

Adapted VPINN for model parameter identification of beam-like structures

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Context: 1D harmonic Euler-Bernoulli equation

Fixed angular frequency ω with clamped-free cantilever BC

[J.-L. Guyader, Vibration in continuous media, John Wiley & Sons, 2013]

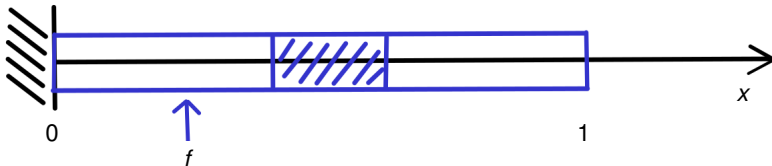
Find w such that

$$\frac{\partial^2}{\partial x^2} \left(E(x)I(x) \frac{d^2 w(x, \omega)}{dx^2} \right) - \rho(x)S(x)\omega^2 w(x, \omega) = f(x, \omega) \quad \text{on } \Omega =]0, 1[,$$

$$w(0) = w'(0) = 0,$$

$$(EIw'')(1) = (EIw'')'(1) = 0,$$

where w is the complex transverse deflection, E is the complex non-homogeneous elastic modulus, ρ the material density, I the flexural moment of inertia, S cross-sectional area and f applied loading respectively.

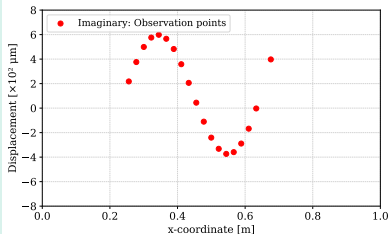
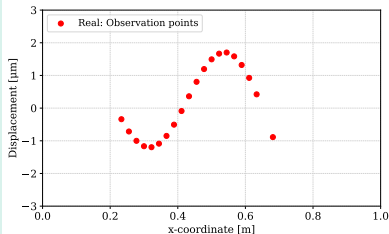


Context: Inverse parameter identification

Linear visco-elastic material: $\eta = E_r/E_i$ loss factor (characterizing material damping)

$$W = \underbrace{w_r}_{\text{transversal deflection}} + i \underbrace{w_i}_{\text{viscoelastic damping}}$$
$$E = \underbrace{E_r}_{\text{storage modulus}} + i \underbrace{E_i}_{\text{loss modulus}}$$

Find (w, E) such that Euler-Bernoulli PDE and measured data on w

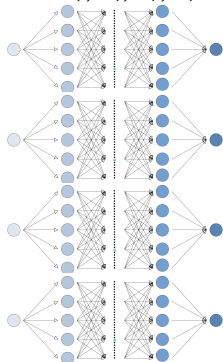


Physic informed approach

[M. Raissi, P. Perdikaris, G. Karniadakis, 2019], [E. Kharazmi, Z. Zhang and G. E. M. Karniadakis, 2021]

Neural Networks:

$$x \mapsto \hat{w}_r, \hat{w}_i, \hat{E}_r, \hat{E}_i$$



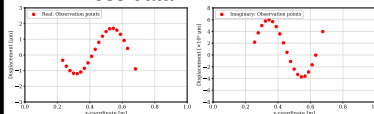
Physics loss

$$\mathcal{L}_{\text{physics},r} = \mathcal{L}_r(\hat{w}_r, \hat{w}_i, \hat{E}_r, \hat{E}_i)$$

$$\mathcal{L}_{\text{physics},i} = \mathcal{L}_i(\hat{w}_r, \hat{w}_i, \hat{E}_r, \hat{E}_i)$$

$$\mathcal{L}_{BC} = \mathcal{L}_{BC}(\hat{w}_r, \hat{w}_i, \hat{E}_r, \hat{E}_i)$$

Loss data

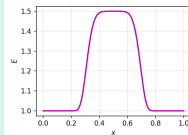
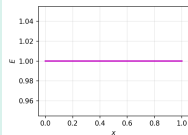


$$\mathcal{L} = \lambda_{\text{data}} \mathcal{L}_{\text{data}} + \lambda_{\text{phys},r} \mathcal{L}_{\text{phys},r} + \lambda_{\text{phys},i} \mathcal{L}_{\text{phys},i} + \lambda_{bc} \mathcal{L}_{bc}$$

Backpropagation

Classical PINN approach

$$E \in \mathcal{C}^2, E > E_{\min} > 0, w \in \mathcal{C}^4, f \in \mathcal{C}$$



$$\mathcal{L}_{\text{total}} = \lambda_{\text{data}} \mathcal{L}_{\text{data}} + \lambda_{\text{phys}} \mathcal{L}_{\text{phys}} + \lambda_{\text{bc}} \mathcal{L}_{\text{bc}}$$

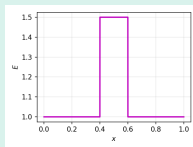
$$\mathcal{L}_{\text{data}} = \frac{1}{N_{\text{obs}}} \sum_{i=1}^{N_{\text{obs}}} (\hat{w}(x_i) - w_{\text{obs},i})^2$$

$$\mathcal{L}_{\text{phys}} = \frac{1}{N_{\text{col}}} \sum_{j=1}^{N_{\text{col}}} \left[\left(\hat{E}(x_j) / \hat{w}''(x_j) \right)'' - \rho S \omega^2 \hat{w}(x_j) - f(x_j) \right]^2$$

$$\mathcal{L}_{\text{bc}} = \left(\hat{E}(1) / \hat{w}''(1) \right)^2 + \left(\left(\hat{E} / \hat{w}'' \right)'(1) \right)^2$$

Drawback

- ▶ Application case : discontinuous E



Strong formulation needs of transmission conditions on discontinuity points ξ

$$\llbracket EI w'' \rrbracket_{\xi} = 0 \quad \llbracket (EI w'')' \rrbracket_{\xi} = 0 \quad (TC)$$

- ▶ NN are smooth $\rightarrow$ (TC) satisfied by construction
- ▶ Automatic differentiation for $\hat{w}^{(4)} \rightsquigarrow$ high- k oscillations in $\hat{w}$ amplified as $k^4 \rightsquigarrow \partial \mathcal{L}_{\text{phys}} / \partial E$ unreliable $\rightsquigarrow$ unstable optimization



Direct variational problem

[Hasanov, Itou, 2019], [Baysal, 2022], [Blommaert, Lazendić, Oparnica, 2021]

Let be $E \in L^\infty(\Omega)$, $E \geq E_{\min} > 0$, $I, \rho, S \in L^\infty(\Omega)$, $I, \rho, S > 0$, $f \in L^2(\Omega)$,

Find $w \in V(\Omega)$ such that $\forall v \in V(\Omega)$:

$$a(w, v) = \ell(v),$$

where

$$a(w, v) := \int_{\Omega} E(x) I(x) w''(x) v''(x) dx - \omega^2 \int_{\Omega} \rho S w(x) v(x) dx$$

$$\ell(v) := \int_{\Omega} f(x) v(x) dx$$

and $V(\Omega) := \{v \in H^2(\Omega) : v(0) = 0, v'(0) = 0\}$.

Classical VPINN approach: weak residuals

Transmission conditions — automatically satisfied

Let be $M := EI w''$, the variational form reads

$$\int_{\Omega} M v'' dx = \int_{\Omega} (f + \rho S \omega^2 w) v dx ,$$

i.e. M solves weakly $-M'' = f + \rho S \omega^2 w \in L^2(\Omega)$. By elliptic regularity: $M \in H^2(\Omega)$.

Sobolev embedding in 1D: $H^2(\Omega) \hookrightarrow C^1(\bar{\Omega}) \Rightarrow M \in C^1(\bar{\Omega})$

$$\Rightarrow \begin{cases} \llbracket M \rrbracket = 0, \\ \llbracket M' \rrbracket = 0 \end{cases}$$

Thus, no explicit (TC) in $\mathcal{L}_{\text{phys}}$ and jump locations ξ need not be known.

Variational loss function with K test functions

$$\mathcal{L}_{\text{phys}} = \frac{1}{K} \sum_{k=1}^K \left[\int_{\Omega} \hat{E}(x) \hat{w}''(x) v_k''(x) dx - \omega^2 \int_{\Omega} \rho S \hat{w}(x) v_k(x) dx - \int_{\Omega} f(x) v_k(x) dx \right]^2$$

VPINN: A smooth NN to approximate a discontinuous signal

Bubble architecture for $\hat{E}$ on $[a, b]$

- ▶ Normalize coordinates: $s = (x - a)/(b - a) \in [0, 1]$
- ▶ Domain partition: $x_0 = 0 < \tilde{x}_1 < \dots < \tilde{x}_{n-1} < x_n = 1$
- ▶ Plateau (bubble) functions:

$$B_1(s) = 1 - \sigma(k(s - \tilde{x}_1))$$

$$B_j(s) = \sigma(k(s - \tilde{x}_{j-1})) - \sigma(k(s - \tilde{x}_j)), \quad 1 < j < n$$

$$B_n(s) = \sigma(k(s - \tilde{x}_{n-1}))$$

where $\sigma(t) = 1/(1 + e^{-t})$ and $k > 0$ is a fixed sharpness parameter

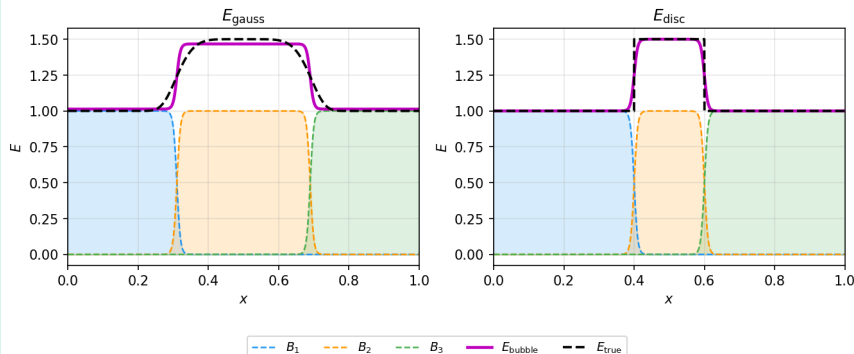
- ▶ Partition of unity at the limit $k \rightarrow \infty$: $\sum_{j=1}^n B_j(s) = 1 \quad \forall s \in [0, 1]$

$\hat{E}$ parameterization

$$\hat{E}(s) = \sum_{j=1}^n E_j B_j(s), \quad E_j = 10^{\alpha_j}, \alpha_j \in \mathbb{R}$$

VPINN: A smooth NN to approximate a discontinuous signal

Exemple of approximation



Remarks

- ▶ Each $B_j \in C^\infty$ approximates $\mathbf{1}_{\Omega_j}$ with transition width of sigmoids $\sim 9.2/k$
- ▶ E.g. $k = 200$: transition over $\sim 4.6\%$ of the domain $\Rightarrow$ sharp enough to resolve “the patch”

Numerical tests PINN-VPINN: $\mathbb{R}$ -case and theoretical context

Reference solutions

- ▶ FEM: cubic Hermite C^1 elements, $N_{el} = 200$, cantilever BC
- ▶ test cases: $E(x) = 1$, $E(x) = 1 + 0.5 e^{-((x-0.5)/0.2)^6}$, $E(x) = 1 + 0.5 \mathbf{1}_{[0.4, 0.6]}(x)$
- ▶ Fine reference: FEM solution evaluated on 2010 equidistant pts (used for error computation)
- ▶ $\omega = 150$ rad/s; $l = 1$, $\rho S = 1$, $f = 1$ (uniform load).

Fixed hyperparameters

- ▶ cantilever BCs: essential ($w(0) = w'(0) = 0$) hard via ansatz $x^2 \mathcal{N}(x)$; natural ($M(1) = V(1) = 0$) soft
- ▶ $\hat{w}$: L3N50 (tanh);
- ▶ $\hat{E}$ (NN): L3N50 (tanh and last softplus for positivity) VS $\hat{E}$ (bubble): $n=3$ plateaux, sharpness $k=200$;
- ▶ 16 observation points
- ▶ PINN: $N_{coll} = 200$ collocation points
- ▶ VPINN: $K = 10$ Legendre basis $v_k = x^2 L_k(2x-1)$, $N_{quad} = 500$ GL pts

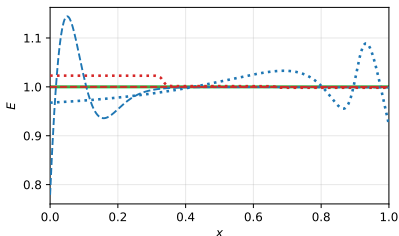
Numerical tests PINN-VPINN: $\mathbb{R}$ -case and theoretical context

E constant

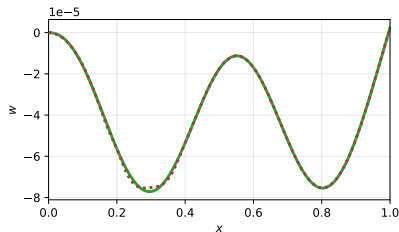
Training: Adam optimizer with $\text{lr} = 10^{-3}$ on 100k epochs:

10k warmup (data only) + 90k full loss with $\lambda_{\text{data}} = \lambda_{\text{phys}} = \lambda_{\text{bc}} = 1$

Solver	Arch	best ep.	$\mathcal{L}$	$\mathcal{L}_{\text{data}}$	$\mathcal{L}_{\text{phys}}$	$\mathcal{L}_{\text{bc}}$
PINN	NN	99811	1.32E-6	1.20E-6	1.23E-7	1.17E-9
PINN	bubble	77267	2.82E-7	2.65E-10	2.82E-7	1.21E-11
VPINN	NN	99363	3.15E-8	6.50E-10	3.62E-9	2.72E-8
VPINN	bubble	99832	8.67E-8	3.54E-8	2.59E-8	2.54E-8



— E true ··· VPINN, DNN - - - PINN, bubble
- - - PINN, DNN



— w true ··· VPINN, DNN - - - PINN, bubble

Solver	Arch	$e_E^{L^2}$ (final)	$e_E^{L^2}$ (best)	$e_w^{L^2}$ (final)	$e_w^{L^2}$ (best)	best ep.
PINN	NN (L3N50)	4.12%	3.82%	0.46%	0.16%	99811
PINN	bubble	0.010%	0.011%	0.003%	0.002%	77267
VPINN	NN (L3N50)	2.31%	2.88%	1.28%	1.18%	99363
VPINN	bubble	1.35%	1.33%	1.32%	1.04%	99832

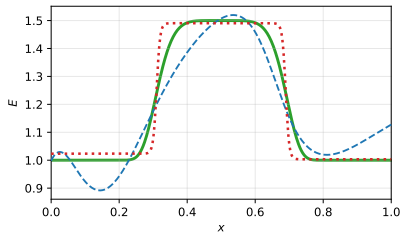
Numerical tests PINN-VPINN: $\mathbb{R}$ -case and theoretical context

E Gaussian

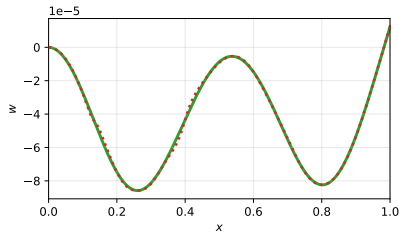
Training: Adam optimizer with $\text{lr} = 10^{-3}$, adaptive_curr (10k warmup on data $\rightarrow$ adaptive_grad);

Architecture: bubble transitions fixed at $[\tilde{x}_1, \tilde{x}_2] = [0.310, 0.690]$ (L^2 -optimal), L^2 floor on E : 3.4%

Solver	Arch	best ep.	$\mathcal{L}$	$\mathcal{L}_{\text{data}}$	$\mathcal{L}_{\text{phys}}$	$\mathcal{L}_{\text{bc}}$
PINN	NN	48332	2.16E-6	1.62E-6	4.96E-7	4.46E-8
x PINN	bubble	10000	2.84E-5	2.84E-5	9.30E+2	3.56E-4
VPINN	NN	49877	3.85E-8	6.91E-9	2.31E-9	2.93E-8
VPINN	bubble	99994	1.93E-6	1.88E-6	1.71E-9	2.21E-8



— E true - - PINN, DNN ··· VPINN, bubble



— w true ··· VPINN, bubble

Solver	Arch	ep.	$e_E^{L^2}$ (f)	$e_E^{L^2}$ (b)	$e_w^{L^2}$ (f)	$e_w^{L^2}$ (b)	best ep.
PINN	NN (L3N50)	150k	5.51%	5.47%	0.50%	0.21%	48332
PINN	bubble	100k	18.42%	23.84%	3.97%	1.52%	10000
VPINN	NN (L3N50)	150k	9.76%	9.30%	1.91%	1.92%	49877
VPINN	bubble	100k	3.66%	3.66%	1.78%	1.78%	99994

Numerical tests PINN-VPINN: $\mathbb{R}$ -case and theoretical context

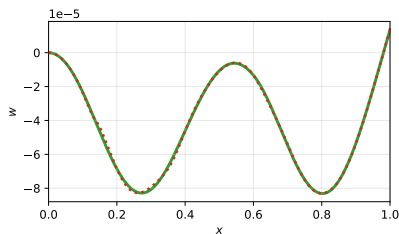
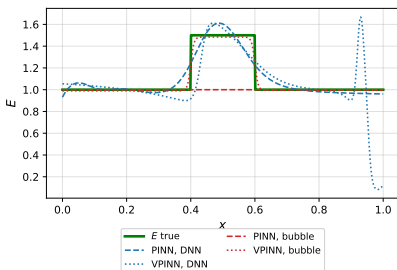
E discontinuous

Training for $\hat{E}$ NN: warmup 100k (λ_{data} only $\rightarrow$ all λ fixed to 1)

Training for $\hat{E}$ bubble: adaptive_curr 100k (10k warmup $\rightarrow$ adaptive_grad);

Architecture: bubble transitions fixed at $[\tilde{x}_1, \tilde{x}_2] = [0.4, 0.6]$, L^2 floor on E : 2.93%

Solver	best ep.	$\mathcal{L}$	$\mathcal{L}_{\text{data}}$	$\mathcal{L}_{\text{phys}}$	$\mathcal{L}_{\text{bc}}$
PINN NN	85 752	1.70E-6	1.45E-6	2.41E-7	1.78E-8
VPINN NN	99 953	9.66E-9	2.62E-9	3.81E-9	3.22E-9
x PINN bubble	9 781	9.81E-6	9.81E-6	1.32E+2	5.30E-4
VPINN bubble	98 152	2.91E-7	2.07E-7	3.46E-9	1.51E-8



Solver	ep.	$e_E^{L^2}$ (f)	$e_E^{L^2}$ (b)	$e_w^{L^2}$ (f)	$e_w^{L^2}$ (b)	best ep.
PINN NN	100k	6.93%	6.90%	0.25%	0.24%	85 752
VPINN NN	100k	20.19%	20.58%	1.95%	1.16%	99 953
PINN bubble	100k	18.34%	19.98%	2.35%	1.08%	9 781 [†]
VPINN bubble	100k	2.93%	2.93%	1.52%	1.50%	98 152

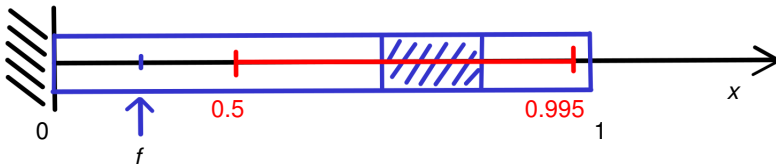
Numerical tests PINN-VPINN: $\mathbb{C}$ -case and application context

Physical framework

- ▶ Steel cantilever beam $\Omega =]0, 1[\text{ m}$, $I = 1.3333e - 11 \text{ m}^4$, $S = 4e - 5 \text{ m}^2$, $\rho = 7080 \text{ kg/m}^3$, with a viscoelastic patch on $[0.7, 0.8]$
- ▶ Observations of the complex displacement $w = w_r + i w_i$ on $\Omega_x = [0.5, 0.995] \text{ m}$ only;
- ▶ fixed frequency at 500 Hz and point load $f = 1 \text{ N } \delta(x - x_f)$, $x_f = 0.25 \text{ m}$
- ▶ $x_f \notin \Omega_x$: forcing term absent from residual on $\Omega_x \Rightarrow$ BC traces at $x = 0.5 \text{ m}$ unknown

Unknowns: $w(x) = w_r(x) + i w_i(x)$ and $E(x) = E_r(x) + i E_i(x)$

- ▶ $E_r = 2.05 \times 10^{11} \text{ Pa}$ (constant — steel)
- ▶ $E_i(x) = \begin{cases} 1.1275 \times 10^9 \text{ Pa} & \text{steel} = \Omega_x \setminus [0.7, 0.8] \\ 1.025 \times 10^{11} \text{ Pa} & \text{patch} = [0.7, 0.8] \end{cases}$
- ▶ Jump factor ≈ 91 (≈ 2 orders of magnitude)
- ▶ Loss factor: $\eta = E_i/E_r$ — from 0.55% (steel) to 50% (patch)



Numerical tests PINN-VPINN: $\mathbb{C}$ -case and application context

Numerical framework

Reference solution

Solution analytically computed on each subdomain where E is constant

Fixed hyperparameters

- ▶ Adam optimizer with $\text{lr} = 10^{-3}$ and epochs: 150k (5k data-only warmup + 5k PDE warmup)
- ▶ $\hat{w}$ **NN**: L3N64 (tanh), two outputs ($\hat{w}_r, \hat{w}_i$); normalized between $[-1, 1]$
- ▶ $\hat{E}$ **NN**: L2N64 (tanh+softplus), two outputs ($\hat{E}_r, \hat{E}_i$), scaled by $E_{\text{scale}} = 10^{11}$ Pa
 $\hat{E}$ **bubble**: $n=3$ plateaux, sharpness $k=300$; separate learnable amplitudes $a_j^{(r)}, a_j^{(i)} \geq 0$ for $\hat{E}_r$ and $\hat{E}_i$; possible learnable transitions points ($\tilde{x}_1, \tilde{x}_2$) on Ω_x
- ▶ Loss weights: $\lambda_{\text{data}} = 1000, \lambda_{\text{phys}} = 1$
- ▶ Observations: $N_{\text{obs}} = 100$ on Ω_x

PINN vs VPINN

- ▶ **PINN**: $N_{\text{coll}} = 512$ collocation points on Ω_x
- ▶ **VPINN**: weak residual $\mathcal{R}_h(\hat{w}, \hat{E}; v_k) = \int_{\Omega_x} \hat{E} \hat{w}'' v_k' dx - \rho S \omega^2 \int_{\Omega_x} \hat{w} v_k dx$;
 $K = 24$ test functions $v_k(s) = b(s) F_m(s)$, where $b = b' = 0$ at $\partial\Omega_x$ and F_m Fourier basis;
 $N_{\text{quad}} = 1024$ GL pts on Ω_x ;

Numerical tests PINN-VPINN: $\mathbb{C}$ -case and application context

Losses

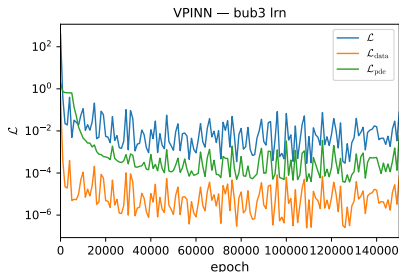
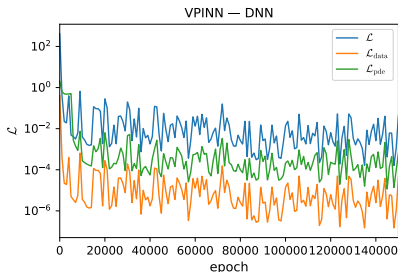
Losses at best epoch

Solver	arch	best ep.	$\mathcal{L}$	$\mathcal{L}_{\text{data}}$	$\mathcal{L}_{\text{pde}}$	$\mathcal{L}_{\text{pde},r}$	$\mathcal{L}_{\text{pde},i}$
PINN	NN	149214	2.34E-4	1.76E-7	5.77E-5	2.32E-5	3.45E-5
VPINN	NN	144496	1.56E-4	1.42E-7	1.37E-5	9.18E-7	1.27E-5
VPINN	bub3 fix	148741	2.37E-4	1.99E-7	3.76E-5	9.26E-6	2.83E-5
VPINN	bub3 lrn	142863	2.65E-4	2.31E-7	3.35E-5	5.25E-6	2.82E-5

Losses at final epoch (150k)

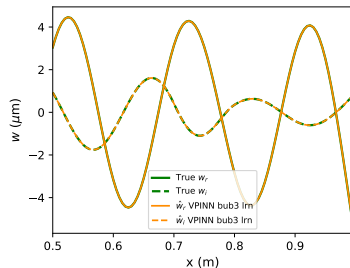
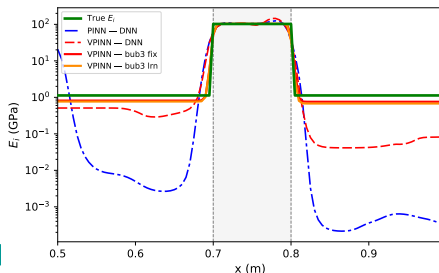
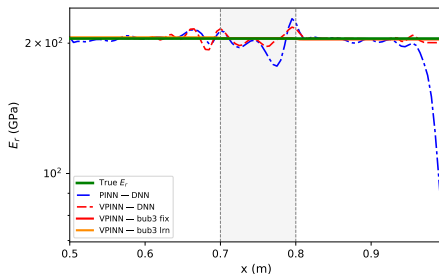
Solver	arch	best ep.	$\mathcal{L}$	$\mathcal{L}_{\text{data}}$	$\mathcal{L}_{\text{pde}}$	$\mathcal{L}_{\text{pde},r}$	$\mathcal{L}_{\text{pde},i}$
PINN	NN	149214	1.01E-2	9.98E-6	1.40E-4	6.89E-5	7.06E-5
VPINN	NN	144496	4.43E-2	4.37E-5	6.45E-4	5.13E-5	5.93E-4
VPINN	bub3 fix	148741	7.60E-4	6.69E-7	9.06E-5	2.27E-5	6.79E-5
VPINN	bub3 lrn	142863	7.83E-2	7.24E-5	5.86E-3	7.13E-4	5.14E-3

$$\lambda_{\text{data}} = 1000, \lambda_{\text{phys}} = 1; \mathcal{L} = 1000 \mathcal{L}_{\text{data}} + \mathcal{L}_{\text{pde}}$$



Numerical tests PINN-VPINN: $\mathbb{C}$ -case and application context

Plots : w , E at best epoch for each method



Deflection w_r , w_i — VPINN bubble learn (best ep. 142863)

Numerical tests PINN-VPINN: $\mathbb{C}$ -case and application context

L^2 -relative errors on w , E

Solver	arch	best ep.	L^2 err on E_r (%)			glob
			steel ₁	patch	steel ₂	
PINN	NN	149214	1.80	6.88	15.84	10.44
VPINN	NN	144496	2.03	3.24	1.12	2.08
VPINN	bub3 fix	148741	0.24	0.21	0.40	0.31
VPINN	bub3 lrn	142863	0.66	0.06	0.35	0.47

Solver	arch	best ep.	L^2 err on E_i (%)			glob
			steel ₁ *	patch	steel ₂ *	
PINN	NN	149214	575.41	19.90	221.83	21.98
VPINN	NN	144496	435.96	23.84	163.90	24.86
VPINN	bub3 fix	148741	67.09	8.04	69.24	15.41
VPINN	bub3 lrn	142863	71.86	9.81	49.27	17.72

Solver	arch	best ep.	L^2 err on w (%)	
			w_r	w_i
PINN	NN	149214	0.037	0.218
VPINN	NN	144496	0.032	0.197
VPINN	bub3 fix	148741	0.050	0.202
VPINN	bub3 lrn	142863	0.068	0.156

Relative errors on transition points: 0.012% - 0.133%

One-frequency VPINN model with bubble parametrization for E

- ▶ ✓ Bubble parametrization: promising architecture to catch discontinuous signals
- ▶ ✓ VPINN approach allows to deal with non-smooth parameter identifications
- ▶ ✗ improve the steel identification when E is discontinuous with great different orders of magnitude

Work in progress

- ▶ Analysis on different families of test functions for VPINN (global vs local test functions)
[Inverse identification of a localized complex modulus in a cantilever beam using VPINNs and localized test functions, M. Auger *et al.*, EURODINAME III, France, 2026]
- ▶ Learning the number of bubble basis
- ▶ Stop criterium for training (coupling different indicators to establish a score linked to a meaningful inverse solution)

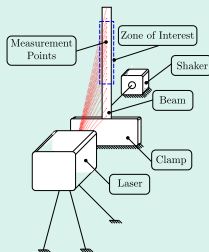
Conclusion and perspectives

Perspectives

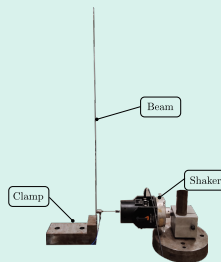
- ▶ Multi-frequency model extension (neural networks: $(x, \omega) \mapsto \hat{w}_r, \hat{w}_i, \hat{E}_r, \hat{E}_i$)
- ▶ Test on noisy data and experimental data

PINN for homogeneous E ✓

A physics-informed neural networks framework for model parameter identification of beam-like structures, Teloli *et al.*, Mechanical Systems and Signal Processing (MSSP), 2025



(a) Schematic representation



(b) Experimental setup



$$\mathcal{R}_f^{(\text{real})}(\mathcal{W}, \mathcal{E}) = \frac{1}{M} \sum_{m=1}^M \left| \frac{l}{\rho S \omega^2} \left(\hat{E}_{r,m} \hat{w}_{r,m}^{(4)} - \hat{E}_{i,m} \hat{w}_{i,m}^{(4)} + \hat{E}_{r,m}'' \hat{w}_{r,m}'' - \hat{E}_{i,m}'' \hat{w}_{i,m}'' \right) + \frac{2l}{\rho S \omega^2} \left(\hat{E}_{r,m}' \hat{w}_{r,m}''' - \hat{E}_{i,m}' \hat{w}_{i,m}''' \right) - \hat{w}_{r,m} \right|^2$$

$$\mathcal{R}_f^{(\text{imag})}(\mathcal{W}, \mathcal{E}) = \frac{1}{M} \sum_{m=1}^M \left| \frac{l}{\rho S \omega^2} \left(\hat{E}_{r,m} \hat{w}_{i,m}^{(4)} + \hat{E}_{i,m} \hat{w}_{r,m}^{(4)} + \hat{E}_{r,m}'' \hat{w}_{i,m}'' + \hat{E}_{i,m}'' \hat{w}_{r,m}'' \right) + \frac{2l}{\rho S \omega^2} \left(\hat{E}_{r,m}' \hat{w}_{i,m}''' + \hat{E}_{i,m}' \hat{w}_{r,m}''' \right) - \hat{w}_{i,m} \right|^2$$



Amplification in k

The physical wavenumber is set by the dispersion relation (constant EI):

$$EI k^4 = \rho S \omega^2 \implies k = \left(\frac{\rho S \omega^2}{EI} \right)^{1/4}.$$

For the considered test cases:

- ▶ **Synthetic case** ($\rho S = 1$, $EI \approx 1$, $\omega = 150$ rad/s): $k \approx 12$ rad/m, $k^4 = 2.25 \times 10^4$ m⁻⁴.
- ▶ **Steel beam** ($\rho S = 0.283$ kg/m, $EI = 2.73$ N·m², $f = 500$ Hz): $k \approx 32$ rad/m, $k^4 \approx 10^6$ m⁻⁴.

Moderate amplification at the physical wavenumber.

Critical issue: neural networks introduce spurious high-frequency spatial components with $k_{nn} \gg k_{phys}$, for which $k_{nn}^4 \gg k_{phys}^4$, and these dominate the PDE residual.



Sigmoid transitions

$\sigma(t) = 1/(1 + e^{-t})$ transitions from 1% to 99% over:

$$\sigma(t) = 0.01 \Leftrightarrow t = -\ln 99$$

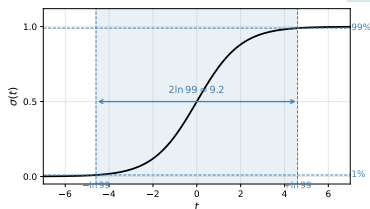
$$\sigma(t) = 0.99 \Leftrightarrow t = +\ln 99$$

$$\Delta t = 2 \ln 99 \approx 9.2$$

Since $B_j(s) = \sigma(k(s - e_{j-1})) - \sigma(k(s - e_j))$, the argument is $k(s - e_j)$, so the transition width in s is:

$$\Delta s = \frac{9.2}{k}$$

E.g. $k = 200$: $\Delta s \approx 4.6\%$ of domain.

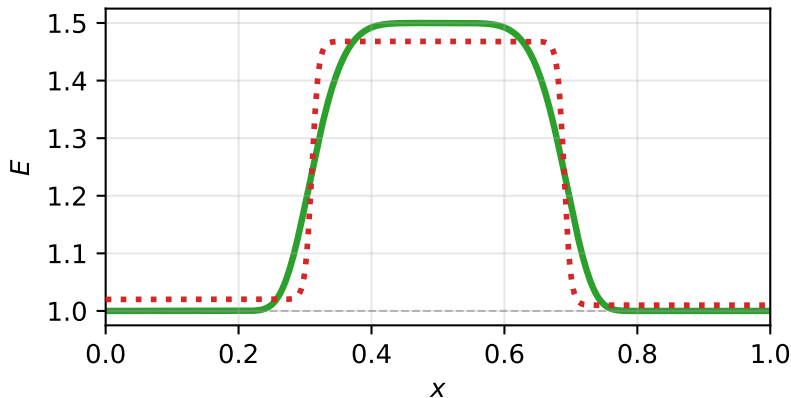


Best approximated solution Real Case

Reference solution

- ▶ FEM: cubic Hermite C^1 , $N_{e1} = 200$, cantilever BC
- ▶ $\omega = 150$ rad/s; $E(x) = 1 + 0.5 e^{-((x-0.5)/0.2)^6}$

L^2 floor = 3.41% ($k=200$, $n=3$)



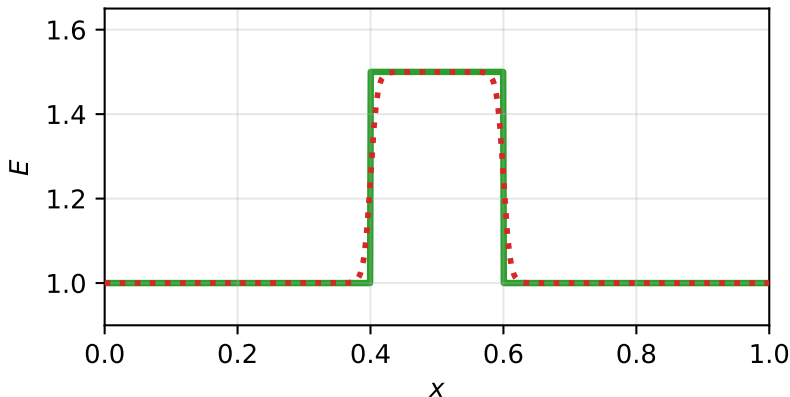
Best approximated solution Real Case

E discontinuous

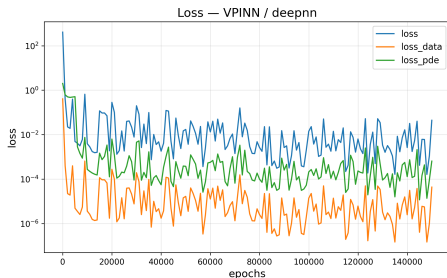
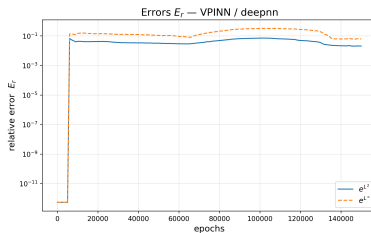
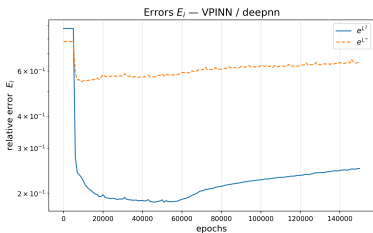
Reference solution

- ▶ FEM: cubic Hermite C^1 , $N_{e1} = 200$, cantilever BC
- ▶ $E(x) = 1 + 0.5 \mathbf{1}_{[0.4, 0.6]}(x)$

sigmoid floor $\approx 2.93\%$ ($k=200$, oracle tr.)



Errors vs Losses: VPINN DNN



Errors vs Losses: VPINN bubble learning transitions points

