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Efficient enumeration of rooted maps of a given orientable genus by number of faces and vertices

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Abstract

We simplify the recurrence satisfied by the polynomial part of the generating function that counts rooted maps of positive orientable genus g by number of vertices and faces. We have written an optimized program in C++ for computing this generating function and constructing tables of numbers of rooted maps, and we describe some of these optimizations here. Using this program we extended the enumeration of rooted maps of orientable genus g by number of vertices and faces to $g = 4$, 5 and 6 and by number of edges to $g = 5$ and 6 and conjectured a further simplification of the generating function that counts rooted maps by number of edges. Our program is documented and available on request, allowing anyone with a sufficiently powerful computer to carry the calculations even further.

Keywords: Efficient enumeration, rooted maps, orientable genus, generating functions

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1 Introduction: definitions and history

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. The *faces* of a map are the connected components of the complement of the graph in the surface. In this article the surface is assumed to be without boundary and orientable, with an orientation already attributed to it (clockwise, say), so that it is completely described by a non-negative integer g, its *genus*. For short, a map on a surface of genus g will be called a *genus-g map*. A *planar map* is a

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genus-0 map (a map on a sphere) and a *toroidal map* is a genus-1 map (a map on a torus or donut). If a map on a surface of genus g has v vertices, e edges and f faces, then by the Euler-Poincaré formula [7, chap. 9]

$$v - e + f = 2(1 - g). \tag{1.1}$$

Two maps are *equivalent* if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other. A *dart* of a map or graph is a semi-edge. A loop is assumed to be incident twice with the same vertex, so that every edge, whether or not it is a loop, is incident to two darts. The *degree* of a vertex is the number of darts incident to it. The face incident to a dart d is the face incident to the edge containing d and on the left of an observer on d facing away from the vertex incident to d and the degree of a face is the number of darts incident to it. A *rooted map* is a map with a distinguished dart, its *root*. Two rooted maps are equivalent if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges, faces and the root of one map into the vertices, edges, faces and the root of the other. A *combinatorial map* is a connected graph with a cyclic order imposed on the darts incident to each vertex, representing the order in which the darts of a (topological) map are encountered during a rotation around the vertex according to the orientation of the imbedding surface. The darts incident to a face are encountered by successive application of the following pair of actions: go from the current dart to the dart on the other end of the same edge and then to the next dart incident to the same vertex according to the cyclic order. In this way the faces of a combinatorial map can be counted, so that its genus can be calculated from (1.1). Two combinatorial maps are equivalent if they are related by a map isomorphism – a graph isomorphism that preserves this cyclic order – with an analogous definition for the equivalence of two rooted combinatorial maps. By *enumerating* maps with a given set of properties, whether rooted or not, we mean counting the number of equivalence classes of maps with these properties. It was shown in [10] that each equivalence class of topological maps is uniquely defined by an equivalence class of combinatorial maps; so for the purposes of enumeration, the term "map" can be taken to mean "combinatorial map".

Let $m_g(v, f)$ be the number of rooted genus-g maps with v vertices and f faces. By face-vertex duality, this number is equal to the number $m_g(f, v)$ of rooted genus-g maps with f vertices and v faces. The generating function that counts rooted genus-g maps is the following formal power series in two variables u and w:

$$M_g(w, u) = \sum_{v,f \geq 1} m_g(v, f) w^v u^f. \tag{1.2}$$

Rooted maps were introduced in [13] because they are easier to count than unrooted maps; this is because only the trivial map automorphism preserves the root [14], so that rooted maps can be counted without considering map automorphisms. In [13], W. T. Tutte found a closed-form formula for the number of rooted planar maps with n edges. In [14], he found a parametric system of equations defining $M_0(w, u)$. In [1] D. Arquès obtained the simpler expression

$$M_0(w, u) = pq(1 - 2p - 2q) \tag{1.3}$$

with the parameters p and q defined by

$$w = p(1 - p - 2q) \tag{1.4}$$

and

$$u = q(1 - 2p - q), \tag{1.5}$$

where $p = q = 0$ when $w = u = 0$. In [14], a recursive formula was found for the number of rooted planar maps given the number of vertices, the number of edges, and the degree of the face containing the root; these numbers of maps were then added over all possible degrees of this face and the result expressed in terms of generating functions. In [15], the first author generalized this method to obtain a recursive formula for the number of maps of genus g with a distinguished dart in each vertex given the number of vertices and the degree of each one; these numbers were then multiplied by the appropriate factor and added over all possible non-increasing sequences of vertex-degrees summing to $2e$ to obtain the number of rooted maps of genus g with e edges and v vertices. A table of these numbers of maps with up to 14 edges appears in [15] (see [17] for a published account of this work and a table of maps with up to 11 edges) but no attempt was made there to express this result in terms of generating functions. We note here that a similar generalization in which the degrees of all the faces are known but only some of them have a distinguished edge on their boundary, and these faces must be of degree at least 3, appears in [8], where it is attributed to Tutte under the name of Tutte's recursion equations.

In [5] an improvement on the method of [15] was introduced: to count rooted genus-g maps it is sufficient to know the degree of the first $g + 1$ vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered. Using doubly-rooted maps, D. Arquès [2] obtained the analogue of (1.3) for toroidal maps:

$$M_1(w, u) = \frac{pq\,(1 - p - q)}{\left[(1 - 2p - 2q)^2 - 4pq\right]^2}. \tag{1.6}$$

From this result, he obtained a closed-form formula for the number of rooted toroidal maps with e edges and another one for the number of rooted toroidal maps with v vertices and f faces. In [6] a generating function was obtained for the number of rooted maps of genus 2 and 3 with e edges.

In [9] the second author generalized (1.6) and obtained a general form for the generating function $M_g(w, u)$ counting rooted maps of any genus $g > 0$:

$$M_g(w, u) = \frac{pq\,(1 - p - q)\,P_g(p, q)}{\left[(1 - 2p - 2q)^2 - 4pq\right]^{5g-3}}, \tag{1.7}$$

where $P_g(p, q)$ is a symmetric polynomial in p and q of total degree bounded by $6g - 6$ with integral coefficients (in what follows, unless otherwise specified, all the polynomials defined here are polynomials in p and q). The polynomial P_g is defined in terms of another polynomial T_g of degree bounded by $10g - 8$ by

$$P_g = \frac{T_g}{(1 - p)^{4g-2}}, \tag{1.8}$$

and that polynomial, in turn, is defined in terms of a family of polynomials $R_g(n_1, \ldots, n_r)$ in p and q by

$$T_g = R_{g-1}(0, 0) + \sum_{j=1}^{g-1} q(1 - p - q)R_j(0)R_{g-j}(0). \tag{1.9}$$

The degree of the polynomial $R_g(n_1, \ldots, n_r)$ is defined by the equation

$$\deg R_g(n_1, \ldots, n_r) = 2(n_1 + \ldots + n_r) + 7r + 10g - 12. \tag{1.10}$$

The polynomials $R_g(n_1, \ldots, n_r)$ are defined recursively in terms of several other families of polynomials and a recursively-defined family of rational functions of p and q. We have two finite families of polynomials in p alone defined by the following two sets of equations.

$$\begin{gathered} K_0(p) = -p; \qquad K_1(p) = -1 - p; \qquad K_2(p) = -1; \\ K_m(p) = 0 \text{ for all } m \geq 3. \end{gathered} \tag{1.11}$$

$$\begin{gathered} L_0(p) = -p; \qquad L_1(p) = -1 - 2p; \\ L_2(p) = -2 - p; \qquad L_3 = -1; \\ L_k(p) = 0 \text{ for all } k \geq 4. \end{gathered} \tag{1.12}$$

In what follows, the parameter p will be omitted, so that these polynomials will be referred to as K_m and L_k. We then have two polynomials H and J (in p and q) defined by

$$J = q(1 - p - q) \tag{1.13}$$

and

$$H = (1 - 2p - 2q)^2 - 4pq. \tag{1.14}$$

Finally we have an infinite family $(E_k)_{k \geq 1}$ of rational functions of p and q, all but the first two of which are polynomials, defined recursively by

$$\begin{gathered} E_1 = \frac{1}{2J(1-p)^2}; \\ E_2 = \frac{-p - 4q + 2p^2 + 4q^2 + 4pq}{2J(1-p)^2}; \\ E_3 = -1; \\ E_k = -J(1-p)^2 \sum_{i=2}^{i=k-1} E_i E_{k+1-i} \text{ for all } k \geq 4. \end{gathered} \tag{1.15}$$

To make the recursive definition of the polynomials $R_g(n_1, \ldots, n_r)$ comprehensible, we first explain the abbreviations we use. For any positive integer r, $[r]$ denotes the sequence $(2, \ldots, r)$ if $r \geq 2$ and the empty sequence if $r = 1$. For any subsequence X of $[r]$, $[r] - X$ denotes the subsequence of the elements of $[r]$ that are not in X. For any sequence $(n_2, \ldots, n_r)$ of integers, N_X denotes the sequence of those n_i such that i is in X and N_j denotes the sequence $(n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r)$.

The polynomials $R_0(n_1)$ are not defined. The anchor of this recursive definition is

$$R_0(0,0) = (1-p)^2. \tag{1.16}$$

If $g = 0$ and $r = 2$ but $(n_1, n_2) \neq (0, 0)$, then we have

$$\begin{aligned} R_0(n_1, n_2) &= (1-p)^2 \left(-n_2 H E_{n_1+n_2+2} - (n_2+1) E_{n_1+n_2+3}\right) \\ &\quad + 2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0,\, k<n_1}} (-1)^{j+1} H^j E_i R_0(k, n_2). \end{aligned} \tag{1.17}$$

We note that (1.16) is a special case of (1.17) where $n_1 = n_2 = 0$.

If $(g, r) \neq (0, 2)$, then

$$R_g(n_1, \ldots, n_r) = \text{term}_1 + \text{term}_2 + \text{term}_3 + \text{term}_4, \tag{1.18}$$

where

$$\text{term}_1 = 2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1 \\ i>0,\, k<n_1}} (-1)^{j+1} H^j E_i R_g(k, n_2, \ldots, n_r) \tag{1.19}$$

(we note that the second line of (1.17) is a special case of (1.19) where $g = 0$ and $r = 2$),

$$\text{term}_2 = J \sum_{\substack{k+l+m=n_1+1 \\ 0\le j\le g \\ X\subseteq[r] \\ (j,X)\neq(0,\emptyset) \\ (j,X)\neq(g,[r])}} K_m H^m R_j(k, N_X) R_{g-j}(l, N_{[r]-X}), \tag{1.20}$$

$$\text{term}_3 = \sum_{i+j+m=n_1+1} K_m H^m R_{g-1}(i, j, N_{[r]}) \tag{1.21}$$

and

$$\text{term}_4 = \sum_{j=2}^{r} \left(\begin{array}{c} n_j \sum_{k+l=n_1+n_j+2} L_k H^{k+1} R_g(l, N_j) \\ +(n_j+1) \sum_{k+l=n_1+n_j+3} L_k H^k R_g(l, N_j) \end{array} \right). \tag{1.22}$$

It was shown in [9] that each polynomial $R_g(n_1, \ldots, n_r)$ is symmetric in all its variables. This was made possible by distinguishing a dart incident to each of the vertices whose degree is considered, which increases the size of the coefficients but does not increase the number of polynomials that have to be calculated.

We note here that in the account of these results published in [3] formula (1.17) and the sum in (1.15) are missing; the formulas are presented correctly in [9]. At that time the second author, programming in Maple, calculated the polynomial P_g and the generating function $M_g(w, u)$ for $g = 2$ and $g = 3$ (these results are published in [3]) and also computed the generating function that counts rooted maps of genus 4 by number of edges. This result was recently included in [11], where it was used to count both rooted and unrooted maps of genus 4 by number of edges.

Recently, the second author extended his enumeration results to genus 5. The first author, programming mainly in C, optimized the calculation of the polynomials $R_g(n_1, \ldots, n_r)$ and thus extended the enumeration by number of vertices and faces, as well as by number of edges, to genus 6. Although each author used a different algorithm and a different programming language, we both obtained the same answers, and the numbers of rooted maps we calculated agree with the tables in [15], providing evidence of the correctness of our results. An account of these extensions is given in Sections 2 and 3 and the polynomials $P_g(p, q)$ appear in Appendix A. A discussion of the enumeration of rooted genus-g maps by number of edges appears in Section 4 and the polynomial part of each of the corresponding generating functions appears in Appendix B. Finally, a discussion of some open problems appears in Section 5.

2 Results from the Maple program

A first version of the Maple code written in 1998 implemented recurrence relations between the rational functions introduced in [4] for the computation of the generating functions M_g. It was not designed for efficiency but for validating formulas from [4]. That code has also been used for validating the formulas from (1.2) through (1.22) for the first values of g, r and $n_1, \ldots, n_r$ (these formulas were first obtained from a long computation that was done by hand and is thus error-prone). When executed in 1998 with Maple V for computing $M_4(w, u)$ that code ran into a fundamental limitation (wired into the Maple kernel) of a maximal number of 65,535 terms in any polynomial.

That old code has been recently replaced by a simpler code implementing directly the recursion between polynomials described by the formulas from (1.2) through (1.22). The code is short (less than 400 lines) and resembles the mathematical formulas as much as possible in order to detect errors. All the results obtained by this new code match known results in rooted map enumeration. For all these reasons, it can be considered as a reference for the debugging of optimized implementations.

With a personal computer running under Windows XP with an Intel Core 2 Duo CPU at 2.19 GHz and 3.5 Gb of memory, and a Maple 14 release supporting larger objects, the next two generating functions $M_4(w, u)$ and $M_5(w, u)$ were successfully computed in 4 minutes and 5 hours, respectively. It was, however, not sensible to continue using this inefficient prototype for computing the next generating functions. A better idea was to write an independent implementation optimizing memory space and execution time.

3 Optimizations and the C program

Aside from the advantage in execution speed that C has over Maple, the first author optimized the calculation of the polynomials $R_g(n_1, \ldots, n_r)$. One of these optimizations was made possible by the following observation.

Proposition 1. For any $(g, r) \neq (0, 1)$ and any sequence $n_1, \ldots, n_r$, the polynomial $R_g(n_1, \ldots, n_r)$ is divisible by $(1 - p)^2$.

Proof. (by generalized induction on the degree of a polynomial of the form $R_g(n_1, \ldots, n_r)$, which we call an *R-polynomial*).

Basic step (degree 2) The only R-polynomial of degree 2 is $R_0(0, 0) = (1 - p)^2$: see (1.16).

Induction step Suppose that the degree d of a given R-polynomial $R_g(n_1, \ldots, n_r)$, as defined by (1.10), is greater than 2 and that every R-polynomial of degree $< d$ is divisible by $(1 - p)^2$. We show that $R_g(n_1, \ldots, n_r)$ is also divisible by $(1 - p)^2$. Since every R-polynomial on the right side of equations (1.17), (1.19), (1.20), (1.21) and (1.22) is of degree $< d$, it follows from the induction hypothesis that each such polynomial is divisible by $(1 - p)^2$. We examine each of these equations in turn. Equation (1.17). The first line contains a factor $(1-p)^2$. The term $E_{n_1+n_2+3}$ is a polynomial for any non-negative n_1 and n_2. The term $E_{n_1+n_2+2}$ is a polynomial unless $n_1 = n_2 = 0$, but in this case $E_{n_1+n_2+2}$ is multiplied by $n_2 = 0$; so the first line of (1.17) is divisible by $(1 - p)^2$. In the second line, each term of the sum contains a polynomial $R_0(k, n_2)$, which, by the induction hypothesis, is divisible by $(1-p)^2$. This factor of $(1-p)^2$ could be cancelled by E_1 or E_2, but the sum is nevertheless a polynomial, and the factor $(1-p)^2$ by which the sum is multiplied ensures that the second line of (1.17) too is divisible by $(1-p)^2$; so the right side of (1.17) is divisible by $(1-p)^2$. Equation (1.19). By an argument similar to the one used for the second line of (1.17), the right side of (1.19) is divisible by $(1-p)^2$.

Equations (1.20)-(1.22). Each term in the sum contains at least one R-polynomial that is divisible by $(1-p)^2$; so the right side of each of these equations is divisible by $(1-p)^2$. It follows from (1.18) that $R_g(n_1,\ldots,n_r)$ is divisible by $(1-p)^2$, which completes the proof. □

We now modify equations (1.8)-(1.10), and (1.16)-(1.22) in the light of Proposition 1. We introduce a new family of polynomials (which we call *S-polynomials*) defined by

$$S_g(n_1,\ldots,n_r) = R_g(n_1,\ldots,n_r)/(1-p)^2 \tag{3.1}$$

and we also let

$$U_g = T_g/(1-p)^2. \tag{3.2}$$

Then U_g is a polynomial of degree $10(g-1)$ and (1.8)-(1.10) become (3.3)-(3.5), respectively.

$$P_g = \frac{U_g}{(1-p)^{4g-4}}, \tag{3.3}$$

$$U_g = S_{g-1}(0,0) + q(1-p-q)\,(1-p)^2 \sum_{j=1}^{g-1} S_j(0)S_{g-j}(0). \tag{3.4}$$

$$\deg S_g(n_1,\ldots,n_r) = 2(n_1+\ldots+n_r) + 7(r-2) + 10g. \tag{3.5}$$

Also, (1.16)-(1.22) become (3.6)-(3.12), respectively.

$$S_0(0,0) = 1, \tag{3.6}$$

$$\begin{aligned} S_0(n_1,n_2) \;&=\; (-n_2 H E_{n_1+n_2+2} - (n_2+1)E_{n_1+n_2+3}) \\ &\quad +2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1\\ i>0,\, k<n_1}} (-1)^{j+1}\, H^j E_i S_0(k,n_2). \end{aligned} \tag{3.7}$$

If $(g,r) \neq (0,2)$, then

$$S_g(n_1,\ldots,n_r) = \text{term}_5 + \text{term}_6 + \text{term}_7 + \text{term}_8, \tag{3.8}$$

where

$$\text{term}_5 = 2J(1-p)^2 \sum_{\substack{i+j+k=n_1+1\\ i>0,\, k<n_1}} (-1)^{j+1}\, H^j E_i S_g(k,n_2,\ldots,n_r), \tag{3.9}$$

$$\text{term}_6 = J(1-p)^2 \sum_{\substack{k+l+m=n_1+1\\ 0\le j\le g\\ X\subseteq[r]\\ (j,X)\neq(0,\emptyset)\\ (j,X)\neq(g,[r])}} K_m H^m S_j(k,N_X)S_{g-j}(l,N_{[r]-X}), \tag{3.10}$$

$$\text{term}_7 = \sum_{i+j+m=n_1+1} K_m H^m S_{g-1}(i,j,N_{[r]}), \tag{3.11}$$

and

$$\text{term}_8 = \sum_{j=2}^{r} \begin{pmatrix} n_j \sum_{k+l=n_1+n_j+2} L_k H^{k+1} S_g(l,N_j) \\ +(n_j+1) \sum_{k+l=n_1+n_j+3} L_k H^k S_g(l,N_j) \end{pmatrix}. \tag{3.12}$$

Since $R_g(n_1,\ldots,n_r)$ is symmetric in all its variables, so is $S_g(n_1,\ldots,n_r)$; so only those polynomials $S_g(n_1,\ldots,n_r)$ with $n_1 \le \ldots \le n_r$ are treated. In all the S-polynomials on the right side of each of the equations (3.9)-(3.12), only the first two variables can violate these inequalities; so they are inserted into their proper slots among the remaining variables to preserve the inequalities. Also, equation (3.4) is symmetric in j and $g-j$, equation (3.10) is symmetric in k and l and equation (3.11) is symmetric in i and j; so the calculations there can be cut almost in half. In equation (3.12), each polynomial $S_g(l,N_j)$ is calculated only once and then used twice. The following easily proved observations can be used to avoid calculating a polynomial that is identically 0: $\text{term}_5 = 0$ if $n_1 = 0$, $\text{term}_6 = 0$ if $g+r \le 2$, $\text{term}_7 = 0$ if $g = 0$, $\text{term}_8 = 0$ if $r = 1$ or $(g,r) = (0,2)$. From these observations, it follows that the only term that could possibly contribute to $S_0(n_1)$ is term_5. From (3.9) it follows by generalized induction on n_1 that $S_0(n_1) = 0$ for all $n_1 \ge 0$; so these polynomials do not have to be defined.

All the S-polynomials are stored in a single one-dimensional array s. A preliminary recursion does not calculate any of these polynomials. Instead, it calculates all the quadruples (d,g,r,c) of parameters of the S-polynomials that will later be calculated, where $d = \deg S_g(n_1,\ldots,n_r)$ and c is an integer coding the sequence $(n_1,\ldots,n_r)$, and stores the list of quadruples in four parallel arrays, one array for each of the four parameters d, g, r, c and one element of all four arrays for each quadruple (d,g,r,c). The program then sorts the four parallel arrays by degree d using bucket sort, computes the number of S-polynomials that have to be calculated and the total number of terms in these polynomials and stores in two arrays the index in s and the one in the four parallel arrays of the first term for each degree d. Then the S-polynomials are calculated in increasing order of their degree and stored in s. This can be done non-recursively because all the S-polynomials that need to be used will have already been stored and need only be found by searching the four parallel arrays, starting with the first index for the appropriate degree d, for the appropriate parameters, and adding $(d+1)(d+2)/2$ to the index in s each time the index in the four parallel arrays is increased by 1. Once the last polynomial $S_{g-1}(0,0)$ has been calculated, first (3.4) is used to calculate U_g and then (3.3) is used to calculate P_g and its coefficients are stored in a text file, which is available from the first author on request. The polynomials P_2 and P_3 appear in [3]. The polynomials P_4, P_5 and P_6 appear in Appendix A. Because these polynomials are symmetric in p and q, to save space we include only those terms in which the exponent of p is at least as great as the exponent of q.

The number of S-polynomials that have to be calculated is roughly the total number of partitions of all the positive integers up to $10(g-1)$. For each of these polynomials, the most expensive calculation is term_6, because the sum there runs over all the partitions of the sequence $[r] = (2,\ldots,r)$, where r can be as great as $g+1$, and involves multiplying two S-polynomials. The time-complexity of calculating P_g is therefore exponential in g, but the optimizations made here nevertheless made it possible to calculate P_g for a greater value of g than was possible previously. Another program computes a table of numbers of rooted genus-g maps counted by number of vertices and faces by reading this file and using (1.2) if $g \geq 1$ or (1.3) if $g = 0$. Tables of numbers of rooted genus-g maps for any $g \leq 6$ and with up to any reasonable number of edges are available from the first author on request.

The programs were written mainly in C. The one that computes the polynomials is about 2000 lines long and the one that computes the tables is about 300 lines long. They both use the C++ library CLN to do arithmetic on big integers because CLN reads arithmetic expressions in C that use only addition, multiplication and subtraction; only statements involving quotients, remainders, input/output of big integers and file management had to be modified. Since CLN requires a GNU compiler, XCODE was downloaded and installed by Jerome Tremblay, a computer technician at UQAM, who also downloaded and installed CLN and wrote sample C++ statements for input/output of big numbers and file management.

The programs were executed on a 2004 Macintosh GR4 computer. The time taken to compute the polynomial P_g varied from run to run. In Table 1 we show, for each g from 1 to 6, the number of S-polynomials that were calculated, the total number of terms in all these S-polynomials, and a typical execution time. Once the S-polynomials had been computed and stored, it took the computer only 48 seconds to make a table of numbers of genus-6 maps with up to 42 edges counted by number of vertices and faces. Source codes of both programs are available from the first author on request.

g	number of S-polynomials	total number of terms	execution time
1	1	1	instantaneous
2	16	507	1 second
3	67	7407	10 seconds
4	205	49796	2 minutes
5	543	235410	20 minutes
6	1314	900114	3,5 hours

Table 1: Evaluation of the computation cost

4 Counting by number of edges

To compute the generating function $M_g(z) = z^{2g-2}M_g(z,z)$ that counts rooted genus-g maps by number of edges alone, we use the substitution obtained in [11], which is a more compact form of the one obtained in [9] and published in [3]. Let

$$p = q = m, \tag{4.1}$$

where

$$z = m(1-3m) \text{ and } m = 0 \text{ when } z = 0. \tag{4.2}$$

By substituting from (4.1) into (1.4) and (1.5) to express w and u in terms of m and then substituting into (1.7), we obtain the following equation for $g \geq 1$:

$$M_g(z) = \frac{m^{2g}(1-3m)^{2g-2}P_g(m,m)}{(1-6m)^{5g-3}(1-2m)^{5g-4}}. \tag{4.3}$$

For $g = 0$, we substitute into (1.3) instead of (1.7) and obtain

$$M_0(z) = (1-3m)^{-2}(1-4m). \tag{4.4}$$

The first author computed $M_g(z)$ from the computed values of $M_g(w,u)$ for $g \leq 6$. The program divides the polynomial $P_g(m,m)$ by $1-2m$ as often as possible. The program then divides the resulting polynomial by 2 and by 3 as often as possible, extracts the appropriate constant factor and then stores the resulting generating function in another text file, also available from the first author. The second author computed $M_g(z)$ directly for $g \leq 6$. We then compared our formulas and verified that they agree. The formulas for $P_g(m,m)$ for $1 \leq g \leq 6$ appear in Appendix B. Now $P_g(m,m)$ is of degree $6g-6$, but we found experimentally that for $1 \leq g \leq 6$, $P_g(m,m)$ is divisible by $(1-2m)^{2g-2}$, so that the quotient is only of degree $4g-4$, and we conjecture that this will be the case for any positive integer g.

5 Some interesting open problems

The recurrences satisfied by the R- and S-polynomials both result from proofs by induction. After the right conjecture has been guessed by observing the first computed terms, these proofs are not difficult to find, but they are tedious and error-prone due to the length of the expressions involved. Thus they are good candidates for automation. We plan to develop a suitable formal framework for assisting this kind of proofs with a computer algebra system. The challenge is to shorten the chain of conjectures and proofs about the general pattern of generating functions for counting rooted maps. This chain has been initiated in [6] and continues in the present work with a new conjecture that the polynomial $P_g(m,m)$ is divisible by $(1-2m)^{2g-2}$ for all $g \geq 1$. Once the numbers of rooted maps of genus up to g are known, the number of unrooted maps up to genus g can be calculated using the methods presented in [12]. As was mentioned above, the second author collaborated with A. Mednykh to count rooted and unrooted maps of genus 4 by number of edges [11]. It would be interesting to count unrooted genus-g maps by number of vertices and faces for as many values of g as possible (see [16] for an account of the progress made on this problem).

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Appendix A

Coefficients of the polynomial $P_g(p,q)$ in the generating function $M_g(w,u)$. The exponent of p is i and the exponent of q is j.

i	j	$g=4$	$g=5$	$g=6$
0	0	225225	59520825	24325703325
1	0	-3447873	-1052857260	-467684337375
2	0	13149279	2996675136	331572725235
1	1	75740652	29236662630	15552480037725
3	0	99376849	84903326508	78629111673210
2	1	-601290577	-278088909474	-164732858445258
4	0	-1364702625	-1167635102634	-1115786805002100
3	1	1323954013	314258237049	-197193320467497
2	2	7798852738	5139190834104	4114016654215968
5	0	7278085815	7383190634820	7679132084813280
4	1	11669094136	16626928198725	22225108689398171
3	2	-44518090957	-38825216035101	-38011804449745265
6	0	-21915530637	-25517944452000	-25194145114920600
5	1	-111691406491	-176205495337005	-258660509951405369
4	2	87176823147	71414891373744	46817554992783206
3	3	394951357187	521245727579217	698821084235510238
7	0	34368350016	27486911684232	-34634849450319300
6	1	467365399116	944779067359587	1599994018015673343
5	2	369316330580	1065131939557443	2325052326558898989
4	3	-1653054602996	-3109774180465479	-5305278050309690436
8	0	5493029256	199121845688766	875431778419760250
7	1	-1150226915904	-2978944697593944	-5427079882725409196
6	2	-3099366738272	-10502687980853538	-26281328772579796556
5	3	2799989555248	6485662227087078	11612050266758295299
4	4	9770469656312	28864236434784750	67234802590102445596
9	0	-168971030800	-1290553718021568	-5250824393143736550
8	1	1603000133328	4319747300224332	2631424256603826266
7	2	10375535302688	49447100947421712	151752654232148634876
6	3	4675439367040	33443054101099032	134785360586359808415
5	4	-29062591554512	-131655002707242864	-403081200640447102025
10	0	446623903440	4175411121161280	19107855057709768350
9	1	-531396192624	7557377263434480	79781498509512839622
8	2	-20288443296752	-140196978107516880	-514882253213415808290
7	3	-37157221449760	-322195198691448864	-1507879430637581935244
6	4	38459634403216	248581069034660208	891983836146643438886
5	5	113455708544944	864416573906739888	3712198575809542074306
11	0	-669316766976	-8761889037807360	-44994838722276643200
10	1	-2677064058240	-62049749603864880	-502584153034345687008
9	2	23203491496832	217239133515491184	728298352379130773279
8	3	95704575490304	1301723014819664880	7881173581586749085040
7	4	25495111214592	573069029641873824	4757786800382576737632
6	5	-232473613207168	-2999701307315295744	-17695909946707419577712
12	0	636872524288	11812486910550144	53771085509744694000
11	1	6233151448832	186781204204153728	1776396207720770666496
10	2	-10474170075648	18744511957965024	2455921681097348996048
9	3	-139300678503936	-3164855575142421984	-24856837687770483118400
8	4	-194528089368832	-5372851934394547296	-51467124780997381162872
7	5	221773985029376	4735380833365872192	35482577674056890378656
6	6	587209040349952	14241231802471091712	123284812684751523348352
13	0	-363019168512	-7139224937714688	63590534180019957600
12	1	-7182531345920	-352660752052764480	-4153379048094569728032

i	j	$g=4$	$g=5$	$g=6$
11	2	-10713306325248	-1042554952469042304	-19187364188750019565152
10	3	117776047833344	4516890628991951232	41344101184760217491296
9	4	358004273099520	17994448820561075520	236076485845255486809152
8	5	55421194708224	5366219028182108736	102780528051948653669760
7	6	-786362512663296	-37365766731378686976	-470331494499995012997600
14	0	78087609600	-8879362292376576	-503264270816231517600
13	1	4962585601792	439159836169197312	6005117202274911079072
12	2	22091079382272	2948070190097081856	65776115947137537556992
11	3	-45065453874432	-1806778237391685888	23807428660187432918496
10	4	-351445117523968	-35859099682771061760	-660157140743706256305248
9	5	-401398904040960	-48629860449837175296	-1063915074157728176550144
8	6	491093270238720	47194241425177354752	820636589635225892801440
7	7	1207155699797760	128635994421962951424	2522653660728373415372256
15	0	45405634560	30437536586096640	1405897564579656192000
14	1	-1987130500096	-313889834629182720	-2020685626130647246336
13	2	-17307386887168	-4812156134775883008	-145688358159205714516480
12	3	-10958309010432	-7595020748159177472	-372207235907475544579328
11	4	190354542085120	43175298148740913920	1069653070552859581622528
10	5	465103051362304	130164330913266000384	4191034331752408859522048
9	6	37722642530304	26373910825745369088	1348526588106895910820608
8	7	-948072226163712	-250823681908999848192	-7692683355481632388033280
16	0	-40611158016	-44077293430410240	-2564721668926444646400
15	1	353838563328	-5940163278028800	-15274501919657531805696
14	2	7138023579648	5117741830087712256	212064524536256523055104
13	3	19519294685184	20212507330103302656	1226553637690902900555264
12	4	-45947746897920	-21482140813979609088	-243040382942721249550464
11	5	-256322714959872	-203161172517235001856	-10071601970923125029169408
10	6	-250719971303424	-231576053975551653888	-13403298477467071272766592
9	7	324137392078848	243040355999985357312	11365401614796151597940480
8	8	754590724208640	616244259549811046400	31892902161216796227875328
17	0	12330160128	40858307934732288	3273804586604498150400
16	1	19965382656	316732084592477184	47937416598433295393792
15	2	-1439829344256	-3340755810744705024	-155152767030038497562624
14	3	-7733071233024	-27057537786195204096	-2469969025869879300964864
13	4	-1529064972288	-24298385693430761472	-3923454526405568393459200
12	5	65721400811520	192393332427115152384	14684112497002422050129408
11	6	136304356098048	473458240345259114496	44474213688772614218402304
10	7	3891353124864	63034150038205237248	10602745811746841269319168
9	8	-262502705516544	-864078559778445296640	-77130360167816877925214720
18	0	-1370017792	-25309570652061696	-2707735007162416358400
17	1	-12330160128	-418385622566006784	-84056393940944669041664
16	2	95137689600	768179107347640320	-122780537702226588027904
15	3	1040585146368	22706940893359718400	3309663787084255470538752
14	4	1986574786560	61057331397679042560	12359357401548728525481984
13	5	-5282409259008	-86028208996517953536	-7168322023226714759682048
12	6	-23295414902784	-546489504712274436096	-89408261564968694578497536
11	7	-20121257705472	-542098581430571433984	-102137784895213554206486528
10	8	27256403496960	608008773914232569856	94051342966594967803089920
9	9	60834856079360	1458371027883359490048	245931328023225286486623232
19	0		9636065496465408	723707611230002380800
18	1		310120924856217600	100676097683364867538944
17	2		834314246568382464	570700302207978850320384
16	3		-11709595470763929600	-2627348914643784470540288
15	4		-62287140959081545728	-21834662412829371652194304
14	5		-30794721181405360128	-22644744973507497254154240
13	6		375565947662073778176	111522176315180495700201472
12	7		791440494610352246784	279143888436091060333703168
11	8		63392468868069912576	48601657178646005372989440
10	9		-1386893855406156496896	-464898279555441038645071872

i	j	$g = 4$	$g = 5$	$g = 6$
20	0		-1314425012060160	1734907897013798707200
19	1		-144756328655585280	-82270304819565116424192
18	2		-1003742680227373056	-950537663871767965327360
17	3		2793526743911620608	123942395082217765601280
16	4		37212677250180292608	25401535424315681802590208
15	5		74033510974972723200	68224472042582041260310528
14	6		-126547074782265335808	-60103610493582286559100928
13	7		-617784335206111223808	-455901258606342321520386048
12	8		-546726907955477200896	-459420163369720948794875904
11	9		649350664927183921152	454685060181345136472375296
10	10		1490642394729913761792	1124259071282640899991023616
21	0		-691618625617920	-3287209913451861196800
20	1		40536738621112320	37757890897078575218688
19	2		525558553546752000	1012421892818336673087488
18	3		537141243831877632	2933439037885385986981888
17	4		-12971512057664028672	-18499038609283674999029760
16	5		-50848326713136218112	-101915380576965190921158656
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14	7		271282055277714407424	465259720654830491776679936
13	8		505564751059841875968	1004168281109519127253991424
12	9		19279289107118014464	123399556004059650470100992
11	10		-858679181948511092736	-1623142957450538470768787456
22	0		422807362338816	3275288152263117619200
21	1		-5245555110445056	5444851329172509179904
20	2		-151512465918394368	-732441188073602443542528
19	3		-600079032863293440	-4616834991352115554598912
18	4		2127180102928957440	4780217912523242048700416
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15	7		-55489682094287486976	-230059791405368439926259712
14	8		-223545638210122285056	-1283832320250295204348280832
13	9		-179740722217994551296	-1164487905229217354305224704
12	10		224480279722662297600	1227775939882047307777605632
11	11		497283663440438231040	2901788578865019448150867968
23	0		-93895550042112	-2191214043993597542400
22	1		-180122816741376	-25945834528510036738048
21	2		22137226308747264	331074763427574885711872
20	3		166230484940881920	4142547710539723494981632
19	4		29854175151390720	6580460553696860428894208
18	5		-3108417189221892096	-59294451733964817490968576
17	6		-9621688907482398720	-242963919760936271213494272
16	7		-288936506265698304	-104046724336661909629304832
15	8		46767719292133441536	1018758498586088458575544320
14	9		78823390451111165952	1952022892302645218017542144
13	10		295628344393334784	158260143999731503140700160
12	11		-130643039199866781696	-3084421320247115490844278784
24	0		7824629170176	1002905366223804825600
23	1		93895550042112	23866267878384050372608
22	2		-1112799110234112	-47763916603770777894912
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20	4		-41493010924044288	-9848254830211976468594688
19	5		179786567324270592	16787453017997301006204928
18	6		1107409870381645824	180200756079958706269814784
17	7		1385183221916368896	297825520442142229904818176
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15	9		-10948740544549355520	-1852715643223710048204619776
14	10		-8105149379104210944	-1536106917047051711260983296
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12	12		22778024644118446080	3900362066433824862261280768

i	j	$g = 4$	$g = 5$	$g = 6$
25	0			-285304295475875020800
24	1			-1301986440416791717478 4
23	2			-5103417421738584440832 0
22	3			9040449577084390407536 64
21	4			69368633766209090233958 40
20	5			57321059192585440405094 40
19	6			-831755327848594753638236 16
18	7			-271158382680239061117566 976
17	8			-647044118074312205657702 40
16	9			106474493774775994879613 3376
15	10			184961950746126272251153 6128
14	11			855050219841166582046392 32
13	12			-287150100667022740361674 7520
26	0			27690864336578150400
25	1			458810634969058941337 6
24	2			4350179596095056550297 6
23	3			-1584437800515737655705 60
22	4			-2984431989395324335947 776
21	5			-8444613336874597020205 056
20	6			19385457184008341525495 808
19	7			14184611952826587991323 4432
18	8			19354428919151430304766 3616
17	9			-31329247407436422891464 2944
16	10			-11948586527603189352987 36128
15	11			-91649409095917073006198 7840
14	12			10739516078567885722433 61792
13	13			23712417763178007648769 14688
27	0			13380773146295009280
26	1			-98917634524785619763 2
25	2			-17138514929224228274176
24	3			-23157227973833071263744
23	4			77189409309206503227392 0
22	5			41450857445424454487244 80
21	6			13200694056839317542666 24
20	7			-43540978794449370439745 536
19	8			-11798532681439328021040 3328
18	9			-10447319791863684478795 776
17	10			44013444374209912795824 1280
16	11			70405353582280674273709 2608
15	12			11679697966843194243022 848
14	13			-10779516390810652043326 25920
28	0			-6190520242793349120
27	1			101016358687762022400
26	2			380790198651308212224 0
25	3			19615951537674393747456
24	4			-976745132202425623511 04
23	5			-10841029135624607839027 20
22	6			-22553481302014752624476 16
21	7			65154949034209477310545 92
20	8			36369531269030571785846 784
19	9			41835756382734171758395 392
18	10			-77589384625987118061060 096
17	11			-25973883428865780135179 0592
16	12			-18604594256238540824562 8928
15	13			22804868246747510323386 7776
14	14			49042347471630304549326 0288

i	j	$g=4$	$g=5$	$g=6$
29	0			1083051875657318400
28	1			2298277762254766080
27	2			-437826375156907376640
26	3			-4211038540274023268352
25	4			-200747762008901812224
24	5			143015960801484793184256
23	6			580006330365817173049344
22	7			-32493650288141295157248
21	8			-5512732263193381158518784
20	9			-12796038146349004392235008
19	10			456361894794872437604352
18	11			45861472268551994896023552
17	12			68348456393778007627530240
16	13			-628640702799589914181632
15	14			-103507964466220510312660992
30	0			-72203458377154560
29	1			-1083051875657318400
28	2			17867338076431319040
27	3			323429243322850344960
26	4			1015274769514512777216
25	5			-6579490900655978053632
24	6			-50868714879266907488256
23	7			-79994609460852264271872
22	8			283830624039163303821312
21	9			1286453941063525028855808
20	10			1265675554473947355414528
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18	12			-7994494268883530252550144
17	13			-5386144726152482176106496
16	14			6882605843718109598318592
15	15			14466113532455038048272384

Appendix B

Polynomial $P_g(m,m)/(1-2m)^{(2g-2)}$ in the generating function $M_g(z)$.

g	$P_g(m,m)/(1-2m)^{(2g-2)}$
1	1
2	$3\,(7-70m+295m^2-636m^3+588m^4)$
3	$3^2\begin{pmatrix} 165-2596m+19835m^2-102138m^3+397742m^4 \\ -1162744m^5+2360496m^6-2918016m^7+1642656m^8 \end{pmatrix}$
4	$3^2\begin{pmatrix} 25025-465894m+4245462m^2-28633200m^3 \\ +178608786m^4-1025233956m^5+4855070265m^6 \\ -17709582732m^7+48202134300m^8-95026128096m^9 \\ +128766120048m^{10}-107657028288m^{11}+41956066368m^{12} \end{pmatrix}$
5	$3^2\begin{pmatrix} 6613425-128153480m+1123286598m^2-7641539820m^3 \\ +68489369190m^4-681945904584m^5+5453799804351m^6 \\ -33175983024306m^7+157025924018370m^8 \\ -590662433458296m^9+1778501684246544m^{10} \\ -4258112783048352m^{11}+7946769062433024m^{12} \\ -11156448512891520m^{13}+11087677481748480m^{14} \\ -6955529138076672m^{15}+2071316467035648m^{16} \end{pmatrix}$
6	$3^3\begin{pmatrix} 900951975-16624244750m+105922471285m^2 \\ -402327939748m^3+9014122899102m^4 \\ -183050473605084m^5+2152106046117936m^6 \\ -17716916701552824m^7+113738504396139378m^8 \\ -602051461456822740m^9+2694620167659984726m^{10} \\ -10264333975933057272m^{11}+33144207748349404248m^{12} \\ -89851078246171110912m^{13}+201700042332545251008m^{14} \\ -368052722019205320960m^{15}+531966143515513800960m^{16} \\ -586003188281237388288m^{17}+462270648384927677952m^{18} \\ -232608604432295245824m^{19}+56102738197832792064m^{20} \end{pmatrix}$