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STRUCTURAL HEALTH MONITORING OF STAY CABLES BY THE SCRUTON NUMBER

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Abstract. Identification of modal parameters: eigenfrequencies, damping ratios and mode shapes from empirical data is of fundamental engineering importance in the dynamical analysis of mechanical structures and particularly in the analysis of stay cables. These modal parameters will serve to perform structural health monitoring, damage detection and safety evaluation. The identification is based on the knowledge of output-only responses, without using excitation information. We propose in this communication a time domain method based on a subspace algorithm to extract the modal parameters of vibrating stay cables from output-only measurements. We can then use these identified parameters for structural health monitoring by the computation of the Scruton number and the cable tension.

1 INTRODUCTION

Identification of modal parameters from empirical data is very important in mechanical vibrations and in the dynamical analysis of mechanical structures excited by external forces. An application constitutes the vibration analysis of stay cables in the civil engineering domain. The identified modal parameters will serve to perform structural health monitoring, damage detection and safety evaluation of such structures. The identification is based on the knowledge of output-only responses, without using excitation information, and is known as Operational Modal Analysis (OMA), also named as ambient or natural excitation or output-only modal analysis [1-6]. The principal advantages of OMA are the ease of use and the fast to conduct since no artificial excitation equipment is required. We propose in this communication a time domain method, based on a subspace algorithm to extract the modal parameters of vibrating stay cables from output-only measurements. We can then use these identified parameters for structural health monitoring.

The subspace identification method is a time domain method and uses the covariance matrices between output data. The subspace method is based on the observability and controllability properties of linear systems and we use shifted properties of the controllability matrix to identify the modal parameters. Unfortunately spurious modes, which are inherent to the subspace method, appear [6, 7]. To eliminate such spurious poles a criterion using modal coherence indicators is used. This criterion describes the coherence between each mode of the state space model and the modes directly inferred from the measured signal. The criterion serves as a model quality measure. Modal parameters are then obtained from stability diagrams. Experimental results of a single tower, double row cable-stayed bridge supported by 112 stay cables are presented, where ambient vibrations of each stay cable are carried out using accelerometers. The subspace method is then applied to determine the eigenfrequencies and damping coefficients of different cables. An estimate of the fundamental frequency f_i for each cable is then obtained. We can then compute the tension force for each cable. For each natural frequency identified a respective damping coefficient is assigned. Thus, a Scruton number for each cable and for each mode of vibration is determined [8, 9]. High values of the Scruton number tend to suppress the oscillation and bring up to the start of instability at high wind speeds. For example, a Scruton number superior to 10 is sufficient to prevent rain, wind and traffic induced vibrations. In a continuous monitoring and modal analysis process, the tension forces and Scruton numbers could be used to assess the health of stay cables in cable-stayed bridges.

2 VIBRATION ANALYSIS OF A CABLE AND THE SUBSPACE IDENTIFICATION METHOD

2.1 Vibration analysis of a cable

The free vibration equation of cable motion is given by [10]

$$EI\,\frac{\partial^4 w(x,t)}{\partial x^4} + \rho S\,\frac{\partial^2 w(x,t)}{\partial t^2} - P\,\frac{\partial^2 w(x,t)}{\partial x^2} = = f(x,t) \qquad (1)$$

where E is the Young's modulus, I the moment of inertia of the cable cross-section, ρ the specific mass, S the cross-sectional area, P the axial load, $f(x,t)$ the external load, $w(x,t)$ the transversal displacement which is assumed to be small, x the position along the cable and t the time variable.

The solution of the free vibration equation can be obtained using the method of separation variables as

$$w(x,t)=q(t)\,W(x) \qquad (2)$$

Assuming $q(t)$ harmonic with natural frequency ω in rad.s^{-1} and substitution of equation (2) into equation (1) gives

$$EI\,\frac{d^4 W}{\partial x^4} - P\,\frac{d^2 W}{\partial x^2} - \rho S\,\omega^2\,W = 0 \qquad (3)$$

By assuming the solution $W(x)$ to be

$$W(x) = C\,e^{s\,x} \qquad (4)$$

in equation (3), the auxiliary equation can be obtained

$$s^4 - \frac{P}{EI}\,s^2 - \frac{\rho A\,\omega^2}{EI} = 0 \qquad (5)$$

and the roots of this equation are

$$s_1^2,\, s_2^2 \;=\; \frac{P}{2EI} \;\pm\; \left(\frac{P^2}{4E^2 I^2} + \frac{\rho A \omega^2}{EI} \right)^{1/2} \tag{6}$$

The solution of $W(x)$ can be expressed as

$$W(x) \;=\; C_1 \cosh s_1 x + C_2 \sinh s_1 x + C_3 \cos s_2 x + C_4 \sin s_2 x \tag{7}$$

where the constants C_1 to C_4 are to be determined from the boundary conditions:

$$W(0) = 0\;; \quad \frac{d^2 W(0)}{dx^2} = 0 \tag{8}$$

$$W(L) = 0 \quad ; \quad \frac{d^2 W(L)}{dx^2} = 0 \tag{9}$$

Equations (8) require that $C_1 = 0$ and $C_3 = 0$, and so

$$W(x) \;=\; C_2 \sinh s_1 x \;+ C_4 \sin s_2 x \tag{10}$$

The application of (9) to (10) leads to

$$\sinh s_1 L \sin s_2 L = 0 \tag{11}$$

Since $\sinh s_1 L > 0$ for all values of $s_1 L \neq 0$, the only roots to this equation are

$$s_2 L = k\,\pi \quad , \quad k = 0, 1, 2, \ldots . \tag{12}$$

Thus equations (12) and (6) give the natural frequencies of vibration:

$$f_k = \frac{\pi}{2L^2} \left(\frac{EI}{\rho S} \right)^{1/2} \left(k^4 + \frac{k^2 P L^2}{\pi^2 EI} \right)^{1/2} \tag{13}$$

Further, note that the smallest Euler buckling load for a simply supported beam is given by $P_{cri} = \pi^2 E\; I / L^2$. Thus equation (13) can be written as

$$f_k = \frac{\pi}{2L^2} \left(\frac{EI}{\rho S} \right)^{1/2} \left(k^4 + \frac{k^2 P}{P_{cri}} \right)^{1/2} \tag{14}$$

We would like to approximate the desired partial differentiate equation solution $w(x,t)$ in a separable form as series expansion of time varying coefficients $q_i(t)$ and spatially varying basis functions $W_i(x)$:

$$\tilde{w}(x,t) = \sum_{i=1}^{n'} q_i(t)\, W_i(x) \tag{15}$$

The Galerkin procedure can then be used to specify the equations of motion for the coefficients $q_i(t)$ and for an n' degrees of freedom system with viscous damping, the motion equation is

$$\boldsymbol{M}\,\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\,\boldsymbol{q} = \boldsymbol{f} \tag{16}$$

where M, C and K are respectively the mass, damping and stiffness matrices. In [11] Barbieri et al. propose a procedure based on experimental and simulated data to identify damping of transmission line cables. The experimental data are collected through accelerometers and the simulated data are obtained using the finite element method. In this communication, we identify eigenfrequencies and damping ratios of line cables from output data only using the subspace method.

2.2 The subspace identification method

The subspace identification method assumes that the dynamic behaviour of a structure excited by ambient forces can be described by a discrete time stochastic state space model [1, 2] :

$$\boldsymbol{z_{k+1}} = \boldsymbol{A}\, \boldsymbol{z_k} + \boldsymbol{w_k} \qquad \text{state equation} \tag{17}$$

$$\boldsymbol{y_k} = \boldsymbol{C}\, \boldsymbol{z_k} + \boldsymbol{v_k} \qquad \text{observation equation} \tag{18}$$

where $\boldsymbol{z_k}$ is the unobserved state vector of dimension $n=2n'$; $\boldsymbol{y_k}$ is the (mx1) vector of observations or measured output vector at discrete time instant k ; $\boldsymbol{w_k}$, $\boldsymbol{v_k}$ are white noise terms representing process noise and measurement noise together with the unknown inputs, it is assumed that the excitation effect appears in the disturbances $\boldsymbol{w_k}$ and $\boldsymbol{v_k}$, since the system input can not be measured; $\boldsymbol{A}$ is the (nxn) transition matrix describing the dynamics of the system and $\boldsymbol{C}$ is the (mxn) output or observation matrix, translating the internal state of the system into observations. The stochastic identification problem deals with the determination of the transition matrix $\boldsymbol{A}$ using output-only data.
The modal parameters of a vibrating system are obtained by applying the eigenvalue decomposition of the transition matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{\Psi}\ \boldsymbol{\Lambda}\ \boldsymbol{\Psi}^{-1} \tag{19}$$

where $\boldsymbol{\Lambda} = diag(\lambda_i)$, i=1,2,…,n, is the diagonal matrix containing the complex eigenvalues and $\boldsymbol{\Psi}$ contains the eigenvectors of $\boldsymbol{A}$ as columns. The eigenfrequencies f_i and damping ratios ζ_i are obtained from the eigenvalues which are complex conjugate pair:

$$f_i = \frac{1}{4\pi\Delta t}\sqrt{\left[\ln\left(\lambda_i\lambda_i^*\right)\right]^2 + 4\left[Ar\cos\left(\frac{\lambda_i+\lambda_i^*}{2\sqrt{\lambda_i\lambda_i^*}}\right)\right]^2} \tag{20}$$

$$\zeta_i = \sqrt{\frac{\left[\ln\left(\lambda_i\lambda_i^*\right)\right]^2}{\left[\ln\left(\lambda_i\lambda_i^*\right)\right]^2 + 4\left[Ar\cos\left(\frac{\lambda_i+\lambda_i^*}{2\sqrt{\lambda_i\lambda_i^*}}\right)\right]^2}} \tag{21}$$

with Δt the sampling period of analyzed signals.

The mode shapes evaluated at the sensor locations are the columns of the matrix $\boldsymbol{\Phi}$ obtained by multiplying the output matrix $\boldsymbol{C}$ with the matrix of eigenvectors $\boldsymbol{\Psi}$:

$$\boldsymbol{\Phi} = \boldsymbol{C}\,\boldsymbol{\Psi} \tag{22}$$

Our purpose is to determine the transition matrix $\boldsymbol{A}$ in order to indentify the modal parameters of the vibrating system. Define the *(mfx1)* and *(mpx1)* future and past data vectors as $\boldsymbol{y}^+_k = [\boldsymbol{y}^T_k, \boldsymbol{y}^T_{k+1}, \ldots, \boldsymbol{y}^T_{k+f-1}]^T$ and $\boldsymbol{y}^-_k = [\boldsymbol{y}^T_k, \boldsymbol{y}^T_{k+1}, \ldots, \boldsymbol{y}^T_{k-p+1}]^T$. The *(mfxmp)* covariance matrix between the future and the past is given by [2] :

$$\boldsymbol{H} = E[\boldsymbol{y}^+_k\, \boldsymbol{y}^{-T}_k] = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 & . & \boldsymbol{R}_p \\ \boldsymbol{R}_2 & \boldsymbol{R}_3 & . & \boldsymbol{R}_{p+1} \\ . & . & . & . \\ \boldsymbol{R}_f & \boldsymbol{R}_{f+1} & . & \boldsymbol{R}_{f+p-1} \end{bmatrix} \tag{23}$$

where E denotes the expectation operator and the superscript T the transpose operation. $\boldsymbol{H}$ is the block Hankel matrix formed with the *(mxm)* individual theoretical auto-covariance matrices $\boldsymbol{R}_i = E[\boldsymbol{y}_{k+i}\, \boldsymbol{y}^T_k] = \boldsymbol{C}\boldsymbol{A}^{i-1}\boldsymbol{G}$, with $\boldsymbol{G}=E[\boldsymbol{x}_{k+1}\, \boldsymbol{y}^T_k]$.

In practice, the auto-covariance matrices are estimated from N data points and are computed by $\boldsymbol{R}_i = N^{-1} \sum_{k=1}^{N-1} \boldsymbol{y}_{k+i}\boldsymbol{y}_k^T$; i = 0,1, . . ., p+f and with these estimated auto-covariance matrices we form the block Hankel matrix $\boldsymbol{H}$. In order to identify the transition matrix $\boldsymbol{A}$ two matrix factorizations of $\boldsymbol{H}$ are employed. The first factorization uses the singular value decomposition (SVD) of $\boldsymbol{H}$

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{U}\,\boldsymbol{\Sigma}^{1/2}\,\boldsymbol{\Sigma}^{1/2}\boldsymbol{V}^T \tag{24}$$

with $\boldsymbol{U}^T\boldsymbol{U}$ and $\boldsymbol{V}^T\boldsymbol{V}$ identity matrices and $\boldsymbol{\Sigma}$ a diagonal matrix of singular values. The second factorization of the block Hankel matrix $\boldsymbol{H}$ considers its *(mfxn)* observability and *(nxmp)* controllability matrices, $\boldsymbol{O}$ and $\boldsymbol{K}$, as

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{CG} & \boldsymbol{CAG} & . & \boldsymbol{CA}^{p-1}\boldsymbol{G} \\ \boldsymbol{CAG} & \boldsymbol{CA}^2\boldsymbol{G} & . & \boldsymbol{CA}^p\boldsymbol{G} \\ . & . & . & . \\ \boldsymbol{CA}^{f-1}\boldsymbol{G} & \boldsymbol{CA}^f\boldsymbol{G} & . & \boldsymbol{CA}^{f+p-2}\boldsymbol{G} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ . \\ \boldsymbol{CA}^{f-1} \end{bmatrix} [\boldsymbol{G}\ \ \boldsymbol{AG}\ldots \boldsymbol{A}^{p-1}\boldsymbol{G}] = \boldsymbol{O}\,\boldsymbol{K} \tag{25}$$

By identification we obtain the observability matrix

$$\boldsymbol{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ . \\ \boldsymbol{CA}^{f-1} \end{bmatrix} \tag{26}$$

and the controllability matrix

$$\boldsymbol{K} = [\boldsymbol{G}\ \ \boldsymbol{AG}\ldots\ \boldsymbol{A}^{p-1}\boldsymbol{G}] \tag{27}$$

The two factorizations of the block Hankel matrix are equated to give

$$\boldsymbol{H} = \boldsymbol{U}\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T} = \boldsymbol{OK} \tag{28}$$

implying O = U $\boldsymbol{\Sigma}^{1/2}$ and K = $\boldsymbol{\Sigma}^{1/2}$ V^T. To determine the transition matrix $\boldsymbol{A}$ we use properties of the controllability matrix (properties of the observability matrix can also be used). We introduce the two following matrices:

$$\boldsymbol{K}\Uparrow\ = [\ \boldsymbol{AG}\ \ \ \boldsymbol{A}^2\boldsymbol{G}.....\boldsymbol{A}^{p-1}\boldsymbol{G}\ ] \quad \text{and} \quad \boldsymbol{K}\Downarrow = [\ \boldsymbol{G}\ \ \boldsymbol{AG}\ \ \ \boldsymbol{A}^2\boldsymbol{G}.....\boldsymbol{A}^{p-2}\boldsymbol{G}\ ] \tag{29}$$

where $\boldsymbol{K}\Uparrow$ is the matrix obtained by deleting the first block column of $\boldsymbol{K}$ and $\boldsymbol{K}\Downarrow$ is the matrix obtained by deleting the last block column $\boldsymbol{K}$. We obtain then:

$$\boldsymbol{K}\Uparrow\ = \boldsymbol{A}\,\boldsymbol{K}\Downarrow \quad \text{or} \quad (\ \Sigma^{1/2}\ \boldsymbol{V}^{T}\ )\Uparrow = \boldsymbol{A}(\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T}\ )\Downarrow \tag{30}$$

We use properties of shifting columns operators '' $\Uparrow$ '' and '' $\Downarrow$ '' : let $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ be two *(axb)* and *(bxc)* matrices, we have the following properties: $(\boldsymbol{\Phi}\ \boldsymbol{\Psi})\Uparrow = (\boldsymbol{\Phi}\ \boldsymbol{\Psi}\Uparrow)$ and $(\boldsymbol{\Phi}\ \boldsymbol{\Psi})\Downarrow = (\boldsymbol{\Phi}\ \boldsymbol{\Psi}\Downarrow)$, consequently we have

$$(\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T}\Uparrow\ ) = \boldsymbol{A}\ (\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T}\Downarrow\ ) \tag{31}$$

The transition matrix is

$$\boldsymbol{A} = (\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T}\Uparrow\ )\ (\ \boldsymbol{\Sigma}^{1/2}\ \boldsymbol{V}^{T}\Downarrow\ )^{+} \tag{32}$$

where ($)^+$ represents the pseudo inverse of a matrix, and the eigenvalues of the transition matrix are given by

$$\lambda(\boldsymbol{A}) = \lambda\,[(\ \boldsymbol{V}^{T}\Uparrow\ )\ (\ \boldsymbol{V}^{T}\Downarrow\ )^{+}] \tag{33}$$

This approach constitutes a subspace modal identification method and in the case of noisy data a problem of model order determination occurs: when extracting mechanical (or physical) modes this algorithm can generate spurious modes. For these reasons, the assumed number of modes, or model order, is incremented over a wide range of values and we plot the stability diagram. The stability diagram involves tracking the estimates of eigenfrequencies and damping ratios as a function of model order. As the model order is increased, more and more modal frequencies and damping ratios are estimated, hopefully, the estimates of the physical modal parameters stabilize using a criterion based on the modal coherence of measured and identified modes. Using this criterion we can separate mechanical modes and spurious modes.

3 APPLICATION OF THE SUBSPACE IDENTIFICATION METHOD

The subspace identification method is applied to the analysis of stay cables of the Jinma cable-stayed bridge (Figure 1), that connects Guangzhou and Zhaoqing in Guangdong Province, China. It is a single tower, double row cable-stayed bridge, supported by 28*4 =112 stay cables. Inputs could evidently not be measured, so only acceleration data from accelerometers in contact with a cable are available.

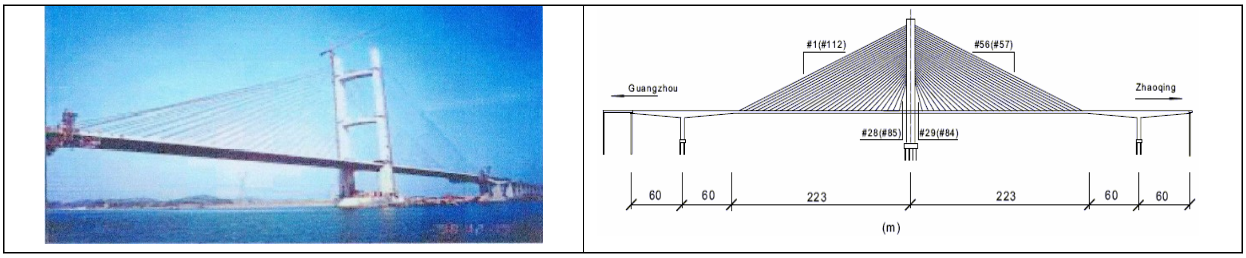

Figure 1: View and schematic of the Jimma cable-stayed bridge

For the ambient vibration measurement of each stay cable an accelerometer was mounted securely to the cables, the sample frequency is 40 Hz and the recording time is 140.8 second, which results in total 5632 data points. Cables 1, 56, 57 and 112 are the longest and cables 28, 29, 84 and 85 are the shortest. A full description of the test can be found in [8]. Figures 2 and 3 show the stabilization diagrams on eigenfrequencies of cable 1 and cable 25 using the subspace method and the modal coherence indicator. These diagrams show remarkable stable eigenfrequencies and from these plots we determine the eigenfrequencies of the two stay cables.

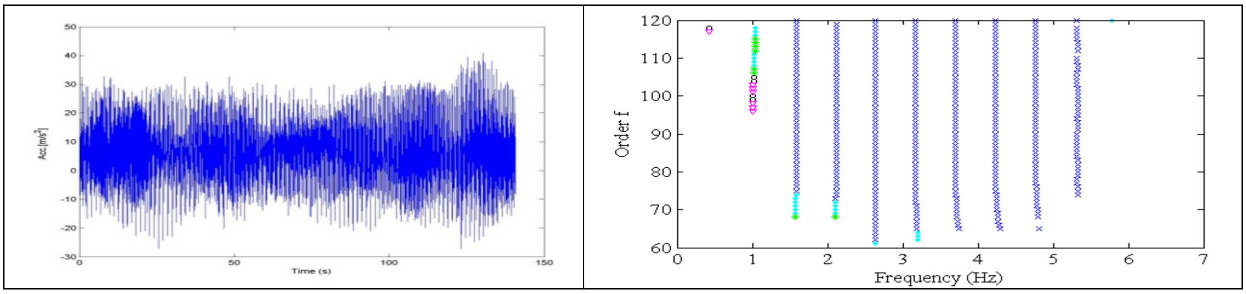

Figure 2: Ambient time response of cable n°1 and its stabilization diagram on eigenfrequencies

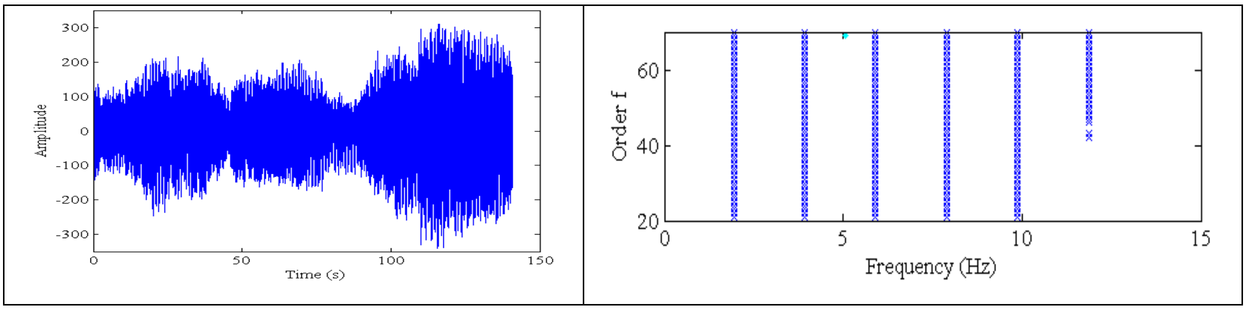

Figure 3: Ambient time response of cable n°25 and its stabilization diagram on eigenfrequencies

The subspace identification procedure is applied to determine the eigenfrequencies of 112 cables of the bridge and we obtain the Figure 4.

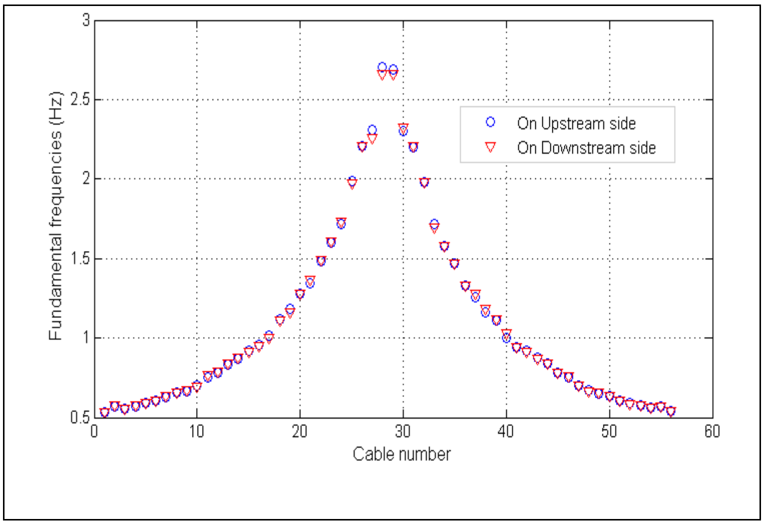

Figure 4: Fundamental frequency of each cable on upstream side and on downstream side

It can be seen that the fundamental frequencies of both bridge sides are almost identical and the fundamental frequency distribution is symmetric with respect to the single tower. The fundamental frequencies vary between 0.533 Hz for the longest cable and 2.703 Hz for the shortest cable. The cable tension can be estimated by the expression $T = 4\, \rho L^2 f_0^2$, where ρ is the linear density of the cable (ρ = 66.94 kg/m), L is the length of the cable and f_0 the fundamental frequency. The maximum and minimum cable forces for the Jinma bridge are then : T_{max} = 5052 kN (cable number 57), T_{min} = 2490 kN (cable number 84). These cable forces can be considered as reference tensions and used as indicators in the field of health monitoring process.

The instability of a cable is assessed by the Scruton number [9] which is defined for each mode of vibration by $S_{c,i} = \dfrac{\zeta_i . \rho}{\rho_a D^2}$, where ζ_i is the damping ratio for each mode, ρ_a is the density of the air and D is the cable diameter. High values of the Scruton number tend to suppress the oscillation and bring up the start of instability at high wind speeds. Considering ρ_a = 1.2 kg/m^3 and D = 0.203m, the Scruton number for each mode of cable 25 is presented in Table 1. In the Scruton number damping is a main factor and most vibration problems in stay cables are subject to low damping values.

Table 1: Natural frequencies, damping ratios and Scruton numbers for cable 25

Modes	1	2	3	4	5	6
f_k (Hz)	1.960	3.938	5.915	7.893	9.856	11.884
ξ_k (%)	0.314	0.046	0.178	0.078	0.081	0.094
Sc_k	4.250	0.622	2.410	1.056	1.096	1.271

4 CONCLUSION

Cable vibrations induced by wind, rain and traffic are observed in a number of cable-stayed bridges and degradation of these cables may have catastrophic effects. The determination of tension and Scruton number of these cables from output only measurements can be used as structural health monitoring indicators. Operational modal analysis applied to the dynamic data of stay cables provides useful information to determine the current condition of stay cables accurately.

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