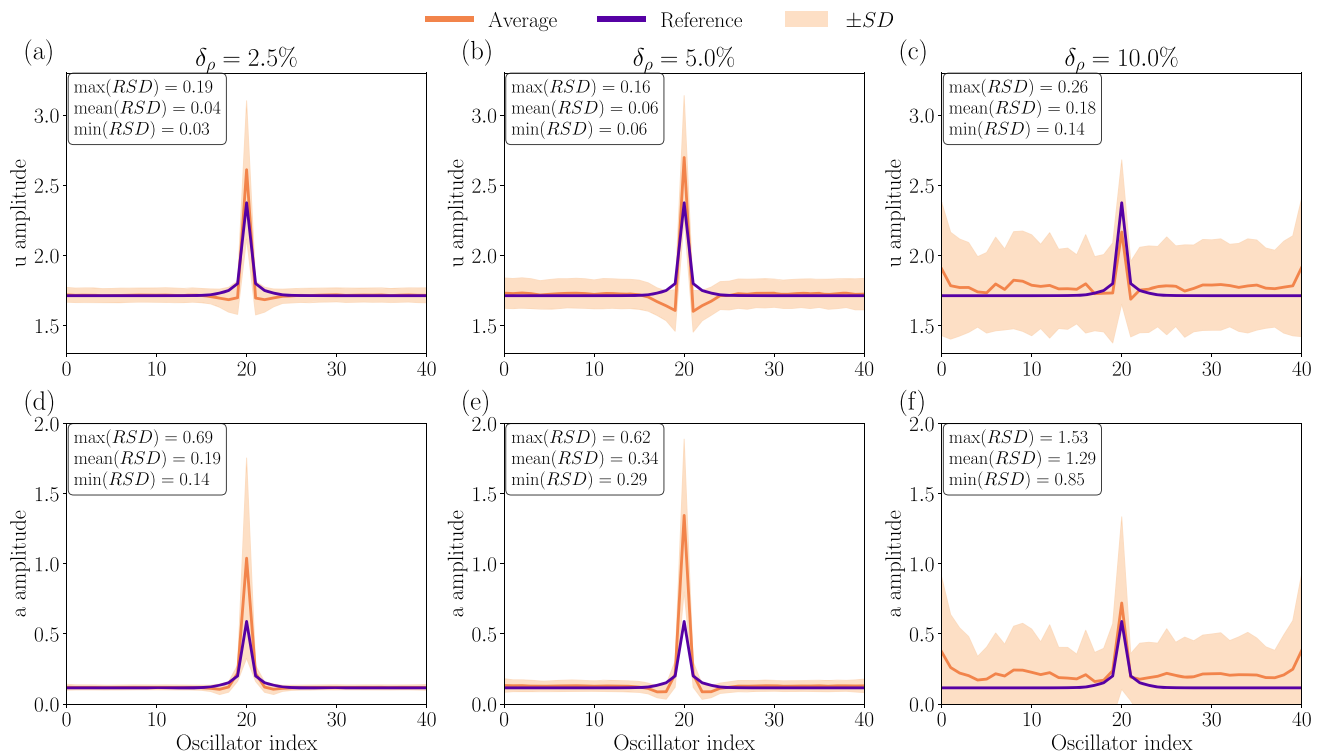


Fig. 11 Simulations under nondeterministic regimes for the softening case ($\pm = -1$). With $\Theta = 0.312$. The damping, natural frequency, and oscillator excitation are subjected to arbitrary variations described by Eq. (29), whereas the remaining parameters are kept periodic. Vari-

ations in the natural frequency are associated with the introduced disorder, with an intended disorder imposed on the central oscillator ($\tilde{f}(20) = -0.1\omega_0^2$, and $\tilde{f}(n) = 0$ for $n \neq 20$)

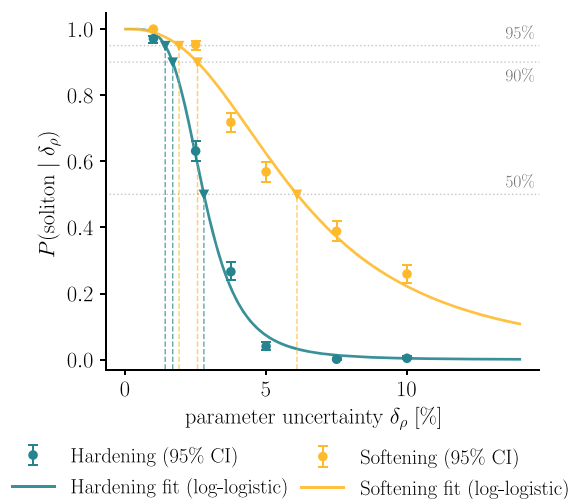


Fig. 12 Soliton nucleation probability as a function of the parameter uncertainty, $P(\text{soliton} | \delta_\rho)$. Symbols are the estimates obtained from the 1000 realizations at each level of δ_ρ , with error bars denoting the 95% confidence interval; solid lines are log-logistic curves. Dotted horizontal lines mark the target probabilities of 95%, 90% and 50%. Variations in the natural frequency are associated with the introduced disorder, with an intended disorder imposed on the central oscillator ($\tilde{f}(20) = -0.1\omega_0^2$, and $\tilde{f}(n) = 0$ for $n \neq 20$)

Table 1 Probability of soliton nucleation along with a confidence interval of 95% under different levels of uncertainty δ_ρ

δ_ρ (%)	Hardening (%)	Confidence interval (%)
1.0	97.1	[95.87, 97.97]
2.5	63.1	[60.06, 66.04]
3.75	26.6	[23.95, 29.42]
5.0	4.0	[2.95, 5.40]
7.5	0.10	[0.00, 0.56]
10.0	0.4	[0.11, 1.02]
δ_ρ (%)	Softening (%)	Confidence interval (%)
1.0	100.0	[99.63, 100.00]
2.5	95.3	[93.81, 96.45]
3.75	71.8	[68.93, 74.50]
5.0	56.8	[53.71, 59.84]
7.5	38.8	[35.83, 41.86]
10.0	25.9	[23.28, 28.70]

tence of the phenomenon, increasing δ_ρ reduces the number of solutions exhibiting soliton dynamics, with a more pronounced decline in the hardening regime than in the softening one, as shown in Fig. 12. Furthermore, increasing δ_ρ also

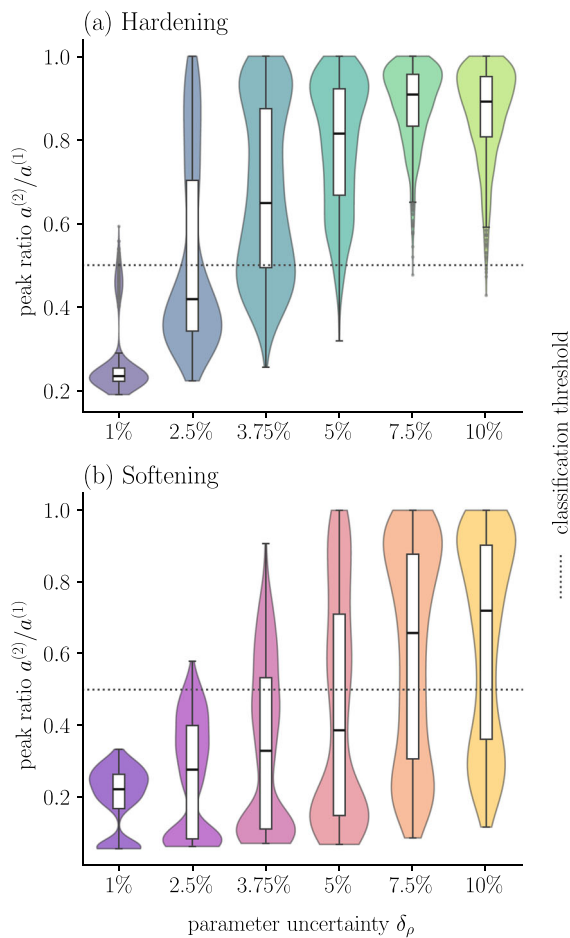


Fig. 13 Distribution of the peak ratio $a^{(2)}/a^{(1)}$ across the 1000 realizations, for **a** the hardening and **b** the softening regime and six levels of parameter uncertainty δ_ρ . The violins show the kernel density estimate of the full sample, the superimposed boxes give the median and the interquartile range, with whiskers extending to 1.5 times the interquartile range and individual points beyond it shown separately. The dotted horizontal line is the classification threshold of 0.5. Variations in the natural frequency are associated with the introduced disorder, with an intended disorder imposed on the central oscillator ($\tilde{f}(20) = -0.1\omega_0^2$, and $\tilde{f}(n) = 0$ for $n \neq 20$)

leads to an increase in the RSD for both regimes. For u amplitude (Figs. 10 and 11, a, b and c), a gradual increase in dispersion is observed for both cases, with comparable values at the highest uncertainty levels. In contrast, the a amplitude (Figs. 10 and 11, d, e and f) exhibits a distinct trend: although the hardening case displays an approximately monotonic increase in variability, the softening case undergoes a stronger increase at the highest value of δ_ρ , reaching a relative standard deviation more than twice that observed for the hardening regime. The analyses centered on Fig. 12 are related to all localization criteria, whereas Fig. 13 provides insight into the degradation of amplitude localization for the amplitude criterion. As δ_ρ increases, the distributions shift across the threshold of 0.5, such that the sample mean

no longer represents any individual realization. Consistently with Fig. 12, the hardening system reaches the region where the peak-ratio criterion fails at a lower δ_ρ than the softening system. Despite this degradation, uncertainties below 2% ensure the robustness of the soliton against energy delocalization.

5 Concluding remarks

This article demonstrates the existence and tunability of stationary solitons in nonlinear metastructures under superharmonic resonance. The system under investigation, consisting of a chain of coupled Duffing oscillators, is subjected to disorder, thereby enabling the extension of the modeling framework to aperiodic configurations. By employing the method of multiple scales, a class of Discrete Nonlinear Schrödinger-type equations is derived. By employing a higher-order multiple-scales expansion, a DNLS-type formulation is derived that captures nonlinear interactions beyond those described by the classical NLS. Compared with the primary-resonance results reported in the literature, which recover the classical Nonlinear Schrödinger Equation, the proposed higher-order formulation extends the predictive capability to excitation regimes beyond those accurately described by the classical model.

The analyses are performed from two complementary perspectives: (i) examination of the analytically derived expressions and (ii) their validation via numerical integration of the lattice equation. Such an approach enables parallels to be drawn between primary and superharmonic mechanical solitons. Notably, the phase of the envelope function constitutes a key parameter. Unlike the primary resonance case, the oscillator response is expressed in terms of sine and cosine components whose direct algebraic superposition is not possible without proper phase consideration, hence precluding a direct mapping between the soliton variable and the measurable displacement field, as demonstrated in the 1-DOF simulations.

Additionally, the selected excitation mechanism introduces a further nuance: the coupling between external forcing and spatial disorder, mediated by the Λ_n term, induces a dependence between forcing and localization mechanisms that is absent in primary resonance scenarios. More specifically, because Λ_n depends on both the excitation amplitude K_n and the excitation frequency Θ , it becomes directly involved in both the forcing and localization mechanisms. In primary resonance, the influence of forcing and disorder can generally be decoupled, enabling an immediate interpretation of the corresponding stability diagrams. In the present case, however, the resulting cross-coupling introduces additional complexity, since modifications in the excitation conditions

simultaneously alter the forcing contribution and the localization conditions.

Regarding the numerical findings on the role of disorder, some conclusions are established. First, the hypothesis of spontaneous nucleation induced by a reduction in the natural frequency of the components is confirmed for both hardening and softening regimes. Second, analogously to primary resonance phenomena, such nucleation exhibits strong sensitivity to excitation frequency, coupling strength, and the number of oscillators, as a result hindering the formulation of a generalized stability diagram. Finally, it is observed that periodicity-breaking mechanisms that increase the natural frequencies of the oscillators favor energy delocalization rather than soliton formation. This is qualitatively consistent with the hypothesis that energy nucleation requires a symmetry-breaking mechanism capable of generating localized high-amplitude states, a condition expected to be satisfied only by reductions in the natural frequency.

From an engineering perspective, operating at submultiples of the fundamental frequency mitigates geometric scaling limitations. In particular, exploiting superharmonic resonances enables access to similar high-frequency dynamics under lower-frequency excitation, despite uncertainties in the physical parameters. For $\delta_\rho = 2.5\%$, the soliton nucleation rate reaches 95.3% in the softening regime and 63.1% in the hardening regime. Although increasing uncertainty progressively reduces these rates and increases the RSD, localized states persist even for $\delta_\rho = 10\%$. Overall, the softening regime exhibits greater persistence of soliton dynamics, whereas the hardening regime is more susceptible to localization loss as uncertainty levels increase. Moreover, the regions in which solitons emerge are consistently characterized by negative

largest Lyapunov exponents, indicating spontaneous convergence toward localized states in the absence of chaotic dynamics. These findings support the feasibility of experimental implementations of solitons in lattices with realistic parametric variability, particularly in applications involving vibration energy harvesting, vibration control, and sensing.

Despite the theoretical findings, several open questions remain. In particular, the influence of the excitation frequency Θ on both the robustness of the phenomenon and the validity range of the first- and second-order approximations in ε remains to be established. Although no sustained soliton nucleation was observed in the periodic chains within the numerical simulations, this result is specific to the investigated parameter range and fully quiescent initial conditions. The absence of boundaries or localized inhomogeneities under superharmonic resonance provides no preferred site for symmetry breaking and localization, making spontaneous nucleation difficult. This observation does not exclude soliton nucleation under other parameter regimes or initial conditions, which would require further analytical and numerical investigations. Furthermore, unlike previously reported cases, the governing equation lacks an associated stability diagram, representing an important direction for future research. The effects of nonlinear coupling in the absence of linear coupling, as well as the extension of the framework to subharmonic regimes, also warrant further investigation.

Appendix A Auxiliary variables employed in the mathematical formulation

See Tables 2, 3, 4 and 5.

Table 2 Mathematical relations defining the lattice parameters for periodic mass

Dimensional parameters associated with Equation (1)		Dimensionless parameters associated with Equation (3)	
Variable	Mathematical relation	Variable	Mathematical relation
ω_0^2	$= \frac{k_l}{m}$	t	$= t_f \omega_0$
ζ_n	$= \frac{b_n}{2\omega_0 m}$	$u_n(t)$	$= \frac{\sqrt{\beta_\varepsilon} U_n(t_f)}{\omega_0}$
f_n	$= \frac{\Delta k_{ln}}{m}$	$\mu(n)$	$= \frac{f_\varepsilon(n)}{\omega_0^2}$
β	$= \frac{k_{nl}}{m}$	Θ	$= \frac{\Omega}{\omega_0}$
$C_l^{(k)}$	$= \frac{k_{cl}^{(k)}}{m}$	$\chi_l^{(k)}$	$= \frac{C_{l\varepsilon}^{(k)}}{\omega_0^2}, \quad k = 1, \dots, \alpha$
$C_{nl}^{(k)}$	$= \frac{k_{nl}^{(k)}}{m}$	$\chi_{nl}^{(k)}$	$= \frac{C_{nl\varepsilon}^{(k)}}{\beta_\varepsilon}, \quad k = 1, \dots, \alpha$
F_n	$= \Omega^2 U_b$	K_n	$= \frac{F_n \sqrt{\beta_\varepsilon}}{\omega_0^3}$

Table 3 Auxiliary variables employed in equation (13)

Variable	Mathematical relation
$\mathcal{L}_1$	$= \frac{3\Lambda_n^3(\Theta^2 - 3\Theta + 18)}{4\Theta(\Theta - 3)}$
$\mathcal{L}_2$	$= -\frac{3\Lambda_n^5(\Theta^2 - 7)}{2(\Theta^2 - 1)} \pm \left(\frac{i\Lambda_n^3\zeta_{n\varepsilon}(\Theta^2 + 12\Theta - 1)}{2(\Theta^2 - 1)} - \frac{\Lambda_n^3\sigma}{2} - \frac{\Lambda_n^3\mu(n)(\Theta^2 - 13)}{4(\Theta^2 - 1)} \right)$
$\mathcal{L}_3$	$= -\frac{3\Lambda_n^3(\Theta^2 - 19)}{2(\Theta^2 - 1)}$
$\mathcal{L}_4$	$= \pm \frac{3\Lambda_n^2\mu(n)(5 - \Theta^2)}{\Theta^2 - 1} - \frac{9\Lambda_n^4(2\Theta^2 - 11)}{2(\Theta^2 - 1)} - \zeta_{n\varepsilon}^2 - \frac{\mu^2(n)}{4}$
$\mathcal{L}_5$	$= -\frac{9\Lambda_n^2(\Theta^4 - 22\Theta^2 + 69)}{(\Theta^2 - 9)(\Theta^2 - 1)} \pm \left(-3i\zeta_{n\varepsilon} - \frac{3\mu(n)}{2} \right)$
$\mathcal{F}_{\chi_l^{(k)}}$	$= \sum_{k=1}^{\alpha} \chi_l^{(k)} \left[\pm \left(\frac{3A_n^2\Delta_k\{\bar{A}\}}{4} - \frac{3\Delta_k\{A A ^2\}}{4} - \frac{3 A_n ^2\Delta_k\{A\}}{2} - 3\Lambda_n^2\Delta_k\{A\} \right) - \frac{\mu(n)\Delta_k\{A\}}{4} - \frac{\Delta_k\{\mu A\}}{4} \right]$
$\mathcal{F}_{\chi_l^{(k)}\chi_l^{(j)}}$	$= \sum_{k=1}^{\alpha} \sum_{j=1}^{\alpha} \chi_l^{(k)} \chi_l^{(j)} \left[\frac{\Delta_{ k-j \{A\}}}{4} + \frac{\Delta_{k+j}\{A\}}{4} - \frac{\Delta_j\{A\}}{2} - \frac{\Delta_k\{A\}}{2} \right]$
$\mathcal{F}_{\chi_{nl}^{(k)}}$	$= \sum_{k=1}^{\alpha} \chi_{nl}^{(k)} \left[3(A_n - A_{n+k})^2(\bar{A}_n - \overline{A_{n+k}}) + 3(A_n - A_{n-k})^2(\bar{A}_n - \overline{A_{n-k}}) \right]$
$\Delta_j\{X\}$	$= 2X_n - X_{n+j} - X_{n-j} \quad \text{for any expression } X$

Table 4 Auxiliary variables employed in equations (22) and (23)

Variable	Mathematical relation
V	$= \frac{a_n \varepsilon \zeta_{n\varepsilon} (\pm 3a_n^2 \varepsilon - 8)}{8}$
W	$= \frac{3\Lambda_n^5 \varepsilon^2 (\Theta^2 - 7)}{2(\Theta^2 - 1)} + \frac{3\Lambda_n^3 \varepsilon^2 a_n^2 (\Theta^2 - 19)}{8(\Theta^2 - 1)} \pm \frac{\Lambda_n^3 (\varepsilon^2 \sigma - 2\varepsilon)}{2}$
X	$= \pm \frac{\Lambda_n^3 \varepsilon^2 \zeta_{n\varepsilon} (-\Theta^2 - 12\Theta + 1)}{2(\Theta^2 - 1)}$
Y	$= -\frac{3a_n^2 \Lambda_n^3 \varepsilon^2 (\Theta^2 - 3\Theta + 18)}{16\Theta(\Theta - 3)}$
Z	$= \frac{15a_n^5 \varepsilon^2}{256} \pm \frac{3\varepsilon a_n (-a_n^2 - 8\Lambda_n^2)}{8} + \frac{9\varepsilon^2 a_n^3 \Lambda_n^2 (\Theta^4 - 22\Theta^2 + 69)}{8(\Theta^2 - 9)(\Theta^2 - 1)} + \frac{9\varepsilon^2 a_n \Lambda_n^4 (2\Theta^2 - 11)}{4(\Theta^2 - 1)} + \frac{\varepsilon^2 \zeta_{n\varepsilon}^2 a_n}{2}$

Table 5 Auxiliary variables employed in Equation (28)

Variable	Mathematical relation
Π_1	$= \pm \frac{3a_n \Lambda_n^2 \sin(\gamma_n)}{4\Theta(\Theta - 1)} - \frac{4\Theta \Lambda_n \zeta_{n\varepsilon}}{\Theta^2 - 1}$
Π_2	$= \pm \frac{3a_n \Lambda_n^2 \cos(\gamma_n)}{4\Theta(\Theta - 1)} + \frac{\pm 3\Lambda_n (a_n^2 + 2\Lambda_n^2) + 2\Lambda_n \mu(n)}{\Theta^2 - 1}$

Appendix B Supplementary numerical simulations

See Figs. 14, 15, 16, 17, 18, 19 and 20.

Appendix C Numerical methods and simulation parameters

Two integration schemes are employed throughout the paper. The DOP853 method is used when integrating analytical amplitude equations in time to obtain the steady state response. The LSODA method is used when integrating

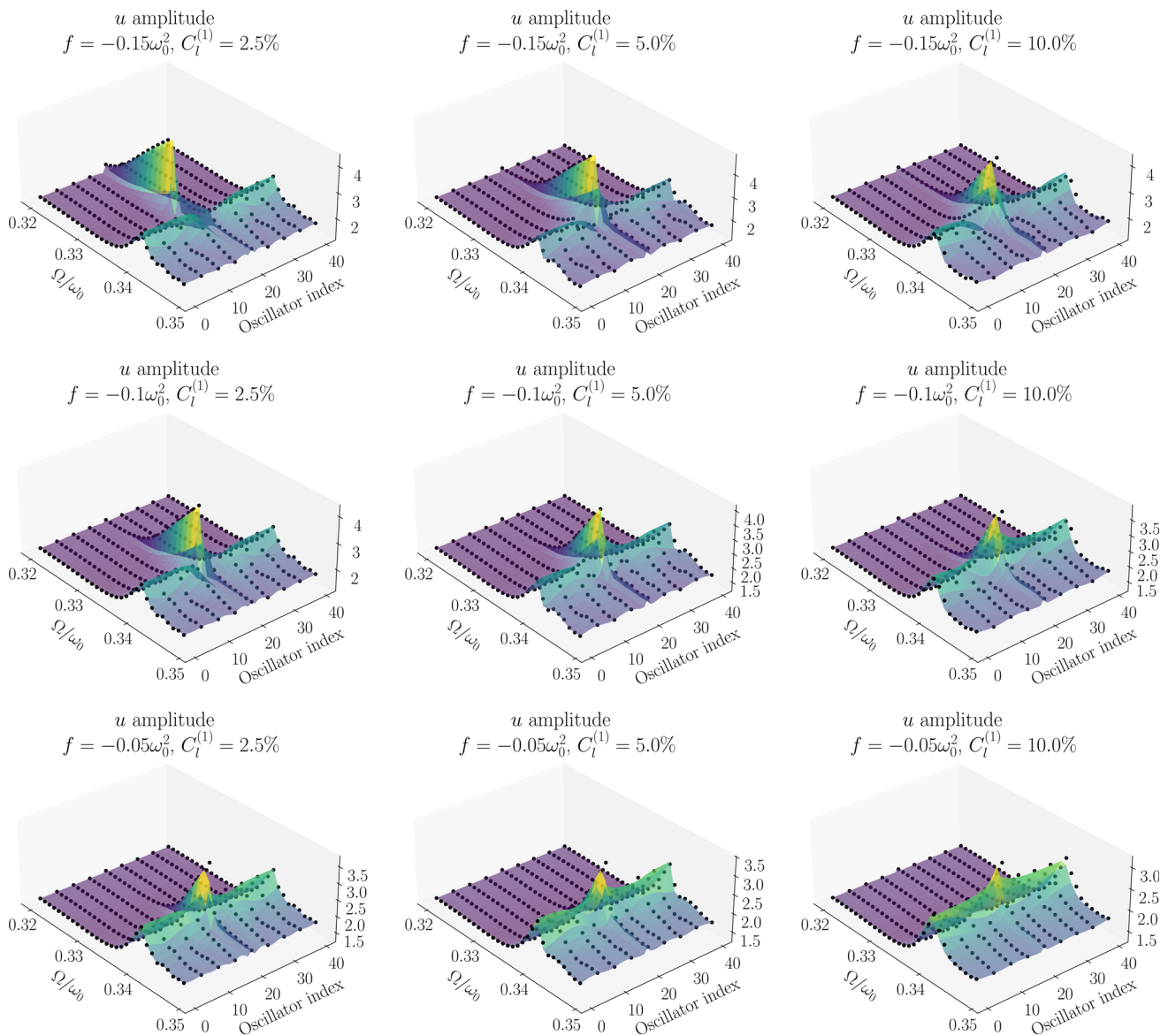


Fig. 14 Numerical validation of the analytical solution given by Eq. (15). Black markers correspond to the numerical integration of the lattice equation (3), while the colored surface represents the analytical solution. The point density was reduced for visualization purposes. Hardening regime ($\pm = 1$)

ODEs directly. Both methods use relative and absolute tolerances of $\text{rtol} = 10^{-6}$ and $\text{atol} = 10^{-8}$.

For the 1-DOF system, the analytical response is obtained from the amplitude-phase equations, using Brent's method for root-finding on the amplitude residual. Spurious branches are discarded by thresholding the RMS relative residual of the governing equation evaluated on each branch, with a tolerance of 8%, applied to the second-order closure only, since the first-order closure yields at most three roots, all physically admissible.

The red and cyan regions are classified by the largest Lyapunov exponent (LLE). The complex envelope A_n and a perturbation vector δA_n are integrated simultaneously

from rest, with $\delta \dot{A} = F_A \delta A + F_{\bar{A}} \bar{\delta A}$, where F_A and $F_{\bar{A}}$ are the Wirtinger derivatives of Eq. (15), obtained analytically. The exponent is evaluated as $\lambda(t) = S(t)/t$, with $S = \int \text{Re} \langle \delta A, \delta \dot{A} \rangle / |\delta A|^2 dt$. Integration is performed with the DOP853 method ($\text{rtol} = 10^{-6}$, $\text{atol} = 10^{-8}$) in successive blocks of 1000 dimensionless time units, with δA renormalized between blocks to prevent overflow; since the integrand is scale-invariant, the renormalization does not affect the accumulated S . Integration stops when Eq. (27) is satisfied or when $\lambda(t)$ stabilizes, defined as a spread of less than 5×10^{-6} among the last 15 block-end values, after a minimum of 15 blocks. The second criterion covers the responses that never reach a fixed point, for which the state

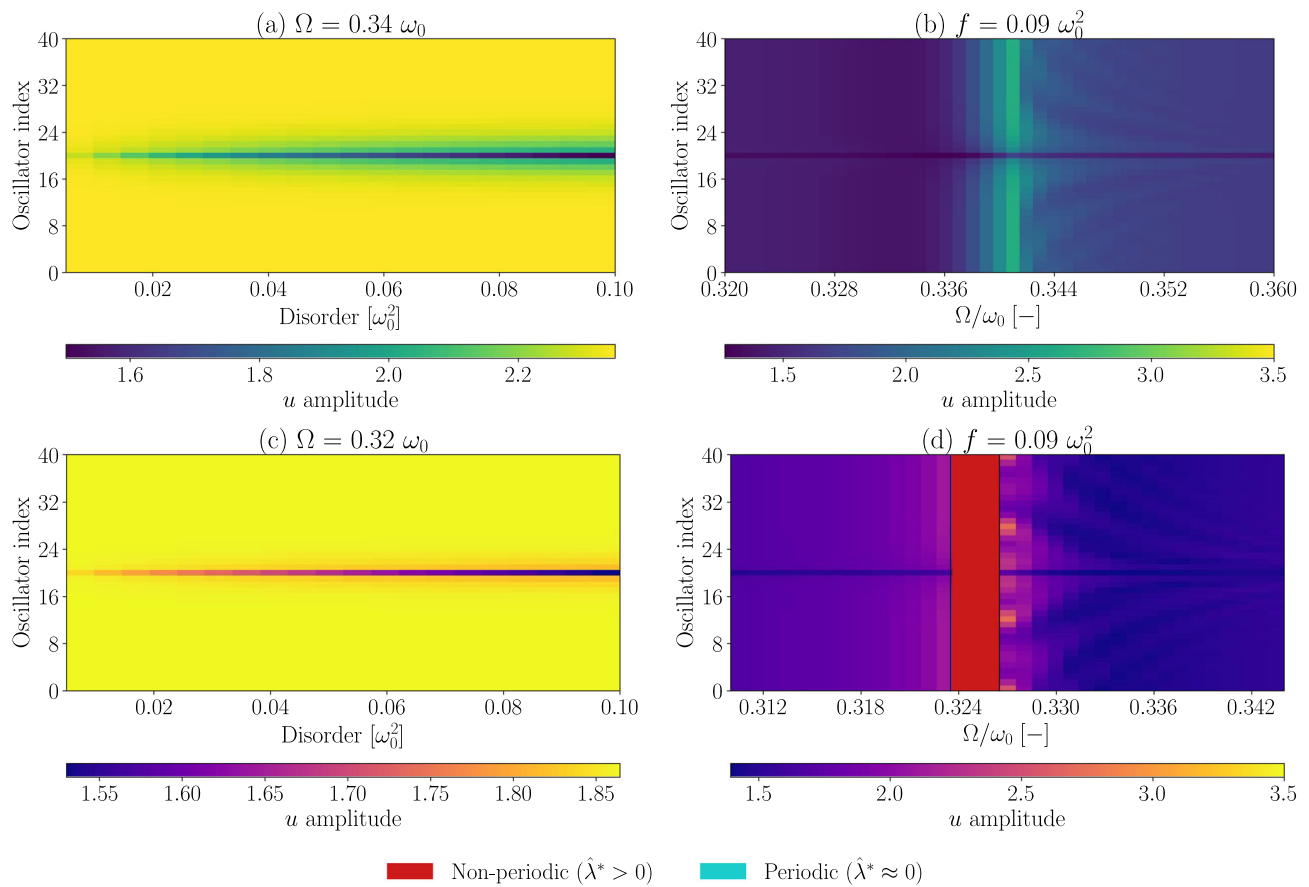


Fig. 15 Numerical sweeps of soliton nucleation for hardening (a and b) and softening (c and d) regimes under positive values of $f(n)$. Panels (a, c): fixed excitation frequency with varying disorder, b, d: fixed disorder

with varying frequency. Red and cyan regions (c and d): Nonperiodic and periodic steady-state responses

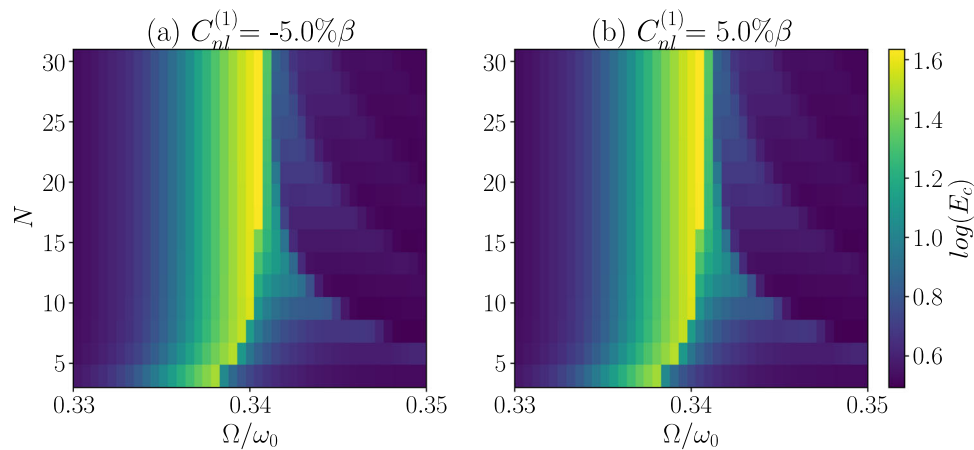


Fig. 16 Parametric sweep addressing the nonlinear coupling. Hardening case ($\pm = 1$). $C_l^{(1)} = 4.0\%$, ω_0^2 . Disorder $f = -0.05\omega_0^2$

Fig. 17 Parametric sweep addressing the nonlinear coupling. Softening case ($\pm = -1$). $C_l^{(1)} = 4.0\%$, ω_0^2 . Disorder $f = -0.05\omega_0^2$

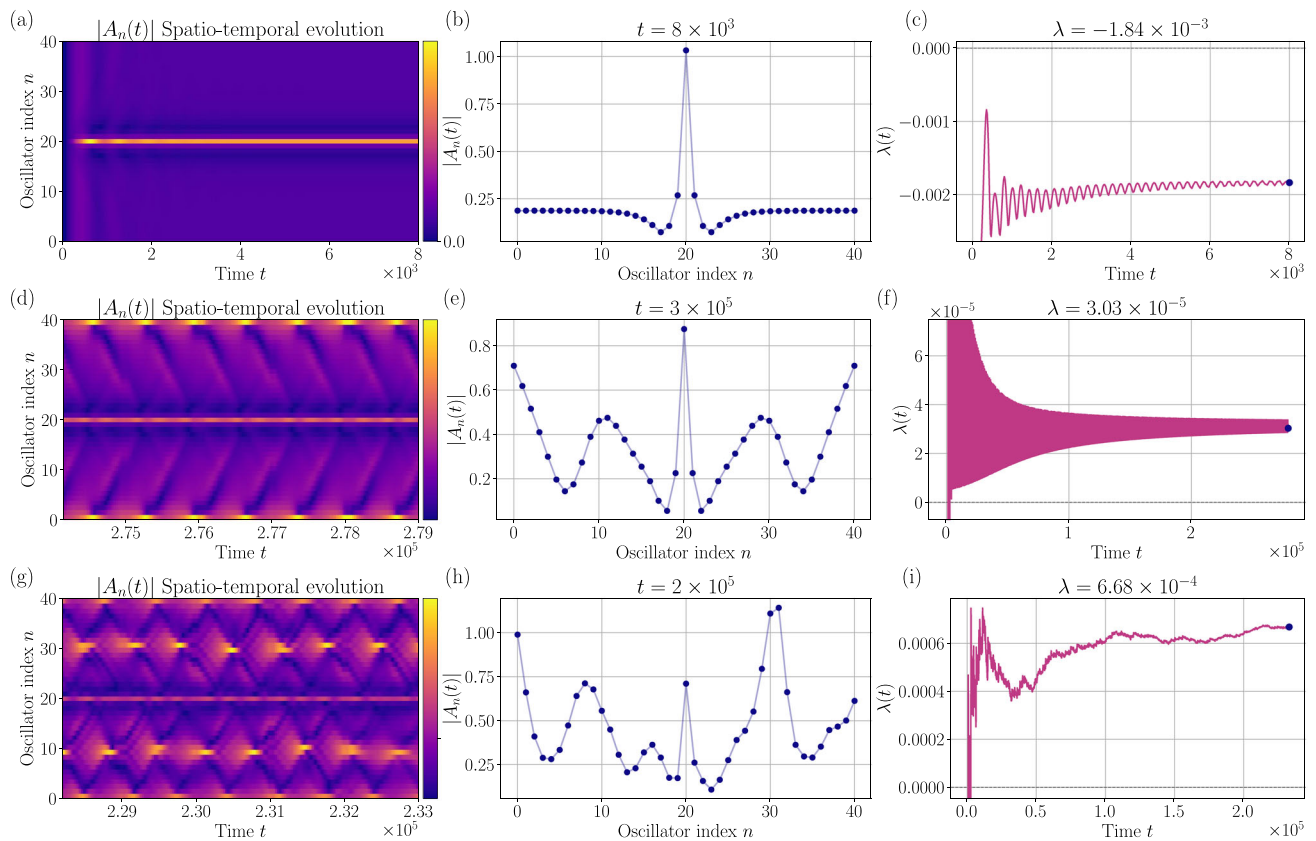
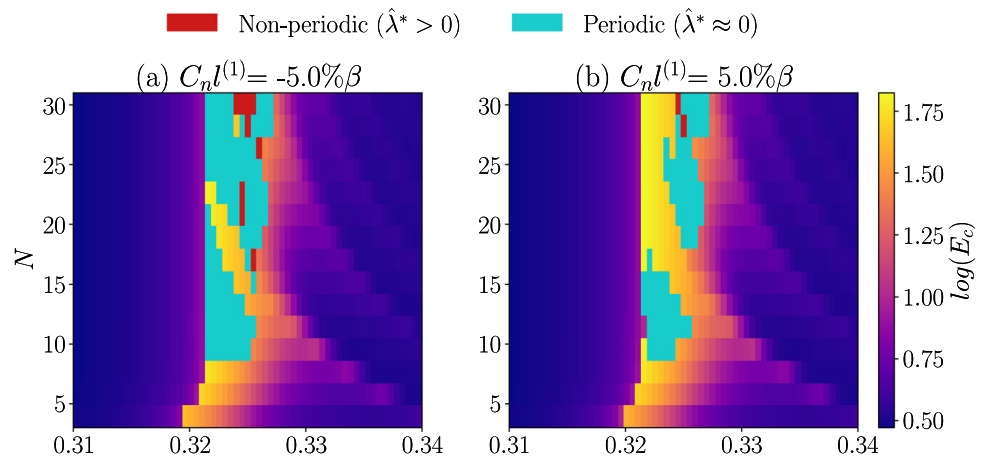


Fig. 18 Space-time evolution obtained from Eq. (15), in the softening regime ($\pm = -1$). Each row corresponds to one response: the spatio-temporal evolution of $|A_n(t)|$ (first column), the amplitude profile along the chain after the transient (second column), and the evolution of the largest Lyapunov exponent $\lambda(t)$ (third column). Panels (a–c) show a response satisfying the convergence criterion defined in Eq.

(27), obtained for $\Theta = 0.322$; panels (d–f) a nonconvergent periodic response, for $\Theta = 0.3238$; and panels (g–i) a nonconvergent nonperiodic response, for $\Theta = 0.326$. The converged exponents are $\lambda = -1.84 \times 10^{-3}$, $\lambda = 3.03 \times 10^{-5}$, and $\lambda = 6.68 \times 10^{-4}$, respectively. Disorder $f = -0.05 \omega_0^2$

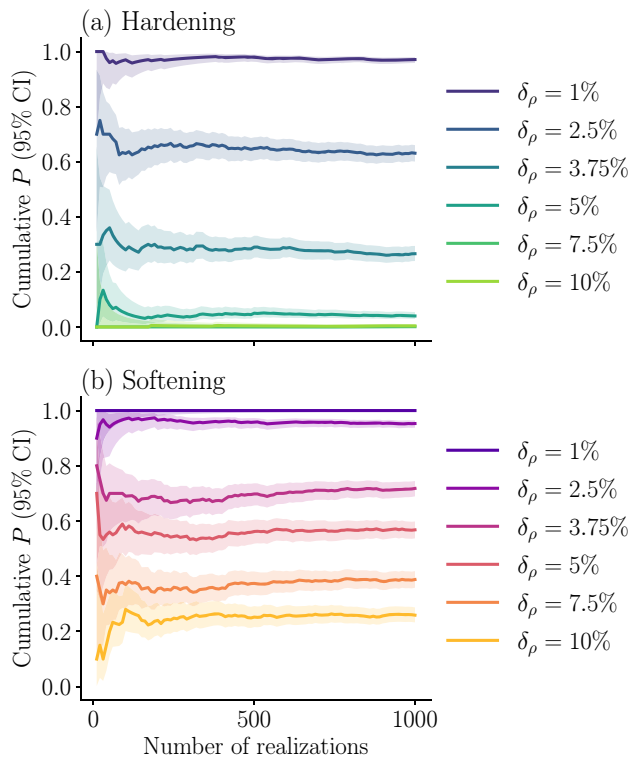


Fig. 19 Convergence of the soliton nucleation probability for **a** the hardening and **b** the softening regime. Each curve shows the running estimate P obtained from the 1000 realizations, evaluated every 10 realizations, for six levels of parameter uncertainty δ_ρ (1% to 10%); the shaded band is the corresponding 95% confidence interval

remains time-dependent while the mean of $\lambda(t)$ converges. The converged exponent then separates the regimes, with a tolerance of 9×10^{-5} around zero: $\lambda < 0$ for asymptotically stable responses, $\lambda \approx 0$ for periodic responses, and $\lambda > 0$ for nonperiodic responses.

Unless stated otherwise, the parameters $N = 41$ oscillators, $\zeta = 0.3/100$, $\omega_0 = 100$ Hz, $\beta = 2.5 \times 10^9$ N/(kg·m³), $\varepsilon = 0.01$, and $F = 700$ m/s² are adopted. In figures involving the coupled lattice, linear coupling is considered up to third order ($\alpha = 3$), with $C_l^{(1)} = 2.5\% \omega_0^2$, $C_l^{(2)} = C_l^{(1)}/4$, and $C_l^{(3)} = C_l^{(1)}/9$. All lattice parameters are periodic except for the introduced disorder. Unless otherwise noted, simulations start from rest initial conditions.

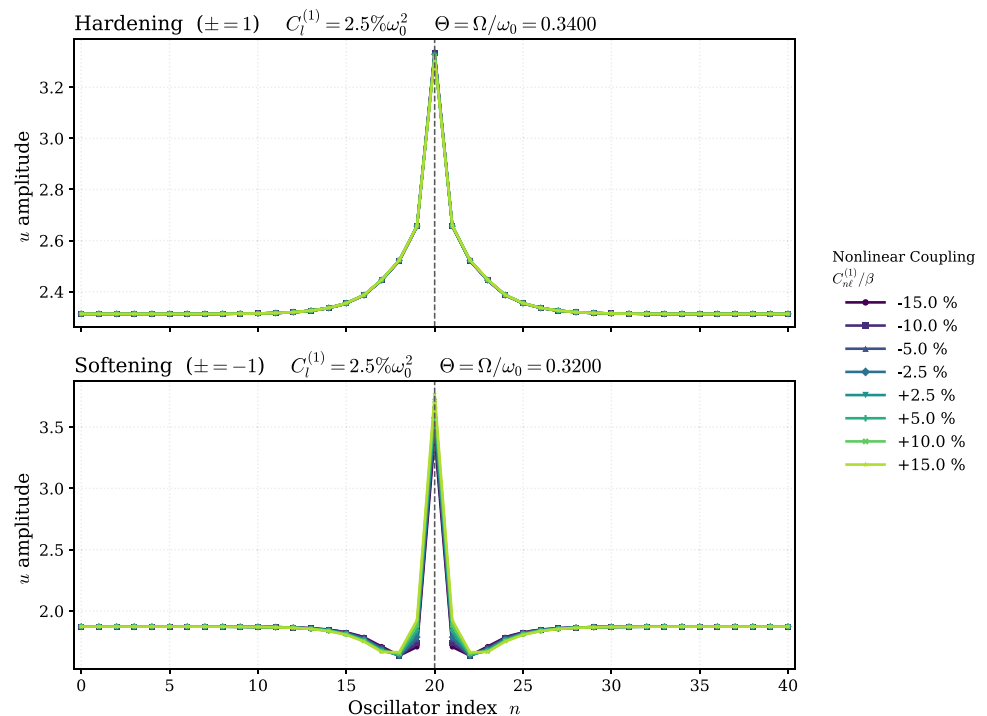
Nonlinear coupling is neglected throughout the simulations because its effect on the lattice is markedly smaller than that of linear coupling, with both couplings typically arising together. To support this choice, Fig. 20 presents a sweep in the nonlinear coupling coefficient for a fixed linear coupling value. For both negative and positive nonlinear

coupling, the effect on the hardening and softening response curves is negligible.

C.1 Figure-specific settings

- **Figs. 2, 3:** Analytical solutions integrated with DOP853 over 100 periods; numerical solution integrated with LSODA over 500 cycles, with the first 200 discarded as transient and 1000 points per cycle.
- **Fig. 4:** Numerical solution integrated with LSODA over 500 cycles, with the first 200 discarded as transient and 1000 points per cycle.
- **Fig. 5:** Numerical solution is obtained by frequency sweep (up and down) with state continuation, integrating Eq. (3) with the LSODA method with tighter tolerances ($\text{rtol} = 10^{-8}$, $\text{atol} = 10^{-10}$); the transient, sized to four damping time constants ($4/\zeta$ excitation cycles), is discarded before averaging the steady-state response over 40 cycles at 128 points per cycle.
- **Fig. 6:** Analytical solutions integrated with DOP853 over 100 periods;
- **Fig. 7:** Numerical solution of Eq. (3) integrated with LSODA. Numerical continuation imposed in the time domain.
- **Figs. 8, 9:** Analytical solutions integrated with DOP853.
- **Figs. 10, 11:** Analytical solutions integrated with DOP853.
- **Fig. 12:** Analytical solutions integrated with DOP853. log-logistic curves fitted by binomial maximum likelihood. The corresponding 95% confidence interval is computed with the Wilson score interval, or with the Clopper-Pearson interval whenever fewer than 10 successes or fewer than 10 failures have accumulated.
- **Fig. 13:** Analytical solutions integrated with DOP853.
- **Fig. 14:** Analytical solution integrated with DOP853, lattice equation with LSODA. Numerical continuation employed for the analytical solution.
- **Fig. 15:** Same procedure as Fig. 6, with the sign of the disorder reversed ($f^{(n)} > 0$).
- **Figs. 16, 17:** Analytical solution integrated with DOP853. Nonlinear coupling is included rather than neglected.
- **Fig. 18:** Analytical solutions integrated with DOP853.
- **Fig. 19:** Analytical solutions integrated with DOP853. The shaded band is the corresponding 95% confidence interval, computed with the Wilson score interval, or with the Clopper-Pearson interval whenever fewer than 10 successes or fewer than 10 failures have accumulated.
- **Fig. 20:** Analytical solutions integrated with DOP853. Nonlinear coupling is considered up to third order ($\alpha = 3$), same scaling applied to linear coupling.

Fig. 20 Amplitude profile along the chain for varying nonlinear coupling $C_{nl}^{(1)}/\beta$, from -15% to $+15\%$, at fixed linear coupling $C_l^{(1)} = 2.5\% \omega_0^2$. Top: hardening regime ($\pm = 1$) at $\Theta = 0.34$. Bottom: softening regime ($\pm = -1$) at $\Theta = 0.32$. Disorder $f = -0.05 \omega_0^2$



Author Contributions All authors contributed to the study conception and design. L.J.D.A.: Data Curation, Formal Analysis, Investigation, Methodology, Software, Validation, Visualization, Writing–Review & Editing. A.S.B.: Conceptualization, Formal Analysis, Investigation, Methodology, Software, Validation, Writing–Original Draft, Review & Editing. R.S.R.: Data Curation, Formal Analysis, Investigation, Writing–Review & Editing. N.K.: Conceptualization, Funding Acquisition, Investigation, Supervision, Writing–Review & Editing. L.P.R.O.: Conceptualization, Funding Acquisition, Supervision, Writing–Review & Editing. N.B.: Conceptualization, Funding Acquisition, Supervision, Writing–Review & Editing. All authors read and approved the final manuscript.

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Data Availability The datasets used and analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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