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STRUCTURAL ACOUSTICS WITH INTERFACE DAMPING: VARIOUS CONSIDERATIONS ABOUT STATIC TERMS FOR EFFICIENT DYNAMIC BEHAVIOR DESCRIPTION

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Abstract. In this paper, we propose an insight into structural acoustics with interface damping problems. A particular attention is paid to the links between static and dynamic at several levels: on the formulation itself, on the impact on physical variables, and on the efficiency of the Reduced Order Models that can be derived from the formulations. Special acknowledgements are due to Prof. Roger Ohayon who inspired a large part of this work through its articles, books and passionate discussions.

1 Formulation of structural-acoustic problem

1.1 Coupled formulation

This part exhibits the basic equations of the coupled problem, which are classically available in literature [1, 2, 3]. The internal vibroacoustic problem which is considered in this paper is presented in figure 1. The whole paper is related to permanent harmonic motion at frequency ω. Let Ω_F be the fluid domain, Ω_S the structural domain. The boundary of the fluid domain is $\Sigma_{FS} \cup \Sigma_F \cup \Sigma_A$, where $\Sigma_{FS} \cap \Sigma_F \cap \Sigma_A = \emptyset$, while the boundary of the structural domain is $\Sigma_{FS} \cup \Sigma_S \cup \Sigma_0$, where $\Sigma_{FS} \cap \Sigma_S \cap \Sigma_0 = \emptyset$. The partition of boundaries is done according to the mechanical conditions: Σ_{FS} is the structural-acoustic coupling surface, Σ_F is the part of the acoustic border on which a Neumann condition is applied, corresponding to a rigid wall (a homogeneous Dirichlet condition could also be considered without loss of generality), Σ_A is the part of the acoustic border on which a Robin condition is considered, corresponding typically to an absorbing material, Σ_S is the structural boundary on which a Neumann condition is applied, corresponding to a prescribed force, Σ_0 is the structural boundary on which a homogeneous Dirichlet condition

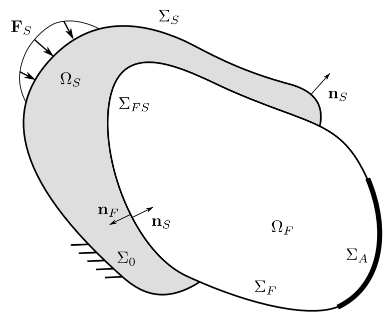

Figure 1: Description of vibroacoustic problem

is applied, corresponding to a clamped area. $\mathbf{n}_S$ and $\mathbf{n}_F$ are respectively the outgoing unit normals of structural and fluid domain.
The physical variables which are used to describe the behavior of the system are the displacement $\mathbf{u}$ for the structure and the acoustic pressure p for the fluid. For the structural part, the linearized strain tensor is denoted as $\varepsilon(\mathbf{u})$ and the associated stress tensor is denoted as $\sigma(\mathbf{u})$. The choice of acoustic pressure p to describe the fluid behavior instead of a fluid displacement vector field reduces the number of degrees of freedom. For the structural part, the displacement field must be regular and verify

$$\begin{cases} -\omega^2\rho_S\mathbf{u} + \boldsymbol{\nabla}\cdot\sigma(\mathbf{u}) = 0 & \text{in } \Omega_S, \quad (a) \\ \mathbf{u} = 0 & \text{on } \Sigma_0, \quad (b) \\ \sigma(\mathbf{u})\cdot\mathbf{n}_S = \mathbf{F}_S & \text{on } \Sigma_S, \quad (c) \\ \sigma(\mathbf{u})\cdot\mathbf{n}_S = p\mathbf{n}_F & \text{on } \Sigma_{FS}, \quad (d) \end{cases} \tag{1}$$

where ρ_S is the structural mass density and $\mathbf{F}_S$ is the complex amplitude of the force exciting the structure at frequency ω. The constitutive law $\sigma = f(\varepsilon)$ is also required to solve the problem.
The fluid cavity must verify the following equations:

$$\begin{cases} \boldsymbol{\nabla}^2 p + \dfrac{\omega^2}{c^2}p = 0 & \text{in } \Omega_F, \quad (a) \\ \dfrac{\partial p}{\partial \mathbf{n}_F} = 0 & \text{on } \Sigma_F, \quad (b) \\ \dfrac{\partial p}{\partial \mathbf{n}_F} = \rho_F\omega^2\mathbf{u}\cdot\mathbf{n}_F & \text{on } \Sigma_{FS}, \quad (c) \\ \dfrac{\partial p}{\partial \mathbf{n}_F} = -\dfrac{i\omega}{Z_a(\omega)}\rho_F p & \text{on } \Sigma_A, \quad (d) \end{cases} \tag{2}$$

where c is the sound speed in the fluid and ρ_F the mass density of the fluid at rest. $Z_a(\omega)$ is the complex impedance of absorbing material.
If the fluid is inviscid and compressible, the constitutive law of a barotropic fluid can be used:

$$p = -\rho_F c^2 \nabla\cdot\mathbf{u}^F, \tag{3}$$

where $\mathbf{u}^F$ is the fluid displacement, linked to the pressure by Euler's equation

$$\nabla p = \rho_F \omega^2 \mathbf{u}^F. \tag{4}$$

In order to obtain a variational formulation of this problem, the admissible spaces $\mathscr{C}_{\mathbf{u}} = \{\mathbf{u} \in [H^1(\Omega_S)]^3 \,/\, \mathbf{u} = \mathbf{0} \text{ on } \Sigma_0\}$ and $\mathscr{C}_p = \{p \in H^1(\Omega_F)\}$ are defined where $H^1(\Omega)$ is the Sobolev space of order 1. One has then to find $\mathbf{u}$ and p in $\mathscr{C}_{\mathbf{u}}$ and $\mathscr{C}_p$ such as, for all $(\delta\mathbf{u}, \delta p) \in \mathscr{C}_{\mathbf{u}} \times \mathscr{C}_p$:

$$\left\{ \begin{aligned} & \int_{\Omega_S} \sigma(\mathbf{u}) : \epsilon(\delta\mathbf{u})\, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u} \cdot \delta\mathbf{u}\, d\Omega \\ & \qquad - \int_{\Sigma_{FS}} p\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u}\, d\Sigma, \qquad (a) \\ & \frac{1}{\rho_F} \int_{\Omega_F} \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} \delta p\, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} p \delta p\, d\Omega \\ & \qquad -\omega^2 \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F\, \delta p\, d\Sigma + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} p \delta p\, d\Sigma = 0. \qquad (b) \end{aligned} \right. \tag{5}$$

The FE discretization of this variational formulation can be written as

$$\left(\begin{bmatrix} K_S & -L \\ 0 & K_F \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & 0 \\ L^T & M_F \end{bmatrix} + \frac{i\omega}{Z_a(\omega)} \begin{bmatrix} 0 & 0 \\ 0 & A_F \end{bmatrix} \right) \begin{Bmatrix} U \\ P \end{Bmatrix} = \begin{Bmatrix} F_S \\ 0 \end{Bmatrix}, \tag{6}$$

where the matrices are obtained by discretization of each corresponding integral of the weak formulation (5) using N_S structural dofs $\{U\}$ associated to $\mathbf{u}$ and N_F acoustic pressure dofs $\{P\}$ associated to p, as:

$$\begin{array}{lcl lcl} [K_S] & \rightarrow & \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \epsilon(\delta\mathbf{u})\, d\Omega, & [M_S] & \rightarrow & \displaystyle\int_{\Omega_S} \rho_S \mathbf{u} \cdot \delta\mathbf{u}\, d\Omega, \\ [K_F] & \rightarrow & \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} \delta p\, d\Omega, & [M_F] & \rightarrow & \displaystyle\frac{1}{\rho_F c^2} \int_{\Omega_F} p \delta p\, d\Omega, \\ [L] & \rightarrow & \displaystyle\int_{\Sigma_{FS}} p\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma, & [A_F] & \rightarrow & \displaystyle\int_{\Sigma_A} p \delta p\, d\Sigma. \end{array} \tag{7}$$

N.B. For a sake of clarity, no structural damping is considered in the weak formulations and associated FE discretizations presented here, but it can naturally be integrated in the formulations.

1.2 Considerations about the static case

As indicated in reference [4], the considered problem is not valid for $\omega = 0$ since when the frequency vanishes, the "movement" of the fluid tends to static irrotational motion. This can be easily seen by taking the rotational of (3) leads to $\rho_F \omega^2 \boldsymbol{\nabla} \times \mathbf{u}_F = 0$, which implies that the movement of the fluid is necessarily irrotational if $\omega \neq 0$.

In that case, the pressure can be decomposed in two terms:

$$p = p^S + \tilde{p}, \tag{8}$$

where $\tilde{p}$ is a dynamic pressure and p^S is a so-called static pressure, which is constant in space and differs from the static solution which could be obtained by extending the considered problem to $\omega = 0$, since it would result in a static pressure which is constant in space but undetermined in amplitude. The uniqueness of $\tilde{p}$ is guaranteed by the condition $\int_{\Omega_F} \tilde{p} d\Omega = 0$. The static pressure can be determined by integration of p in the fluid domain:

$$\int_{\Omega_F} p d\Omega = \int_{\Omega_F} p^S d\Omega + \int_{\Omega_F} \tilde{p} d\Omega = V_F p^S, \tag{9}$$

in which V_F is the measure of the volume occupied by Ω_F. On the other side, if the fluid is inviscid and compressible, the constitutive law of the fluid can be used: $p = -\rho_F c^2 \nabla \cdot \mathbf{u}^F$. The integral of the pressure is

$$\int_{\Omega_F} p d\Omega = -\rho_F c^2 \int_{\Omega_F} \nabla \cdot \mathbf{u}^F d\Omega = -\rho_F c^2 \int_{\Sigma_F \bigcup \Sigma_{FS} \bigcup \Sigma_A} \mathbf{u}^F \cdot \mathbf{n}_F d\Sigma. \tag{10}$$

It is then clear that:

$$p^S = -\rho_F \frac{c^2}{V_F} \int_{\Sigma_F \bigcup \Sigma_{FS} \bigcup \Sigma_A} \mathbf{u}^F \cdot \mathbf{n}_F d\Sigma. \tag{11}$$

In particular, if Σ_F is rigid, the static pressure can be directly derived from the normal displacement of the structure:

$$p^S = -\rho_F \frac{c^2}{V_F} \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F d\Sigma - \rho_F \frac{c^2}{V_F} \int_{\Sigma_A} \mathbf{u}^F \cdot \mathbf{n}_F d\Sigma. \tag{12}$$

2 A displacement-pressure formulation valid for the static case

Above considerations mean that several ways can be considered to solve the considered problem using a displacement/pressure formulation. The variable which describes the movement of the structure is the displacement $\mathbf{u}$, while for the fluid one can use the following strategies:

- use of pressure p for fluid description. This formulation has been presented in section 1.1, it is not valid for $\omega = 0$.

- use of dynamic pressure $\tilde{p}$ for fluid description. This formulation is valid for $\omega = 0$. The constraint (**??**) has to be considered to solve the problem. In particular, in a finite elements context, this constraint has to be discretized and included in the final system. The total pressure p is then obtained by summing $\tilde{p}$ and p^S which comes from equation (12). This requires in particular the discretization of p^S from equation (12) in which $\mathbf{u}^F \cdot \mathbf{n}_F$ must be expressed on Σ_A in function of the dynamic pressure $\tilde{p}$ using the impedance condition. The corresponding system has large expressions and exhibits no special interest comparing to other strategies.

- use of both dynamic pressure $\tilde{p}$ and static pressure p^S for fluid description. This formulation is valid for $\omega = 0$. The authors did not found any mention of this possibility in the literature, even if a $(\mathbf{u}, \tilde{p}, \phi, p^S)$ formulation can be found in [5]. To obtain this formulation, the constraint (**??**) and the equation (12) have to be considered and discretized. The total pressure p is then obtained by summing $\tilde{p}$ and p^S. Following this approach, one can define the subspace $\mathscr{C}_{\tilde{p}}$:

$$\mathscr{C}_{\tilde{p}} = \left\{ \tilde{p} \in \mathscr{C}_p \; / \; \int_{\Omega_F} \tilde{p} \, d\Omega = 0 \right\}. \tag{13}$$

In a weak form, one has then to find $\mathbf{u}$, $\tilde{p}$ and p^S in $\mathscr{C}_{\mathbf{u}}$, $\mathscr{C}_{\tilde{p}}$ and $\mathcal{R}$ such as, for all $(\delta\mathbf{u}, \delta\tilde{p}, \delta p^S) \in \mathscr{C}_{\mathbf{u}} \times \mathscr{C}_{\tilde{p}} \times \mathcal{R}$:

$$\left\{ \begin{array}{ll} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \epsilon(\delta\mathbf{u}) \, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u} \cdot \delta\mathbf{u} \, d\Omega & \\ \qquad \displaystyle - \int_{\Sigma_{FS}} \tilde{p} \, \mathbf{n}_F \cdot \delta\mathbf{u} \, d\Sigma - \int_{\Sigma_{FS}} p^S \, \mathbf{n}_F \cdot \delta\mathbf{u} \, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u} \, d\Sigma, & (a) \\ \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \nabla\tilde{p} \cdot \nabla\delta\tilde{p} \, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} \tilde{p}\delta\tilde{p} \, d\Omega - \omega^2 \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F \, \delta\tilde{p} \, d\Sigma & \\ \qquad \displaystyle + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} \tilde{p}\delta\tilde{p} \, d\Sigma + \frac{i\omega}{Z_a(\omega)} p^S \int_{\Sigma_A} \delta\tilde{p} \, d\Sigma = 0, & (b) \\ \displaystyle\frac{1}{\rho_F c^2} p^S V_F \delta p^S + \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F \, d\Sigma \, \delta p^S & \\ \qquad \displaystyle + \frac{1}{i\omega Z_a(\omega)} \int_{\Sigma_A} \tilde{p} \, d\Sigma \, \delta p^S + \frac{1}{i\omega Z_a(\omega)} p^S S_A \delta p^S = 0, & (c) \end{array} \right. \tag{14}$$

where V_F is the volume of the fluid domain and S_A the surface of the absorption area. The corresponding finite element formulation can be written as:

$$\left(\begin{bmatrix} K_S & -L & -\ell \\ 0 & K_F & 0 \\ \ell^T & 0 & \frac{V_F}{\rho_F c^2} \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & 0 & 0 \\ L^T & M_F & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{i\omega}{Z_a(\omega)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_F & a_F \\ 0 & \frac{-1}{\omega^2} a_F^T & \frac{-1}{\omega^2} S_A \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ \tilde{P} \\ p^S \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \\ 0 \end{array} \right\}, \tag{15}$$

where the acoustic dynamic pressure dofs $\tilde{P}$ are associated to $\tilde{p}$ and:

$$[a_F] \;\rightarrow\; p^S \int_{\Sigma_A} \delta\tilde{p} \, d\Sigma, \qquad [\ell] \;\rightarrow\; p^S \int_{\Sigma_{FS}} \delta\mathbf{u} \cdot \mathbf{n}_F \, d\Sigma. \tag{16}$$

One should emphasize that this system must be solved under the constraint $\int_{\Omega_F} \tilde{p} d\Omega = 0$, which corresponds to $[C]^T\{\tilde{P}\} = 0$ where $[C] \to \int_{\Omega_F} \tilde{p} d\Omega$. A useful information can be derived from this system concerning the frequency evolution of acoustic impedance: as indicated in [6], in order that the model is compatible with static behavior, the impedance Z_a must verify the conditions

$$\left\{ \begin{array}{l} \lim_{\omega\to 0} \omega \,\mathrm{Re}(Z_a(\omega)) = 0, \\ \lim_{\omega\to 0} \omega \,\mathrm{Im}(Z_a(\omega)) < \infty, \end{array} \right. \tag{17}$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ correspond respectively to real and imaginary part of complex number. These two conditions are required to ensure that the system has a static solution which is real and finite. This system can be symmetrized by dividing the equations related to dynamic fluid dofs by ω^2 and changing sign in the last line.

3 Alternative formulation: displacement potential

3.1 Definition of the displacement potential

An alternative formulation to the previous one is the $(\mathbf{u}, \varphi)$ formulation where φ is the displacement potential. The relation between p and φ is [1]:

$$p = \rho_F \omega^2 \varphi. \tag{18}$$

This relation is valid only when $\omega \neq 0$. The static case will be discussed later.
The admissible space $\mathscr{C}_\varphi$ is then defined as:

$$\mathscr{C}_\varphi = \left\{ \varphi \in H^1(\Omega_F) \right\}. \tag{19}$$

The weak form of this problem consists in searching $\mathbf{u}$ and φ in admissible spaces $\mathscr{C}_\mathbf{u}$ and $\mathscr{C}_\varphi$ such as, for all $(\delta\mathbf{u}, \delta\varphi) \in \mathscr{C}_\mathbf{u} \times \mathscr{C}_\varphi$:

$$\left\{ \begin{array}{ll} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \varepsilon(\delta\mathbf{u})\, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u}\cdot\delta\mathbf{u}\, d\Omega & \\ \displaystyle\qquad -\rho_F\omega^2 \int_{\Sigma_{FS}} \varphi\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u}\, d\Sigma, & (a) \\ \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \boldsymbol{\nabla}\varphi \cdot \boldsymbol{\nabla}\delta\varphi\, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} \varphi\delta\varphi\, d\Omega & \\ \displaystyle\qquad -\frac{1}{\rho_F} \int_{\Sigma_{FS}} \mathbf{u}\cdot\mathbf{n}_F\, \delta\varphi\, d\Sigma + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} \varphi\delta\varphi\, d\Sigma = 0. & (b) \end{array} \right. \tag{20}$$

The FE discretization of this variational formulation can be written as:

$$\left(\begin{bmatrix} K_S & 0 \\ -\frac{1}{\rho_F} L^T & K_F \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & \rho_F L \\ 0 & M_F \end{bmatrix} + \frac{i\omega}{Z_a(\omega)} \begin{bmatrix} 0 & 0 \\ 0 & A_F \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ \Phi \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \end{array} \right\}, \tag{21}$$

where the acoustic displacement potential dofs Φ are associated to φ.

3.2 Considerations about the static case

As indicated above, the definition of displacement potential φ by equation (18) is not valid when $\omega = 0$. One can use $\tilde{\varphi}$ instead of φ where [1]:

$$p = \rho_F \omega^2 \tilde{\varphi} + p^S. \tag{22}$$

The uniqueness of $\tilde{\varphi}$ is guaranteed by the constraint [1] $\int_{\Omega_F} \tilde{\varphi}\, d\Omega = 0$. This constraint leads to the definition of the subspace $\mathscr{C}_{\tilde{\varphi}}$:

$$\mathscr{C}_{\tilde{\varphi}} = \left\{ \tilde{\varphi} \in \mathscr{C}_{\varphi} \;/\; \int_{\Omega_F} \tilde{\varphi}\, d\Omega = 0 \right\}. \tag{23}$$

In the literature, it has been proposed to include the static pressure in $(\mathbf{u}, \tilde{p}, \tilde{\varphi})$ formulation by keeping p^S as a degree of freedom in the model [5]. Here, a direct $(\mathbf{u}, \tilde{\varphi}, p^S)$ formulation is derived in presence of absorption area, which limits the size of the final system compared with $(\mathbf{u}, \tilde{p}, \tilde{\varphi})$ formulation.

In a weak form, one has then to find $\mathbf{u}$, $\tilde{\varphi}$ and p^S in $\mathscr{C}_{\mathbf{u}} \times \mathscr{C}_{\tilde{p}} \times \mathcal{R}$ such as, for all $(\delta\mathbf{u}, \delta\tilde{\varphi}, \delta p^S) \in \mathscr{C}_{\mathbf{u}} \times \mathscr{C}_{\tilde{p}} \times \mathcal{R}$:

$$\left\{ \begin{array}{ll} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \varepsilon(\delta\mathbf{u})\, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u} \cdot \delta\mathbf{u}\, d\Omega & \\ \displaystyle\qquad -\rho_F \omega^2 \int_{\Sigma_{FS}} \tilde{\varphi}\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma - \int_{\Sigma_{FS}} p^S\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u}\, d\Sigma, & (a) \\ \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \boldsymbol{\nabla}\tilde{\varphi} \cdot \boldsymbol{\nabla}\delta\tilde{\varphi}\, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} \tilde{\varphi}\delta\tilde{\varphi}\, d\Omega - \frac{1}{\rho} \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F\, \delta\tilde{\varphi}\, d\Sigma & \\ \displaystyle\qquad + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} \tilde{\varphi}\delta\tilde{\varphi}\, d\Sigma + \frac{i}{\rho_F \omega Z_a(\omega)} p^S \int_{\Sigma_A} \delta\tilde{\varphi}\, d\Sigma = 0, & (b) \\ \displaystyle\frac{1}{\rho_F c^2} p^S V_F \delta p^S + \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F\, d\Sigma\, \delta p^S & \\ \displaystyle\qquad - \frac{i\omega\rho_F}{Z_a(\omega)} \int_{\Sigma_A} \tilde{\varphi}\, d\Sigma\, \delta p^S + \frac{1}{i\omega Z_a(\omega)} p^S S_A \delta p^S = 0, & (c) \end{array} \right. \tag{24}$$

The FE discretization of this variational formulation can be written as:

$$\left(\begin{bmatrix} K_S & 0 & -\ell \\ -\frac{1}{\rho_F} L^T & K_F & 0 \\ \ell^T & 0 & \frac{V_F}{\rho_F c^2} \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & \rho_F L & 0 \\ 0 & M_F & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{i\omega}{Z_a(\omega)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_F & \frac{1}{\rho_F \omega^2} a_F \\ 0 & -\rho_F a_F^T & \frac{-S_A}{\omega^2} \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ \tilde{\Phi} \\ p^S \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \\ 0 \end{array} \right\}, \tag{25}$$

where the acoustic displacement potential dofs $\tilde{\Phi}$ are associated to $\tilde{\varphi}$. This system can be symmetrized by multiplying the equations related to dynamic fluid dofs by $\omega^2 \rho_F^2$ and changing sign in the last line. One should emphasize that this system must be solved under the constraint $[C]^T \{\tilde{\Phi}\} = 0$.

4 Alternative formulation: velocity potential

4.1 Definition of the velocity potential

An alternative formulation to the previous ones is the $(\mathbf{u}, \psi)$ formulation where ψ is the velocity potential. The relation between p and ψ is [7]:

$$p = -i\omega\rho_F\psi. \tag{26}$$

This relation is valid only when $\omega \neq 0$. The static case will be discussed later.
The admissible space $\mathscr{C}_\psi$ is then defined as:

$$\mathscr{C}_\psi = \left\{\psi \in H^1(\Omega_F)\right\}. \tag{27}$$

The weak form of this problem consists in searching $\mathbf{u}$ and ψ in admissible spaces $\mathscr{C}_\mathbf{u}$ and $\mathscr{C}_\psi$ such as, for all $(\delta\mathbf{u}, \delta\psi) \in \mathscr{C}_\mathbf{u} \times \mathscr{C}_\psi$:

$$\left\{\begin{array}{ll} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \epsilon(\delta\mathbf{u})\, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u} \cdot \delta\mathbf{u}\, d\Omega & \\ \qquad \displaystyle + \rho_F i\omega \int_{\Sigma_{FS}} \psi\, \mathbf{n} \cdot \delta\mathbf{u}\, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u}\, d\Sigma, & (a) \\ \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\delta\psi\, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} \psi\delta\psi\, d\Omega & \\ \qquad \displaystyle - \frac{i\omega}{\rho_F} \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n}_F\, \delta\psi\, d\Sigma + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} \psi\delta\psi\, d\Sigma = 0. & (b) \end{array}\right. \tag{28}$$

The FE discretization of this variational formulation can be written as:

$$\left(\begin{bmatrix} K_S & 0 \\ 0 & K_F \end{bmatrix} + i\omega \begin{bmatrix} 0 & \rho_F L \\ -\frac{1}{\rho_F} L^T & \frac{1}{Z_a(\omega)} A_F \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & 0 \\ 0 & M_F \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ \Psi \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \end{array} \right\}, \tag{29}$$

where the acoustic velocity potential dofs Ψ are associated to ψ. A symmetric version of this equation is obtained by multiplying the fluid equations by $-\rho_F^2$:

$$\begin{bmatrix} K_S - \omega^2 M_S & i\omega\rho_F L \\ i\omega\rho_F L^T & -\rho_F^2 K_F + \omega^2 \rho_F^2 M_F - \rho_F^2 \frac{i\omega}{Z_a(\omega)} A_F \end{bmatrix} \left\{ \begin{array}{c} U \\ \Psi \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \end{array} \right\}. \tag{30}$$

In some industrial codes (e.g. *MSC.Nastran* [8]), this is the version which is implemented, in some cases it eventually differs from ρ_F factors in the definition of velocity potential. In all cases, a final symmetric system is obtained. This is of particular interest for modal analysis purposes when Σ_A does not exist, since dedicated eigenvalue solvers for generalized symmetric quadratic problems can be used. However, even in the case of direct harmonic response estimation, a formulation based on symmetric matrices leads to calculation time reduction. One can note that the previously presented FE formulations can sometimes be easily transformed to symmetric problems for frequency response using division of acoustic equations by proper terms (including frequency).

4.2 Considerations about the static case

The remarks evoked for the displacement potential regarding the static case are still valid for the velocity potential. One can use $\tilde{\psi}$ instead of ψ where [1]:

$$p = -i\omega\rho_F\tilde{\psi} + p^S. \tag{31}$$

The uniqueness of $\tilde{\psi}$ is guaranteed by the constraint [1] $\int_{\Omega_F} \tilde{\psi}\, d\Omega = 0$. This constraint leads to the definition of the subspace $\mathscr{C}_{\tilde{\psi}}$:

$$\mathscr{C}_{\tilde{\psi}} = \left\{ \tilde{\psi} \in \mathscr{C}_\varphi \;/\; \int_{\Omega_F} \tilde{\psi}\, d\Omega = 0 \right\}. \tag{32}$$

In a weak form, one has then to find $\mathbf{u}$, $\tilde{\psi}$ and p^S in $\mathscr{C}_{\mathbf{u}} \times \mathscr{C}_{\tilde{\psi}} \times \mathcal{R}$ such as, for all $(\delta\mathbf{u}, \delta\tilde{\psi}, \delta p^S) \in \mathscr{C}_{\mathbf{u}} \times \mathscr{C}_{\tilde{\psi}} \times \mathcal{R}$:

$$\left\{ \begin{array}{ll} \displaystyle\int_{\Omega_S} \sigma(\mathbf{u}) : \varepsilon(\delta\mathbf{u})\, d\Omega - \omega^2 \int_{\Omega_S} \rho_S \mathbf{u}\cdot\delta\mathbf{u}\, d\Omega & \\ \displaystyle\qquad + \rho_F i\omega \int_{\Sigma_{FS}} \tilde{\psi}\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma - \int_{\Sigma_{FS}} p^S\, \mathbf{n}_F \cdot \delta\mathbf{u}\, d\Sigma = \int_{\Sigma_S} \mathbf{F}_S \cdot \delta\mathbf{u}\, d\Sigma, & (a) \\ \displaystyle\frac{1}{\rho_F} \int_{\Omega_F} \nabla\tilde{\psi}\cdot\nabla\delta\tilde{\psi}\, d\Omega - \frac{\omega^2}{\rho_F c^2} \int_{\Omega_F} \tilde{\varphi}\delta\tilde{\psi}\, d\Omega - \frac{i\omega_F}{\rho} \int_{\Sigma_{FS}} \mathbf{u}\cdot\mathbf{n}_F\, \delta\tilde{\psi}\, d\Sigma & \\ \displaystyle\qquad + \frac{i\omega}{Z_a(\omega)} \int_{\Sigma_A} \tilde{\psi}\delta\tilde{\psi}\, d\Sigma - \frac{1}{\rho_F Z_a(\omega)} p^S \int_{\Sigma_A} \delta\tilde{\psi}\, d\Sigma = 0, & (b) \\ \displaystyle\frac{1}{\rho_F c^2} p^S V_F \delta p^S + \int_{\Sigma_{FS}} \mathbf{u}\cdot\mathbf{n}_F\, d\Sigma\, \delta p^S & \\ \displaystyle\qquad - \frac{\rho_F}{Z_a(\omega)} \int_{\Sigma_A} \tilde{\psi}\, d\Sigma\, \delta p^S + \frac{1}{i\omega Z_a(\omega)} p^S S_A \delta p^S = 0, & (c) \end{array} \right. \tag{33}$$

The FE discretization of this variational formulation can be written as:

$$\left(\begin{bmatrix} K_S & 0 & -\ell \\ 0 & K_F & 0 \\ \ell^T & 0 & \frac{V_F}{\rho_F c^2} \end{bmatrix} + i\omega \begin{bmatrix} 0 & \rho_F L & 0 \\ -\frac{1}{\rho_F} L^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \omega^2 \begin{bmatrix} M_S & 0 & 0 \\ 0 & M_F & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{i\omega}{Z_a(\omega)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_F & -\frac{1}{i\omega\rho_F} a_F \\ 0 & -\frac{\rho_F}{i\omega} a_F^T & -\frac{S_A}{\omega^2} \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ \tilde{\Psi} \\ p^S \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \\ 0 \end{array} \right\}, \tag{34}$$

where the acoustic velocity potential dofs $\tilde{\Psi}$ are associated to $\tilde{\psi}$. This system can be symmetrized by multiplying the equations related to dynamic fluid dofs by $-\rho_F^2$ and changing sign in the last line. One should emphasize that this system must be solved under the constraint $[C]^T\{\tilde{\Psi}\} = 0$.

5 Model reduction of structural-acoustic problem

5.1 Classical reduction using decoupled basis

In a generic way, all previous problems can be written as:

$$[K - \omega^2 M + \frac{i\omega}{Z_a(\omega)} A]\{Y\} = \{F\}, \tag{35}$$

where $\{Y\}$ includes a partition of structural and acoustic dofs. In this generic notation, the matrices can possibly depend on frequency. A very classical way to project the size of the harmonic problem is to search the response on a given vectorial space, typically built from the associated undamped problem. In our case, one classically define :

- the *in vacuo* structural modes, which are the normal modes of the structure without wet surface (i.e. for which Σ_{FS} is replaced with a free surface), these modes have shapes that can be stored in the structural modal matrix T_S;
- the *blocked* acoustic modes, which are the normal modes of the cavity in which both Σ_{FS} and Σ_A are replaced by rigid wall conditions. The associated shapes are stored in the acoustic modal matrix T_F.

One should emphasize that the decoupled mode shapes are identical in all formulations providing that the static pressure is not included in the formulation. The global projection matrix is then built as:

$$[T] = \begin{bmatrix} T_S & 0 \\ 0 & T_F \end{bmatrix} \tag{36}$$

One can reduce now the initial problem using the projection $\{Y\} = [T]\{q\}$ with $\{q\} = \begin{Bmatrix} q_S \\ q_F \end{Bmatrix}$:

$$[\bar{K} - \omega^2 \bar{M} + \frac{i\omega}{Z_a(\omega)} \bar{A}]\{Y\} = \{\bar{F}\}, \tag{37}$$

where:

$$[\bar{K}] = [T^T K T], \quad [\bar{M}] = [T^T M T], \quad [\bar{A}] = [T^T A T], \quad \{\bar{F}\} = [T^T]\{F\}. \tag{38}$$

5.2 Considerations about the static case

Concerning the remarks presented in the previous sections about the static case, one should underline that in literature, the constraints (**??**), (**??**) or (**??**) are generally omitted in the FE formulation. This is valid since the acoustic modes are calculated with rigid boundary conditions, which implies that they automatically verify the constraints. On the other side, for a full model computation, the constraints must be taken into account for proper estimation of the low frequency content of the responses.

Several strategies are available to take into account static response of the fluid domain in the projection:

- The mode p_0 can simply be added in the Ritz basis, even if this is not correct in a mathematical point of view, as indicated in [4].

- One of the proposed $(\mathbf{u}, p, p^S)$ $(\mathbf{u}, \varphi, p^S)$ or $(\mathbf{u}, \psi, p^S)$ formulations can be used without condensation of p^S.

- The impact of p^S on the structure can be evaluated using elastic modes, as indicated in [9]. In this case, its contribution is interpreted in terms of added mass and stiffness. The projection of acoustic part leads to:

$$\left(\begin{bmatrix} K_S + K_C & 0 \\ 0 & diag\left(\frac{1}{\rho_F}\right) \end{bmatrix} - \omega^2 \begin{bmatrix} M_S + M_C & LT_F diag\left(\frac{1}{\omega_\alpha^2}\right) \\ sym & diag\left(\frac{1}{\rho_F \omega_\alpha^2}\right) \end{bmatrix} \right) \left\{ \begin{array}{c} U \\ q_F \end{array} \right\} = \left\{ \begin{array}{c} F_S \\ 0 \end{array} \right\}, \quad (39)$$

where $M_C = \sum_{\alpha=1}^{n} \frac{\rho_F}{\omega_\alpha^2} L P_\alpha P_\alpha^T L^T$ is the added mass matrix and K_C is the added stiffness matrix obtained by the discretization of $p_0^2 \int_{\Sigma_{FS}} \mathbf{u} \cdot \mathbf{n} d\Sigma \int_{\Sigma_{FS}} \mathbf{n} \delta \mathbf{u} dS$.
The reduction of the structural part is then performed using the modified structural eigenvalue problem [1, 9]:

$$\left(K_s + K_c\right) U_\beta = \omega_\beta^2 \left(M_s + M_c\right) U_\beta, \quad (40)$$

where U_β is a structural mode of the structure including added mass and stiffness effects of fluid, associated to K_c and M_c. The reduced problem is then:

$$\left(\begin{bmatrix} diag(\omega_\beta^2) & 0 \\ 0 & diag\left(\frac{1}{\rho_F}\right) \end{bmatrix} - \omega^2 \begin{bmatrix} I & T_S^T L T_F diag\left(\frac{1}{\omega_\alpha^2}\right) \\ sym & diag\left(\frac{1}{\rho_F \omega_\alpha^2}\right) \end{bmatrix} \right) \left\{ \begin{array}{c} q_S \\ q_F \end{array} \right\} = \left\{ \begin{array}{c} T_S^T F_S \\ 0 \end{array} \right\} \quad (41)$$

It should be emphasized that the above equations have been obtained without considering the acoustic absorption surface Σ_a. When Σ_a, characterized by Z_a, is present, the methodology used to derive the previous system leads to complex relationships which has no special interest compared to the alternative ways to take into account the static behavior of the fluid cavity.

6 What's next?

Due to the limited number of pages in the WCCM proceedings papers, the next chapter of the story will be available during the oral presentation (and also in next papers!). Nevertheless, the main comments associated to the numerical simulations associated to the above formulations are that in all cases, the convergence rate is very low; moreover the direct reduction by using only "elastic" acoustic modes leads to large errors, it is then clear that including static contribution for the fluid part is necessary for correct estimation of low-frequency content of the responses. Another point is that considering static contribution through added mass and stiffness is efficient for a low number of modes, but the convergence is very low, due to the fact that the added mass is evaluated from modal reduction and that no information is provided in the fluid domain to improve convergence. Finally, the best convergence rate corresponds to direct inclusion of constant vector p_0 in the fluid basis and also to projection using the $(\mathbf{u}, p, p^S)$ formulation. In the last part of the talk, it will be shown that using alternative bases enrichment strategies can save a lot of computation time.

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