Using System Theory to prove Existence of Non-Linear PDE's ⋆

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Abstract: In this discussion paper we present an idea of using techniques known from systems theory to show existence for a class of non-linear partial differential equations (pde's). At the end of the paper a list of research questions and possibe approaches is given.

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1. INTRODUCTION

Consider the heat equation on the spatial interval $[0,1]$

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) \tag{1}$$

with boundary conditions

$$\frac{\partial x}{\partial \zeta}(0,t) = 0, \qquad \frac{\partial x}{\partial \zeta}(1,t) = f\left(x(1,t)\right) \tag{2}$$

where f represents the boiling curve, see van Gils (2010). The boiling curve describes the non-linear relation between the temperature and the heat flux at the boundary.

We write the non-linear pde (1)–(2) as the linear system

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) \tag{3}$$

$$\frac{\partial x}{\partial \zeta}(0,t) = 0, \qquad \frac{\partial x}{\partial \zeta}(1,t) = u(t) \tag{4}$$

$$y(t) = x(1,t) \tag{5}$$

with the non-linear feedback

$$u(t) = f(y(t)). \tag{6}$$

The linear system (3)–(5) has nice properties, such as existence of unique solutions for any initial condition in $L^2(0,1)$ and any input in $L^2_{loc}(0,\infty)$. The non-linear boiling curve is (uniformly) Lipschitz continuous, and by using Logemann and Ryan (2003) (see also Section 2) we conclude that the non-linear pde (1)–(2) possesses a unique solution for any initial condition.

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In the following section we shall make the above assertion more precise. In Section 3 we show how this idea can be further extended.

2. LOCAL EXISTENCE

The theory of well-posed infinite-dimensional systems is now well-established and can be found in books, see e.g. Staffans (2005) and (Jacob and Zwart, 2012, Chapter 13). In essence well-posedness states that the (abstract) differential equation on the Hilbert space X

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{7}$$

$$y(t) = Cx(t) + Du(t) \tag{8}$$

possesses for every $x_0 \in X$, $u \in L^2_{loc}([0,\infty);U)$ a unique solution with $x(t)$ continuous w.r.t. t, and $y \in L^2_{loc}([0,\infty);Y)$. Hence if the system is well-posed, then we can write the solution as

$$x(t) = T(t)x_0 + \left(\mathcal{B}(u)\right)(t) \tag{9}$$

$$y = \mathcal{C}(x_0) + \mathcal{F}(u), \tag{10}$$

where the first equality holds point-wise in t, and the second is an equality in L^2.

We will assume that our linear pde gives a well-posed linear system. On the non-linearity we assume that it is *locally Lipschitz continuous* from Y to U. Thus for every $r > 0$ there exists a $L(r)$ such that for all $y_1, y_2 \in Y$ satisfying $\|y_1\|, \|y_2\| \leq r$ there holds

$$\|f(y_1) - f(y_2)\| \leq L(r)\|y_1 - y_2\|. \tag{11}$$

If $L(r)$ can be chosen independently of r, then the mapping f is called *uniformly Lipschitz continuous*. Before we can formulate the existence result we need to define the *feed-through bound*

$$\delta := \inf_{t>0} \sqrt{\left(\sup_{u \in L^2([0,t);U)} \frac{\int_0^t \| (\mathcal{F}(u))(\tau)\|^2 d\tau}{\int_0^t \|u(\tau)\|^2 d\tau}\right)} \tag{12}$$

For a finite-dimensional systems $\delta = \|D\|$. This also holds for infinite-dimensional systems if B or C is a bounded operator, see (7) and (8), respectively. If the transfer function $G(s)$ of the system (7)–(8) has the property that

$$\lim_{Re(s)\to\infty} G(s) = 0, \tag{13}$$

then $\delta = 0$ as well. We make the claim that δ equals $\limsup_{Re(s)\to\infty} \|G(s)\|$.

Theorem 1. Consider the well-posed system (9)–(10), and let $L(r)$ denote the Lipschitz constant, see (11), and let δ denote the feed-through bound. If $\delta L(r) < 1$, then there exits a solution of the abstract non-linear differential equation

$$\dot{x}(t) = A + B(f(h(Cx))), \qquad x(0) = x_0 \tag{14}$$

on the time interval $[0, t_{\max})$. Here $y_1 = h(y_2)$ if and only if $y_2 + Df(y_1) = y_1$. If f is uniformly Lipschitz continuous and $\delta = 0$, then $h(y_2) = y_2$ and $t_{\max} = \infty$.

Proof. The proof is done by applying a fixed point argument on (10) with u replaced by $f(y)$. Under the above conditions there will exists a fixed point $y_{fix} \in L^2([0,t_1);Y)$ for sufficiently small t_1. Now substituting $u = f(y_{fix})$ into (9) gives the desired solution of the non-linear equation.

Using this theorem we look once more at the example in Section 1. It is well-known that (3)–(5) defines a well-posed system with $\delta = 0$. Furthermore, the boiling curve is uniformly Lipschitz continuous, and so by Theorem 1 we have that (1)–(2) possesses a unique solution for every initial condition.

3. EXAMPLES

3.1 Heat equation with a non-linear term

Suppose we want to use the above presented technique to prove that the following non-linear pde possesses a unique solution.

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \alpha x(\zeta,t)^2 \tag{15}$$

with homogeneous boundary conditions

$$x(0,t) = 0, \qquad x(1,t) = 0. \tag{16}$$

If we would choose $U = X = Y = L^2(0,1)$ and $f(y) = y^2$, then we have a problem, since f maps outside the input space $L^2(0,1)$. So to use the theory we have to choose other spaces, and/or operators. We discuss two options.

Choosing other spaces If we want to keep $f(y) = y^2$, then we have to change the input and output. One option is the following. As state space and output space we take $X = Y = L^2(0,1)$, and as input space we take $U = L^1(0,1)$. For these spaces we write the non-linear (15) with the homogeneous boundary conditions as the abstract system (7)–(8) with

$$Ag = \frac{\partial^2 g}{\partial \zeta^2}, \tag{17}$$

$$D(A) = \{g \in X \mid \dot{g}, \ddot{g} \in X, \text{ and } g(0) = 0 = g(0)\} \tag{18}$$

$$B = \imath d, C = \alpha I, \tag{19}$$

where $\imath d$ denotes the inclusion from $L^1(0,1)$ in $L^2(0,1)$, which is only densely defined. Furthermore, we define

$$f(y) = y^2.$$

By the Cauchy-Schwarz inequality it is not hard to show that f is (locally) Lipschitz continuous from Y to U. Hence to use the idea presented in Section 2 we must have that the system (17)–(19) is well-posed.

Since the input space has become a Banach space, Theorem 1 has to be formulated in general Banach spaces. Once this is done, there is another possibility for formulation (15).

As input, state, and output space we choose $L^\infty(0,1)$. Furthermore, A remains the same as in (17) and (18), and $B = C = I$. It is clear that f is (locally) Lipschitz continuous on L^∞, and since we can easily check that Theorem 1 holds on general Banach spaces, when B and C are bounded, we obtain that (15) possesses a unique solution on $L^\infty(0,1)$ for every initial condition in $L^\infty(0,1)$. In the next part we show that by choosing another f we can obtain existence on $L^2(0,1)$

Choosing another f As spaces we choose $U = X = Y = L^2(0,1)$ and we choose A with its domain as in (17) and (18), respectively. The input operator we choose the identity. However, we choose the output operator as

$$Cg = \frac{\partial g}{\partial \zeta}, \tag{20}$$

on the domain

$$D(C) = \{g \in X \mid \dot{g} \in X\}$$

and we choose

$$(f(y))(\zeta) = \left(\int_0^\zeta y(\eta) d\eta\right)^2. \tag{21}$$

It is well-known that the system $\Sigma(A,B,C)$ is well-posed, and furthermore, it is not hard to show that f is (locally) Lipschitz continuous from $L^2(0,1)$ to $L^2(0,1)$. Since $f(C(x)) = x^2$, Theorem 1 gives that (15) possesses a unique solution on $L^2(0,1)$ for every initial condition in $L^2(0,1)$.

3.2 Non-linear Hamiltonian systems

A general Hamiltonian system can be written as

$$\frac{\partial x}{\partial t}(\zeta,t) = \frac{\partial}{\partial \zeta}(\mathcal{N}(x))(\zeta,t), \tag{22}$$

where $\mathcal{N}$ is the variational derivative of the Hamiltonian with respect to x. For such a system Theorem 1 cannot be applied directly. Therefor we perturb the pde and consider

$$\frac{\partial x}{\partial t}(\zeta,t) = \varepsilon \frac{\partial^2 x}{\partial \zeta^2}(\zeta,t) + \frac{\partial}{\partial \zeta}(\mathcal{N}(x))(\zeta,t). \tag{23}$$

We are going to study this equation assuming homogeneous boundary conditions. Again we have the two options.

Choosing other spaces The easiest choice seems to be to choose all spaces to be $L^\infty(0,1)$ and A as ε times the operator in (17) with domain (18), $C = I$ and $B = \frac{\partial}{\partial\zeta}$. If $\mathcal{N}$ is (locally) Lipschitz continuous on L^∞ and the system is well-posed, then we find a unique solution on $L^\infty(0,1)$.

Choosing another f Here we can apply a similar trick as in the previous example. We choose A as ε times the operator in (17) with domain (18), $B = \frac{\partial}{\partial\zeta}$ and C as (20). Furthermore,

$$(f(y))(\zeta) = \mathcal{N}\left(\int_0^\zeta y(\eta)d\eta\right).$$

It is known that the system $\Sigma(A,B,C)$ is well-posed on $L^2(0,1)$, but the feed-through bound is non-zero. Hence if the above defined f is locally Lipschitz, then we still need to check whether the Lipschitz constant and δ is smaller than 1. Once that is checked, we can conclude by Theorem 1 that (23) possesses a unique solution on $L^2(0,1)$.

4. CONCLUSION

In this paper we discussed the possibility of using feedback theory for showing existence of solutions. As may be clear from the presented work several steps still need additional analysis. We present them in a list

(1) Prove Theorem 1.
The idea of the proof is presented here, but the details all need to be worked out.
(2) Formulate and prove Theorem 1 on a general Banach space.
(3) Find standard examples of linear systems that are well-posed in Banach spaces, especially in L^∞. In particular, investigate the well-posedness of the system in *Choosing other spaces* in Subsection 3.2.
(4) Fill in the details of the argument presented in *Choosing another f* in Subsection 3.2.
(5) Investigate the behavior of the solution of (23) when ε approaches zero. In particular, is this solution converging to a solution of (23)?

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