Control of input-output contact systems

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Abstract: Control input-output contact systems are the representation of open irreversible Thermodynamic systems whose geometric structure is defined by Gibbs' relation. These systems are called conservative if furthermore they leave invariant a particular Legendre submanifold defining their thermodynamic properties. In this paper we address the stabilization of controlled input-output contact systems. Firstly it is shown that it is not possible to achieve stability on the complete Thermodynamic Phase Space. As a consequence, the stabilization is addressed on some invariant Legendre submanifold of the closed-loop system. For structure preserving feedback of input-output contact systems, i.e., for the class of feedback that renders the closed-loop system again a contact system, the closed-loop invariant Legendre submanifolds have been characterized. The stability of the closed-loop system has then been proved using Lyapunov's second method. The results are illustrated on the classical thermodynamic process of heat transfer between two compartments and an exterior control.

Keywords: Contact systems, Irreversible thermodynamics, Feedback stabilization, Heat exchanger.

1. INTRODUCTION

The Thermodynamic Phase Space, consisting of the $n+1$ extensive variables and n intensive variables defining a thermodynamic system, is structured by Gibbs' equation which endows it with a canonical differential-geometric structure called *contact structure* (Hermann, 1973; Libermann and Marle, 1987; Arnold, 1989; Mrugala et al., 1991).

Controlled control systems have been introduced in Eberard et al. (2005) and further developed in Eberard et al. (2007); Favache et al. (2010); Ramirez et al. (2013) leading to define input-output contact systems (Ramirez et al., 2011b; Ramirez, 2012). They are the analogue for the controlled irreversible thermodynamic systems of the input-output Hamiltonian systems (Brockett, 1977; van der Schaft, 1986, 1989; Marsden, 1992) developed for mechanical systems. However concerning the structure preserving feedback, the situation is quite different for input-output contact systems. It has been shown in (Ramirez et al., 2011a; Ramirez, 2012, chap.3) that by state-feedback one cannot preserve the differential-geometric structure, the contact form, of these systems but only transform it to another one, in a very similar way as assigning closed-loop structure matrices of Port Hamiltonian Systems in the IDA-PBC method (Ortega et al., 2002).

The problem of stabilization of input-output contact systems by output-feedback is addressed. Few papers exist on the dynamic properties of contact vector fields. Using linearisation techniques, a local stability study of dynamical systems defined by contact vector fields has been performed in Favache et al. (2009) in the case when these vector field leaves invariant some Legendre submanifolds. In the sequel, we show that it is impossible, by output feedback, to achieve the asymptotic stabilization of an equilibrium point but only stability with respect to some Legendre submanifold rendered invariant by feedback. To this end the possible invariant Legendre submanifolds are characterized and then by using Lyapunov's second method on the restriction of the closed-loop contact vector field to some invariant Legendre submanifold, stability assured.

The paper is organized as follows. In the Section 2 we recall the definition of input-output contact systems and of the main mathematical tools. The main results are presented in Section 3, where we characterize the closed-loop invariant Legendre submanifolds. A thermodynamic process, namely the heat exchanger, is used to illustrate the results in Section 4. Finally in Section 5 some closing remarks and lines of future work are given.

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2. INPUT-OUTPUT CONTACT SYSTEMS

In this section we shall briefly recall the definition and main properties of a class of nonlinear control systems, called *input-output contact systems*, that arise when modelling control systems in chemical engineering or any process where the internal energy (or entropy) balance equation is written (Eberard et al., 2007; Favache et al., 2009, 2010; Ramirez, 2012).

2.1 Contact manifolds, contact vector fields and Legendre submanifolds

The Thermodynamic Phase Space, consisting of the $n+1$ extensive variables and n intensive variables defining a thermodynamic system, is structured by Gibbs' equation which endows it with a canonical differential-geometric structure called *contact structure* (Hermann, 1973; Mrugala et al., 1991). In the sequel we shall recall briefly the main definitions and properties of contact geometry used in this paper; the reader is referred to the following textbooks for a detailed definition (Arnold, 1989, app. 4.), (Libermann and Marle, 1987, chap. 5).

We shall consider systems defined on state-spaces which are *contact manifolds*, that are $(2n+1)$-dimensional differential manifolds $\mathcal{M} \ni \tilde{x}$ equipped with a *contact form* denoted by θ which, according to Darboux's theorem (Libermann and Marle, 1987, pp. 288), is defined in a set of *canonical coordinates* $(x_0, x, p) \ni \tilde{x} \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ by

$$\theta = dx_0 - \sum_{i=1}^{n} p_i \, dx_i.$$

In order to define the input-output contact systems, we shall use *strict contact vector fields* (Libermann and Marle, 1987, chap. 6), that is vector fields X which leaves the contact form invariant (i.e. $L_X \theta = 0$ where L_X denotes the Lie derivative with respect to X). These strict contact vector fields are uniquely defined by a function $K \in C^\infty(\mathcal{M})$, called *contact Hamiltonian* and are expressed, in a set of canonical coordinates $(x_0, x, p) \ni \mathbb{R}^{(2n+1)}$, by

$$X = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -p^\top \\ 0 & 0 & -I_n \\ p & I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial K}{\partial x_0} \\ \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial p} \end{bmatrix}, \qquad (1)$$

where I_n denotes the identity matrix of order n and the function K satisfies $\frac{\partial K}{\partial x_0} = 0$.

Furthermore the contact form defines a set of distinguished submanifolds, called *Legendre submanifolds* which are defined by the Pfaffian equation: $\theta = 0$.

Definition 1. (Libermann and Marle, 1987) A Legendre submanifold of a $(2n+1)$-dimensional contact manifold $(\mathcal{M}, \theta)$ is an n-dimensional integral submanifold $\mathcal{L} \subset \mathcal{M}$ of θ.

In some set of canonical coordinates, the Legendre submanifold is defined by a generating function $U \in C^\infty(\mathbb{R}^n)$ as follows

$$\mathcal{L}_U = \left\{ x_0 = U(x), x = x, p = \frac{\partial U}{\partial x}(x), \ x \in \mathbb{R}^n \right\}.$$

The Legendre submanifolds characterize the properties of thermodynamic systems and are the differential-geometric formulation of Equilibrium Thermodynamics, i.e. Gibbs' equation which is at the root of contact geometry (Arnold, 1989, app. 4.)(Hermann, 1973; Mrugała, 1978; Mrugala et al., 1991).

Contact vector fields may satisfy an additional condition, namely that they leave some Legendre submanifold invariant (defining for instance the equilibrium properties of a system) and may be checked using the following proposition.

Proposition 1. (Mrugala et al., 1991) Let $\mathcal{L}$ be a Legendre submanifold. Then X_K is tangent to $\mathcal{L}$ if and only if K vanishes on $\mathcal{L}$, i.e., $\mathcal{L} \subset K^{-1}(0)$.

2.2 Input-output contact systems

Using the definition of contact vector fields, we may define input-output contact systems as follows.

Definition 2. (Ramirez, 2012) A *(single) input - (single) output contact system* affine in the scalar input $u(t) \in L_1^{\text{loc}}(\mathbb{R}_+)$ is defined by the two functions $K_0 \in C^\infty(\mathcal{M})$, called the *internal contact Hamiltonian* and $K_c \in C^\infty(\mathcal{M})$ called the *interaction (or control) contact Hamiltonian*, the state equation

$$\frac{d\tilde{x}}{dt} = X_{K_0} + X_{K_c} u, \qquad (2)$$

where X_{K_0} and X_{K_c} are the contact vector fields generated by K_0 and K_c with respect to the contact form θ, and the output relation

$$y = K_c(\tilde{x}). \qquad (3)$$

Such systems may represent the dynamical models of open thermodynamic systems. However they should satisfy an additional assumption: they should leave invariant the thermodynamic properties of the system. This may be expressed by the fact that both the internal and the interaction contact Hamiltonians satisfy Proposition 1. This motivates the definition of *conservative controlled contact systems.*

Definition 3. (Eberard et al., 2007) A *conservative* input-output contact system is an input-output contact system with the contact Hamiltonians satisfying Proposition 1: $K_0|_{\mathcal{L}} = 0, \quad K_c|_{\mathcal{L}} = 0$ with respect to some Legendre submanifold $\mathcal{L}$.

Conservative input-output contact systems allow to represent open physical systems subject to irreversible phenomena. In this case, the contact Hamiltonian may be interpreted as a virtual power associated with the reversible and irreversible phenomena inducing the dynamics. The Legendre submanifold corresponds to the definition of the total energy of the system (or any Legendre transformation of it). The reader may find the construction of these systems from sets of balance equations in Eberard et al. (2007); Favache et al. (2010) as well as different examples such as the Continuous Stirred Tank Reactor or a gas-piston system in Eberard et al. (2007); Favache et al. (2010); Favache (2009); Ramirez et al. (2013).

3. STRUCTURE PRESERVING STABILIZATION OF INPUT-OUTPUT CONTACT SYSTEMS

In this section we consider the problem of stabilizing structure preserving state-feedback of input-output contact systems. That is, state-feedbacks which lead to a closed-loop system which is a again a contact system and also stabilize the system at some desired equilibrium point. However it has been shown that structure preserving feedback, i.e. a feedback that renders the closed-loop system again a contact system, necessarily changes the closed-loop contact form (Ramirez et al., 2011a; Ramirez, 2012). This is quite different from structure preserving control of Hamiltonian system for which there exist non trivial state-feedbacks which preserve the symplectic bracket in closed-loop. But this means that, under some conditions which are recalled in the subsection that follows, one may assign the contact form in closed-loop (Ramirez et al., 2011a; Ramirez, 2012).

3.1 Structure preserving feeeback

In this subsection we briefly recall the results presented in Ramirez (2012); Ramirez et al. (2011a) on the state feedbacks $u = \alpha(\tilde{x})$ such that the closed-loop vector field

$$X = X_{K_0} + X_{K_c}\alpha(\tilde{x}) \tag{4}$$

is a contact vector field with respect to some closed-loop contact form θ_d which may be different from the open-loop one, θ. For the sake of brevity, we express all results using a set of canonical coordinates $(x_0, x, p) \ni \tilde{x} \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ (a global, i.e., coordinate free, derivation is given in Ramirez (2012); Ramirez et al. (2011a)). In a first instance we shall restrict the class of achievable closed-loop contact forms with the following assumption.

Assumption 4. The closed-loop contact form θ_d is defined as

$$\theta_d = \theta + dF, \tag{5}$$

with $F \in C^\infty(\mathcal{M})$ satisfying $\frac{\partial F}{\partial x_0} = 0$.

Proposition 2. (Ramirez, 2012; Ramirez et al., 2011a) The 1-form defined in Assumption 4 is a contact form.

By Assumption 4, the function F depends only on the variables (x, p) and hence the closed-loop contact form θ_d in (5) may be written

$$\begin{aligned}\theta_d = \theta + dF &= \Big(dx_0 - \sum_{i=1}^{n} p_i dx_i\Big) + dF(x,p)\\ &= dx_0' - \sum_{i=1}^{n} p_i dx_i\end{aligned} \tag{6}$$

with $x_0' = x_0 + F(x,p)$. Hence a set of canonical coordinates for the closed-loop contact form θ_d is now given by (x_0', x, p). One may interpret this new set of coordinates as the feedback changing the direction of the x_0-axis which is, in the differential-geometric representation of thermodynamic systems, the coordinate of a thermodynamic potential such as the energy U or the entropy S. This interpretation is in accordance to the one provided in Hermann (1973) and Hermann (1974) for the isothermal interaction of thermodynamic systems using contact geometry.

Proposition 3 characterizes the admissible state feedbacks that render the closed-loop vector field a contact vector field with respect to a modified contact form.

Proposition 3. (Ramirez, 2012; Ramirez et al., 2011a) Consider the contact manifold $(\mathcal{M}, \theta)$ and the smooth real functions $K_0, K_c, F \in C^\infty(\mathcal{M})$, such that $\frac{\partial K_0}{\partial x_0} = \frac{\partial K_c}{\partial x_0} = \frac{\partial F}{\partial x_0} = 0$. Then the closed-loop vector field $X = X_{K_0} + \alpha X_{K_c}$, with $\alpha \in C^\infty(\mathcal{M})$, is a strict contact vector field with respect to the shaped contact form θ_d and the shaped contact Hamiltonian K, respectively, $\theta_d = \theta + dF$ and $K = K_0 + \Phi(y) + c_F$, where $\Phi(y) \in C^\infty(\mathbb{R})$ and $c_F \in \mathbb{R}$, if and only if $\alpha = \frac{d\Phi}{dy}(y) = \Phi'(y)$ and the matching equation

$$X_{K_0}(F) + \Phi'(y)[K_c + X_{K_c}(F)] - \Phi(y) = c_F \tag{7}$$

is satisfied. The closed-loop vector field is then denoted by $X = \hat{X}_K$, where $\hat{X}_K$ denotes the contact vector field generated by K with respect to the contact form θ_d.

The function Φ *shapes* the closed-loop contact Hamiltonian in a very similar manner as for the feedback of input-output Hamiltonian systems (van der Schaft, 1986) or the Casimir method for port-Hamiltonian systems (van der Schaft, 2000). As it has been discussed in Ramirez (2012); Ramirez et al. (2011a) the linear PDE (7) may be solved by using classical methods such as the method of characteristics (Abbott, 1966; Evans, 1998; Myint-U and Debnath, 2007).

3.2 Characterization of closed-loop invariant Legendre submanifolds

In this section we shall develop the previous results by imposing an additional constraint on the closed-loop contact system: it should also be a *conservative* contact system. Hence it should leave invariant some desired Legendre submanifold, denoted by $\mathcal{L}_d$, with respect to the closed-loop contact form θ_d. In this way the closed-loop system may again be interpreted in terms of a thermodynamic system. As already mentioned, the Legendre submanifold is defined by the energy of the system (or some Legendre transform of it) thus shaping the invariant Legendre submanifold in closed-loop may be interpreted as shaping the energy of the system in closed-loop.

In general the Legendre submanifolds with respect to the closed-loop contact form θ_d are not Legendre submanifolds with respect to the open-loop contact form θ. Proposition 4 characterizes the conditions for $\mathcal{L}_d$ to be an invariant of the closed-loop vector field X.

Proposition 4. Under the conditions of Proposition 3, the closed-loop contact vector field X in (4) leaves a Legendre submanifold $\mathcal{L}_d$ (with respect to the closed-loop contact form θ_d) invariant if and only if

$$K_0\big|_{\mathcal{L}_d} + \Phi(y)\big|_{\mathcal{L}_d} = -c_F. \tag{8}$$

Proof. From the expression of the closed-loop contact Hamiltonian in Proposition 3 we have, $K\big|_{\mathcal{L}_d} = K_0\big|_{\mathcal{L}_d} + \Phi(y)\big|_{\mathcal{L}_d} + c_F$, which proves the result. ■

Proposition 4 defines a criteria for the choice of the function Φ, namely the condition such that the contact vector field leaves $\mathcal{L}_d$ invariant. Once a suitable Φ is chosen the state feedback is obtained as $\alpha = \Phi'(y)$ (Proposition 3).

The desired closed-loop Legendre submanifold may be related with some specified performance property, as for instance a reference temperature. Denote the generating function of the closed-loop Legendre submanifold U_d and the closed-loop Legendre submanifold $\mathcal{L}_{U_d}$. Then, in the new canonical coordinates

$$\mathcal{L}_{U_d}: \left\{ x_0^d = U_d(x), \quad x = x, \quad p = \frac{\partial U_d}{\partial x}(x) \right\}.$$

In order to design a feedback law that leaves $\mathcal{L}_{U_d}$ invariant, (8) should be evaluated on the restriction defined by $p = \frac{\partial U_d}{\partial x}(x)$ and solved for $\Phi(y)$. Note that (8) is parametrized by the contact Hamiltonian function and $\mathcal{L}_{U_d}$. Furthermore, the control contact Hamiltonian K_c defines the argument of $\Phi(y) = \Phi \circ K_c(x, p)$, hence the achievable closed-loop Legendre submanifolds are essentially depending on the control contact Hamiltonian.

3.3 About stability of equilibrium points

Favache et al. (2009) have given conditions for the existence of an equilibrium point of a contact vector field and then, for a conservative contact Hamiltonian leaving invariant some Legendre submanifold $\mathcal{L}$, they have given conditions for the stability of such an equilibrium on the restriction of the contact vector field to the invariant Legendre submanifold $\mathcal{L}$. In this section we shall show that one cannot expect to stabilize any larger invariant submanifold than such a Legendre submanifold.

Therefore we shall analyse the linearisation of a contact field around some equilibrium point. Let us first recall the condition for the existence of an equilibrium point.

Lemma 5. (Favache et al., 2009) Consider a contact manifold $(\mathcal{M}, \theta)$ with canonical coordinates $(x_0, x, p) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ and a contact vector field X_K generated by the contact Hamiltonian function $K(x_0, x, p)$ satisfying $\frac{\partial K}{\partial x_0} = 0$. A point (x_0^*, x^*, p^*) is an equilibrium point if and only if it is a zero and a critical point of the contact Hamiltonian K, that is satisfies: $K(x_0^*, x^*, p^*) = 0$ and $\frac{\partial K}{\partial x} = \frac{\partial K}{\partial p} = 0$.

Let us now go further and analyse the stability around the equilibrium by studying the Jacobian DX_K of the contact field X_K expressed in some canonical coordinates.

Proposition 6. Under the conditions of Lemma 5, let us consider an equilibrium state (x_0^*, x^*, p^*) of the contact vector field X_K. Then zero is eigenvalue of DX_K and the remaining $2n$ eigenvalues are symmetrical with respect to the imaginary axis.

Proof. Since $\frac{\partial K}{\partial x_0}(x_0, x, p) = 0$, the Jacobian is given by

$$DX_K = \begin{bmatrix} 0 & \left(\frac{\partial^\top K}{\partial x} - {p^*}^\top \frac{\partial^2 K}{\partial x \partial p}\right) & -{p^*}^\top \frac{\partial^2 K}{\partial p^2} \\ 0 & -\frac{\partial^2 K}{\partial x \partial p} & -\frac{\partial^2 K}{\partial p^2} \\ 0 & \frac{\partial^2 K}{\partial x^2} & \left(\frac{\partial^2 K}{\partial x \partial p}\right)^\top \end{bmatrix}. \quad (9)$$

According to Lemma 5, $\frac{\partial K}{\partial x}(x_0^*, x^*, p^*) = 0$ and hence we may rewrite (9) at the equilibrium point as

$$DX_K(x_0^*, x^*, p^*) = \begin{bmatrix} 0 & -{p^*}^\top A & -{p^*}^\top B \\ 0 & -A & -B \\ 0 & C & A^\top \end{bmatrix} \quad (10)$$

with $A = \frac{\partial^2 K}{\partial x \partial p}(x_0^*, x^*, p^*)$, $B = B^\top = \frac{\partial^2 K}{\partial p^2}(x_0^*, x^*, p^*)$, $C = C^\top = \frac{\partial^2 K}{\partial x^2}(x_0^*, x^*, p^*)$. The characteristic polynomial of DX_K is given by $\det(DX_K - \lambda I)$, where $\det(\cdot)$ denotes the determinant, and may be evaluated by using cofactor expansion with respect to the first column and the properties of the determinant of block matrices (Meyer, 2000).

$$\begin{aligned} \det(DX_K - \lambda I) &= -\lambda \det\left(\begin{bmatrix} -(A - \lambda I) & -B \\ C & (A - \lambda I)^\top \end{bmatrix}\right), \\ &= \lambda \det(A - \lambda I) \det\left((A + \lambda I) - B(A - \lambda I)^{-1} C\right), \\ &= \lambda \det(A + \lambda I) \det\left((A - \lambda I) - C(A + \lambda I)^{-1} B\right), \end{aligned}$$

where it has been assumed that the inverse matrices $(A - \lambda I)^{-1}$ and $(A + \lambda I)^{-1}$ are computed for values of λ where they exist. It follows that $\lambda = 0$ is always an eigenvalue and that the remaining $2n$ eigenvalues are symmetrical with respect to the imaginary axis. ■

The immediate consequence of Proposition 6 is that an equilibrium point of a contact vector field cannot be asymptotically stable on the complete Thermodynamic Phase Space. Furthermore at most n eigenvalues may have strictly negative real part and the asymptotically stable submanifold at the equilibrium point may be of dimension at most n. This is precisely similar to the case of Hamiltonian systems (van der Schaft, 2000, chap. 8). This justifies to study stability of input-output contact system by structure preserving feedback on the restriction to an invariant Legendre submanifold $\mathcal{L}_d$.

4. EXAMPLE: THE HEAT EXCHANGER

Control contact systems corresponding to physical models are a subclass of the control contact systems of Definition 2 and may be considered as the lift of systems of balance equations to the complete Thermodynamic Phase Space (Eberard et al., 2007; Favache et al., 2010). We illustrate the results of this paper with the example of the control contact system associated to the model of two compartments exchanging heat flow through a heat conducting wall and one of the compartments exchanging heat flow with the environment. For simplicity the system is briefly called "heat exchanger". In the sequel we shall give briefly the definition of the control contact system representing the heat exchanger, which is obtained by lifting the two entropy balance equations to the complete Thermodynamic Phase Space. It should be noted that these lifts are not unique (Favache et al., 2009) and that the one given below is different from the one suggested in (Eberard et al., 2007, pp. 190) but still defines the same dynamics on the invariant Legendre submanifold defining the thermodynamic properties of the system.

The physical system consists in two simple thermodynamic systems, indexed by 1 and 2 (for instance two ideal gases), which may interact only through a heat conducting wall and where compartment 2 exchanges heat flow with a controlled environment. The dynamic of this system is given by the following entropy balance equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \lambda \left(\frac{1}{\frac{\partial U}{\partial x_2}} - \frac{1}{\frac{\partial U}{\partial x_1}} \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \end{bmatrix} + \lambda_e \begin{bmatrix} 0 \\ \frac{u}{\frac{\partial U}{\partial x_2}} - 1 \end{bmatrix}, \quad (11)$$

where the state variable $x = (x_1, x_2)^\top \in \mathbb{R}^2$ is the vector of the entropies x_1 and x_2 of subsystem 1 and 2, $U(x_1, x_2) = U_1(x_1) + U_2(x_2)$ is the internal energy of the overall system composed of the addition of the internal energies of each subsystem, the gradient of the total internal energy $\frac{\partial U}{\partial x_i} = T_i(x_i)$ being the temperatures of each compartment with $T(x_i) = T_0 \exp\left(\frac{x_i}{c_i}\right)$, where T_0 and c_i are constants (Couenne et al., 2006), $\lambda, \lambda_e > 0$ denote Fourier's heat conduction coefficients of the internal and external walls respectively and the controlled input $u(t)$ is the temperature of the external heat source. The Thermodynamic Phase Space is $R^5 \ni (x_0, x_1, x_2, p_1, p_2)^\top$ and its elements correspond respectively to the total internal energy, the entropies and the temperatures. These coordinates are equal to the physical variables only on the Legendre submanifold $\mathcal{L}_U$ generated by the potential $U = U_1 + U_2$,

$$\mathcal{L}_U : \left\{ \begin{array}{ll} x_0 & = U(x_1, x_2) \\ x & = [x_1, x_2]^\top \\ p & = \left[\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right]^\top = [T_1(x_1), T_2(x_2)]^\top \end{array} \right\}. \quad (12)$$

Consider the control contact system defined by the internal and control contact Hamiltonians

$$K_0 = -Rp^\top JT - (T_2 - p_2)\lambda_e \frac{p_2}{T_2}, \quad K_c = e^{-\lambda_e\left(\frac{p_2}{T_2}-1\right)} - 1 \quad (13)$$

with $R(x,p) = \lambda\left(\frac{p_1 - p_2}{T_1 T_2}\right)$ and $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. It may be checked that $K_0\big|_{\mathcal{L}_U} = 0$, $K_c\big|_{\mathcal{L}_U} = 0$, hence according to Definition 3 (and Proposition 1) the contact vector field $X_{K_0} + X_{K_c}u$ leaves the Legendre submanifold $\mathcal{L}_U$ invariant, (i.e. the thermodynamic properties) and its restriction to $\mathcal{L}_U$ is equivalent to the entropy balance equations (11).

4.1 The closed-loop system

Let us first recall that the control contact Hamiltonian is defined as the function

$$\begin{array}{l} K_c : \mathbb{R}^2 \to \,]-1, \infty[\subset \mathbb{R} \\ (x_2, p_2) \mapsto e^{-\lambda_e\left(\frac{p_2}{T_2}-1\right)} - 1 = e^{-\lambda_e\left(\frac{p_2 - T_2}{T_2}\right)} - 1, \end{array}$$

with $T_2(x_2)$ a strictly positive (and increasing) real function. Hence any admissible output feedback $\alpha = \Phi'(y)$ is expressed as a function of (x_2, p_2) as follows:

$$\alpha(x_2, p_2) = \Phi' \circ K_c(x_2, p_2) = \Gamma(T_2(x_2), p_2)$$

and is hence only a function of the co-state p_2 and the temperature of the second compartment which is the only one in direct contact with the environment. This restricts quite a lot the possible feedbacks as a consequence of the requirement that they should be "structure preserving". Note that the control may be expressed in a more simple way as follows. As on its domain of definition K_c may be partially inverted, using $\ln(K_c+1) = \ln\left(e^{-\lambda_e\left(\frac{p_2-T_2}{T_2}\right)}\right) = -\lambda_e\frac{p_2-T_2}{T_2}$, the feedback may actually be expressed more simply as:

$$\alpha(x_2, p_2) = \beta\left(\lambda_e \frac{p_2 - T_2}{T_2}\right) \quad (14)$$

The actual state feedback is restricted to the closed-loop Legendre submanifold $\mathcal{L}_{U_d}$ defined with respect to the generating function $U_d(x)$:

$$u(x_1, x_2) = \alpha\left(x_2, \frac{\partial U_d}{\partial x_2}(x_1, x_2)\right) = \beta\left(\lambda_e \frac{\frac{\partial U_d}{\partial x_2}(x_1, x_2) - T_2(x_2)}{T_2(x_2)}\right) \quad (15)$$

which may be interpreted as a nonlinear function of a "virtual" entropy flux into the compartment 2 induced by a control temperature $\frac{\partial U_d}{\partial x_2}(x_1, x_2)$ defined by the closed-loop Legendre submanifold. But note that the function β may not be chosen freely as its definition depends on the function Φ which should satisfy the invariance conditions (8).

4.2 Stabilizing control

Let us design a control such that the closed-loop contact Hamiltonian associated to the heat exchanger leaves invariant the desired Legendre submanifold generated by the function $U_d(x_1, x_2) = (x_1 + x_2)T^*$, where T^* is a desired temperature. Let us choose the desired Legendre submanifold given by

$$\mathcal{L}_{U_d} : \left\{ \begin{array}{ll} x_0^d & = U_d(x) \\ x & = x \\ p & = \frac{\partial U_d}{\partial x}(x) = [T^* \; T^*]^\top \end{array} \right\}. \quad (16)$$

To verify Proposition 4 using the expression of the contact Hamiltonians (13), we check the invariance condition (8): $K_0\big|_{\mathcal{L}_d} + \Phi(y)\big|_{\mathcal{L}_d} = -c_F$ and we obtain the condition

$$-\lambda_e(T_2 - T^*)\frac{T^*}{T_2} + \Phi(y)\big|_{\mathcal{L}_{U_d}} = 0.$$

This implies that on $\mathcal{L}_{U_d}$, $\Phi(y)\big|_{\mathcal{L}_{U_d}} = \lambda_e(T_2 - T^*)\frac{T^*}{T_2}$.

Remark 5. It should be noticed that the invariance condition is satisfied for the internal Hamiltonian as, on the closed-loop Legendre submanifold $p_1 - p_2 = T^* - T^* = 0$.

Using the results of the previous subsection, one may find a Φ that satisfies the invariance condition as follows

$$\Phi(y) = T^* \ln(y+1) = \lambda_e(T_2 - p_2)\frac{T^*}{T_2}$$

with the restriction to the closed-loop Legendre submanifold being $\Phi(y)\big|_{\mathcal{L}_d} = \Phi(x, T^*) = \lambda_e(T_2 - T^*)\frac{T^*}{T_2}$. The state feedback on the whole TPS is obtained by derivation: $\alpha(x_2, p_2) = \Phi'(y) = \frac{T^*}{y+1} = T^* e^{\lambda_e\left(\frac{p_2}{T_2}-1\right)}$ and the actual control is its restriction to the closed-loop Legendre submanifold

$$u(x_2) = \Phi'(y)\big|_{\mathcal{L}_d} = T^* e^{\lambda_e\left(\frac{T^*}{T_2}-1\right)}.$$

In this case the function β defining the nonlinear control (14) is $\beta(\zeta) = T^* e^{\zeta}$.

The stability of the closed-loop contact vector field may be verified directly on the system (11) as (x_1, x_2) are coordinates for the open- and closed-loop system. Consider the function $V(x_1, x_2) = \frac{1}{2}\sum_{i=1}^{2}(U_i - U_i^*)^2$ where $U_i^* = U_i(x_i^*)$ with $T^* = \frac{\partial U_i}{\partial x_i}(x_i^*)$: it has a global strict minimum at (x_1^*, x_2^*). Furthermore its differential is: $\left[\begin{smallmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{smallmatrix}\right] = \left[\begin{smallmatrix} (U_1 - U_1^*)T_1(x_1) \\ (U_2 - U_2^*)T_2(x_2) \end{smallmatrix}\right]$ and one obtains

$$\frac{dV}{dt} = \lambda\left(\tfrac{1}{T_2} - \tfrac{1}{T_1}\right)\Big(-(U_1 - U_1^*)T_1T_2 + (U_2 - U_2^*)T_2T_1\Big)$$
$$- \lambda_e\big(U_2 - U_2^*\big)\left(T_2 - T^* e^{\lambda_e\left(\frac{T^*}{T_2} - 1\right)}\right)$$
$$= -c\lambda\left(T_1 - T_2\right)^2 - c\lambda_e\big(T_2 - T^*\big)\left(T_2 - T^* e^{\lambda_e\left(\frac{T^*}{T_2} - 1\right)}\right)$$

Here it has been assumed that the two gases have the same properties and that: $U_i = c\exp\left(\frac{x_i}{c}\right) = cT_i$. The Lyapunov stability follows by nothing that $(T_2 - T^*)\left(T_2 - T^*\exp\left(\lambda_e\left(\frac{T^*}{T_2} - 1\right)\right)\right) \geq 0$, with equality only if $T_1 = T_2 = T^*$.

5. CONCLUSIONS

In this paper we have addressed the problem of the stabilization of input-output contact systems by structure preserving output-feedback. These feedback laws are structure preserving in the sense of Ramirez et al. (2011b); Ramirez (2012), meaning that the closed-loop system is again a contact system, with respect to some different contact structure.

It has been shown that it is in general not possible, by structure preserving output-feedback, to achieve asymptotic stabilization of input-output contact systems to some equilibrium point, but only on some invariant Legendre submanifold with respect to the closed-loop contact form. This result fits well with physical-based control design, in the sense that it implies that the system in closed-loop again has a physical interpretation in the sense that it should admit a closed-loop invariant Legendre submanifold, corresponding to a closed-loop energy (or more generally thermodynamic potential).

The results have been illustrated on the classical process of heat transfer between two compartments and an exterior control. Future work with deal with the application to processes such as the Continuous Stirred Tank Reactor.

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