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*Abstract

Model reduction methods for viscoelastic sandwich structures in frequency and time domains

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Abstract

This paper deals with modeling and model reduction methods intended to sandwich structures with viscoelastic materials. For the modeling step, it is carried out by combining the First Order Shear Deformation Theory (FSDT) with the Golla-Hughes-Mc Tavish (GHM) model. The GHM model introduces auxiliary coordinates to take into account the frequency dependence of viscoelastic materials which combined to the Finite Element Method (FEM) leads to large order models. This paper focuses on the use of model reduction methods. The reduced models compared to the full model are illustrated by three numerical examples in order to outline the performance, the practical interest of these methods and their validity domains.

Keywords: Sandwich, GHM Viscoelastic model, FEM, Model reduction methods, Frequency analysis, Time analysis.

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To: John E. Dolbow, Editor-In-Chief
Finite Elements in Analysis and Design Journal

Dear Professor,

Please find enclosed the corrections concerning the manuscript entitled "*Model reduction methods for viscoelastic sandwich structures in frequency and time domains*" submitted for publication in Finite elements in analysis and design Journal (**D3375R1**).

The authors hope that the paper in its present form will satisfy the requirements to ensure its publication.

Should you need any further information, please do not hesitate to contact me back.

Yours sincerely

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Highlights

- The combination of First order shear deformation theory (FSDT), the Golla Hughes Mc-Tavish (GHM) viscoelastic model with the model reductions methods is developed.
- Proposed reduction methods for viscoelastic sandwich structures are implemented in the Finite Element codes.
- A focus on the Guyan reduction method shows the practical interest of this method expanded to viscoelastic sandwich structures.
- The potential of the GHM model is higlighted in time domain analysis notably with introduction of local nonlinearities.

1. Introduction

The use of viscoelastic [1, 2] sandwich structures [3] has been regarded as a convenient strategy for many industries such as aeronautics, marines and automotives. In fact, these structures present a high way of vibration control in term of lightweight and high specific stiffness especially when they incorporated viscoelastic materials.

Several theories [4-7] were developed in order to approximate the displacement and the mechanical deformation of such structures. One of the well-known and useful theories is the classical theory of plates (CPT) which assume that a plane section initially normal to the midsurface before deformation remains plane and normal to that surface after deformation. Hence, this theory neglects the effect of shear deformations and leads to inaccurate results for laminated plates. So, it is obvious that transverse shear deformations have to be taken into account in the analysis. Thus, the first order shear deformation theory (FSDT) introduced by Reissner and Mindlin [4, 7] takes into account this effect and assumes a linear variation of the midplane displacements through the thickness of the structure. This method has a significant advantage due to its simple implementation and low computational cost. Another laminated theory based on Reddy's refined [8] high order shear deformations theory (HSDT) which includes both bending and shear effects was been carried out in Ferreira et al. [9], Chugal and Shimpi [10] studies. Unfortunately, this method requires a prohibitive computational time which is undesirable of such applications. Some others researchers [11, 12] have used the Layerwise theory for modeling the sandwich structures. Indeed, this theory assumes a displacement field in the form of zig-zag along the thickness of the structure allowing a kinematic description of each layer as a piecewise linear functions. In addition, this theory is applicable for both thin and thick structures. Nevertheless, when the study is intended to thin structures, the first order shear deformation theory (FSDT) presents a suitable choice for the modeling of sandwich structures favored by its simple implementation in the most of finite elements codes.

However, these structures exhibit viscoelastic damping which combine viscous and elastic character. Hence, this dual character leads to a complicated behavior which requires a correct modeling approach. More recently, Golla, Hughes and Mc Tavish [13, 14] have proposed the so called GHM model. This model provides an effective method which includes viscoelastic damping through the addition of auxiliary coordinates called dissipation coordinates as a sum of elementary mini-oscillators.

Furthermore, the GHM model combined to the finite element method (FEM) [15], allows the introduction of viscoelastic material properties through element mass, stiffness, and damping matrices. The addition of internal mini-oscillators for each viscoelastic finite element allows a general description of frequency-dependent viscoelastic materials properties behavior. The main advantage of this method consists in its efficient modeling of viscoelastic material behavior; but its major lack is the largely finite element dimension system which requires a prohibitive computing time. Consequently, a model reduction should be applied to the augmented GHM model.

The present paper proposes an alternative of model reduction such Dynamic [16, 17], Guyan [18, 19], modal and modal in physical space (SEREP) [20-23] reduction methods for this problem. The first one based on the elimination of unwanted variables; partitioned the full degree-of-freedom (dofs) into master and slave dofs; uses the modal properties of the slave part of the structure when the master dofs are grounded. Hence, the derived slave modes are operated to enrich the dynamic basis leading to a drastic reduction method. The simplest, yet very useful model reduction method is the well-known Guyan reduction method. It is a particular case of dynamic reduction method according to which the inertia associated to the slave coordinates is neglected; only master dofs are retained. Thereby, the unwanted variables are removed leading to reduced model which is a subset of the original system in a restricted range of frequency. However this method is limited by its validity domain [24, 25]. Another reduction method is the frequently used modal reduction method according to which the derived modes associated to the undamped structure are incorporated in the GHM damped model yielding to an exact transformation basis. This basis restitutes correctly the undamped modes of the original system leading to a drastic reduction. The modal reduction method can be expanded the projection from generalized coordinates system to the physical coordinates system leading to another strategy of reduction called modal reduction in physical space method. This method restitutes also the first modes of the undamped structure and partition the modal basis into master and slave dofs. This leads to several cases which will be tested examining both the number of retained modes and the number of master dofs.

In other hand, the modeling of viscoelastic sandwich structures has attracted many researchers, but only a few papers have dealt with the GHM model [26, 27]. However, these papers remain limited in the most to frequency domain analysis with major uses of the space state modal reduction method for model reduction. In fact, Trindade et al. [28], De Lima and Rade [29] was used frequently the modal reduction in their studies. It consists to transform the second order equation of motion into an equivalent first-order form (space-state model). Unfortunately, this method leads generally to a space state model of dimension at least the double of the total dimension of the GHM model (2N) and the quadruple dimension of the structural dofs which requires a prohibitive time of calculations.

Therefore, the application of the proposed reduction methods, which are often used with the undamped structures, combined to the GHM model is an ability to add the effects of viscoelastic components to the sandwich structures without increasing the order of the finite element models. Furthermore, these reduction methods can be applied to sandwich structures described kinematically by the others mentioned theories.

In this paper, both the theory related to the implementation of the FSDT theory combined to the GHM method and the theory related to its reduction methods is presented. Numerical simulations applied to beam, plate and non-linear assembled beams in both frequency and time domains are also illustrated. These examples will highlight the performance of reduction methods and its practical interest in the dynamic analysis of viscoelastically damped sandwich structures.

2. Three-layer viscoelastic finite element model

Multilayer structures are typically used for its light-weight, high specific stiffness and strength values in many engineering fields. In fact, there are attempts to replace components with classical materials (steel, concrete) by laminated materials notably sandwich structures. Hence, the modeling of such structures has been a particular interest of many studies [7, 9-11]. In this paper, the considered sandwich structure is constituted by three laminated materials: a core generally formed by viscoelastic material of thickness h_c, incorporated between two elastic faces of thickness h_{f_1} and h_{f_2} respectively. The studied sandwich panel is assumed to have a length L, width b and total thickness h as shown in Fig.1.

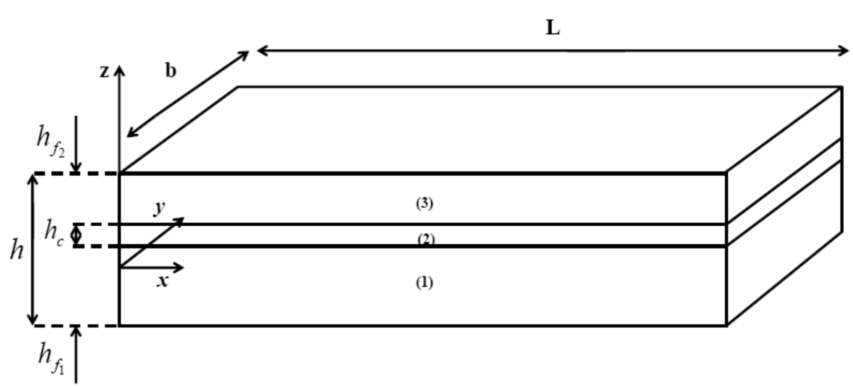


Fig.1. Sandwich structure geometry

The modeling process of the three-layer sandwich structure is based on the following assumptions:

- The sandwich is a laminate made of a stack of permanently combined layers. No slip or delamination between the layers, they are perfectly bonded. Consequently, the continuity of displacements along the interfaces between the layers is considered.
- The sandwich is a homogeneous material on a macro scale level, but its properties depend in turn on the properties of each layer.

  The lamina are macroscopically homogeneous, isotropic and linearly elastic.
- Both extensional and bending deformations are considered.

It should be noted that when the lamina core is made of viscoelastic material, an appropriate model will be used to model such behavior.

The kinematic model of the sandwich structure is based on the first shear deformation theory (FSDT) of Reissner–Mindlin [4, 7], which assumes that a plane section and perpendicular to the midplane of the structure before deformation remains plane but not necessary perpendicular to the midplane after deformation. This theory takes into account the effect of transverse shear deformations in both faces and core. Hence, the displacement field of a sandwich laminate structure can be expressed as:

$$\left\{\begin{array}{l} u(x,y,z,t)=u_0(x,y,t)+z\psi_x(x,y,t) \\ v(x,y,z,t)=v_0(x,y,t)+z\psi_y(x,y,t) \\ w(x,y,z,t)=w_0(x,y,t) \end{array}\right\} \tag{1}$$

Where:

u , v and w represent the displacements along the axes x, y and z respectively; $u_0(x,y,t)$, $v_0(x,y,t)$ and $w_0(x,y,t)$ denote the corresponding midplane displacements in the (x , y , z) directions.

z is generally the thickness of the structure along the axis (z).

$\psi_x(x,y,t)$ and $\psi_y(x,y,t)$ are the rotations of normals to midplane about the y and x axes.

This theory is well applicable for thin and moderately thick plates and allows the compromise: good capacity of prediction/moderate computational time for large manufactories investigations. Besides, it offers the feasibility of easy implementation in many Finite Elements codes.

Thus, the Finite element formulation uses an eight-node shell finite element with five dofs per node called Serendip element [15]. The choice of this element is based on the investigations realized by Chee [30] which proved that this element provide an excellent performance for the modeling of composites structures notably sandwich structures. Furthermore, this element is adapted for the majority of laminated theories, especially First order Shear Deformation theory (FSDT).

It is a quadratic element belonging to isoparametric elements family and it uses a bilinear shape functions whose coordinates in elementary and local system are presented as follows:

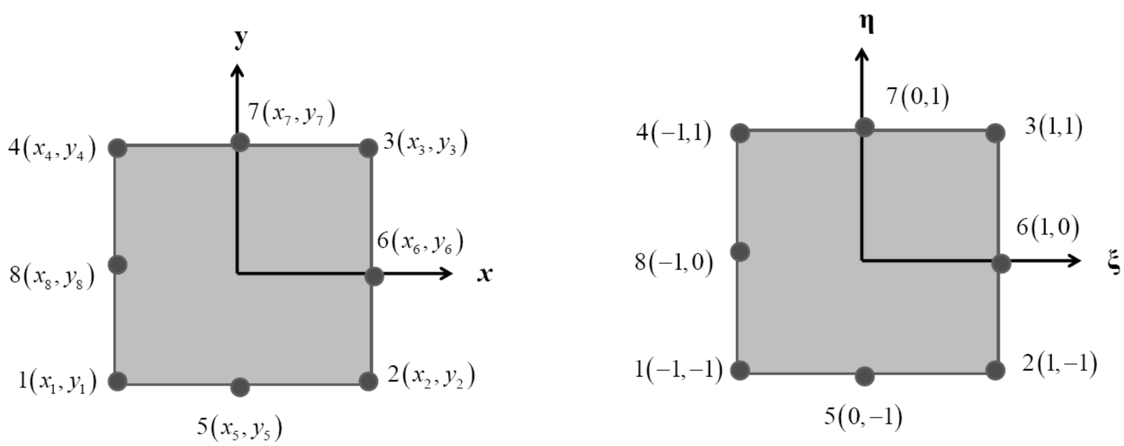


Fig.2. Serendip finite element in: (A) elementary coordinates, (B) local coordinates

Therefore, the displacement field can be discretized in local coordinates as follows:

$$\begin{Bmatrix} u(\xi,\eta,z,t) \\ v(\xi,\eta,z,t) \\ w(\xi,\eta,z,t) \end{Bmatrix} = [A(z)]_{(3\times5)} [N(\xi,\eta)]_{(5\times40)} \{u_e(t)\}_{(40\times1)} \tag{2}$$

Where: $[A(z)]_{(3\times5)}$ is the matrix of z coordinates along the thickness axe; $[N(\xi,\eta)]_{(5\times40)}$ is the shape functions matrix; $\{u_e(t)\}_{(40\times1)}$ is the elementary nodal displacement vector.

Based on the hypotheses of the stress-states assumed for each layer, the stress–strain relations can be obtained and kinetic energies of the sandwich plate finite element are formulated [31]. Then, the variational Hamilton principle is used considering the nodal displacements and rotations as generalised coordinates leading to derive the elements stiffness and mass matrices as follows:

$$[M_e] = \int_{V_e} \rho^{(k)} [N]^T [A]^T [A][N] dV_e \tag{3a}$$

$$[K_e] = \sum_{k=1}^{N_c} \int_{\xi=-1}^{1} \int_{\eta=-1}^{1} \int_{z=z_k}^{z_{k+1}} \left([B_b]^T [C_b^e]^{(k)} [B_b] + [B_s]^T [C_s^e]^{(k)} [B_s] \right) J dz d\eta d\xi \tag{3b}$$

Where: the subscript k denotes the k^{th} layer of the laminated structure; $\rho^{(k)}$ is the corresponding lamina density; N_c denotes the number of laminated layers. Herein, N_c is considered equal to 3. z_k denotes the thickness of the k^{th} sandwich layer; $[B_b]$ and $[B_s]$ refers to the strain–displacement matrices where the extensional, bending, plan shear and transverse shear effects are uncoupled separately.

$[C_b^e]^{(k)}$ and $[C_s^e]^{(k)}$ refers to the strain–stress matrices associated to each layer k where the extensional, bending, plan shear effect (subscript b) and transverse shear effect (subscript s).

dV_e Indicates the elementary variation volume and J is the Jacobian matrix.

After derivation of the element mass and stiffness matrices using the Gaussian quadrature integration, the corresponding global matrices are assembled accounting for the connectivity using the standard assembling procedure and the equation of motion of undamped structure is established as follows:

$$[M]\{\ddot{q}\}+[K]\{q\}=\{F\} \tag{4}$$

Where: $[M]\in R^{N\times N}$ is the mass matrix (symmetric and positive definite), $[K]\in R^{N\times N}$ is the stiffness matrix (symmetric and nonnegative definite), $\{q\}$ is the displacements vector and $\{F\}\in R^{N\times 1}$ is the external load vector.

Nevertheless, when the sandwich structure is made of viscoelastic material, this equation of motion is unable to describe the frequency-dependence of such materials. Indeed, it omitted the damping effect. Hence, the use of consistent model across a broad range of frequencies should be considered.

Several approaches are presented in the literature to describe this behavior such as the Anelastic Displacement Fields model proposed by Leisuture [32], Fractional derivatives models proposed by Bagley and Torvik [33] and especially the Golla-Hughes-Mc Tavish (GHM) model [13, 14]. Hence, the GHM model can be developed for direct incorporation into the finite element method.

3. GHM viscoelastic approach

For a sandwich structure incorporating viscoelastic materials, the stiffness matrix can be decomposed as follows:

$$[K]=[K_e]+[K_V(s)] \tag{5}$$

Where $[K_e]$ is the stiffness matrix corresponding to the purely elastic layers and $[K_V(s)]$ is the stiffness matrix associated to the viscoelastic layer. The inclusion of the frequency-dependent behavior of the viscoelastic material can be made by generating $[K_V(s)]$ for specific types of elements (beams, plates...) considering initially constant moduli ($E(s)$ or $G(s)$). Then, using the elastic-viscoelastic correspondence principle [34, 35], these moduli are factored out of the stiffness matrix reflecting the frequency dependence of viscoelastic materials.

Hence, the viscoelastic stiffness can be written as:

$$[K_V(s)] = G(s)\left[\bar{K}_V\right] \tag{6}$$

Golla-Hughes and Mc Tavish [13, 14] introduced the so called GHM model to describe the shear modulus of viscoelastic structure as a serie of mini-oscillator terms:

$$G(s) = G_0\left(1 + \sum_{i=1}^{N_G} \alpha_i \frac{s^2 + 2\zeta_i\omega_i s}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}\right) \tag{7}$$

Where: G_0 is the static modulus; s is the Laplace complex variable; ($\alpha_i, \zeta_i, \omega_i$) are the parameters of the ith mini-oscillator, and N_G is the number of mini-oscillators. The parameters ($\alpha_i, \zeta_i, \omega_i$) are identified from the experimental fit curves of the corresponding viscoelastic material [26, 34]. In fact, different viscoelastic materials have different frequency dependence and so have a different number of terms N_G of the GHM fit.

Substituting Eq. (6) into Eq. (5) and replacing $G(s)$ by its expression, the equation of motion can be written as follows:

$$\left(s^2[M] + s[D] + [K_e] + \left[K_V^0\right]\right)\{q(s)\} + \left[K_V^0\right]\left(\sum_{i=1}^{N_G} \alpha_i \frac{s^2 + 2\xi_i\omega_i s}{s^2 + 2\xi_i\omega_i s + \omega_i^2}\right)\{q(s)\} = \{F(s)\} \tag{8}$$

Now, by adding extra-coordinates $\{z_i\}(1,...,N_G)$ called dissipation coordinates as:

$$\{z_i(s)\} = \left\{ \frac{\omega_i^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \right\} \{q(s)\} \tag{9}$$

The equation of motion may be rewritten, in the Laplace domain, as two coupled second order equations:

$$\left(s^2[M] + s[D] + [K_e] + \left[K_V^\infty\right]\right)\{q\} - \alpha\left[K_V^0\right]\{z\} = \{F\} \tag{10a}$$

$$s^2\{z\} + 2s\zeta\omega\{z\} - \omega^2\{q\} + \omega^2\{z\} = \{0\} \tag{10b}$$

With $\left[K_V^0\right] = G_0\left[\bar{K}_V\right]$ and $\left[K_V^\infty\right] = \left[K_V^0\right]\left(1 + \sum_{i=1}^{N_G} \alpha_i\right)$ are respectively the static or low frequency stiffness matrix and the dynamic or high frequency stiffness matrix corresponding to the viscoelastic layer.

The matrix $[D]$ represents the structural damping of the structure without the viscoelastic effect.

After some manipulations and back to time domain, the following equation of motion in the Laplace domain is obtained:

$$\left\{ s^2 \begin{bmatrix} [M] & 0 \\ 0 & [M_z] \end{bmatrix} + s \begin{bmatrix} [D] & 0 \\ 0 & [D_z] \end{bmatrix} + \begin{bmatrix} [K_q] & [K_{qz}] \\ [K_{qz}^T] & [K_z] \end{bmatrix} \right\} \begin{Bmatrix} q(s) \\ z(s) \end{Bmatrix} = \begin{Bmatrix} F(s) \\ 0 \end{Bmatrix} \tag{11}$$

Or in compact form:

$$\left\{s^2[M_G] + s[D_G] + [K_G]\right\}\{q_G\} = \{F_G(s)\} \tag{12}$$

The derived second order time domain equation of motion is expressed as:

$$[M_G]\{\ddot{q}_G\} + [D_G]\{\dot{q}_G\} + [K_G]\{q_G\} = \{F_G\} \tag{13}$$

Where: $[M_G]$; $[D_G]$ and $[K_G] \in R^{n_G \times n_G}$, with $n_G = N(1 + N_G)$, are respectively the mass, damping and stiffness matrices of the global viscoelastic GHM model expressed as follows:

$$[M_G]=\begin{bmatrix}[M] & 0 & \cdots & 0 \\ 0 & \frac{\alpha_1}{\omega_1^2}\left[K_V^0\right] & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\alpha_{N_G}}{\omega_{N_G}^2}\left[K_V^0\right]\end{bmatrix} \quad [D_G]=\begin{bmatrix}[D] & 0 & \cdots & 0 \\ 0 & \frac{2\alpha_1\zeta_1}{\omega_1}\left[K_V^0\right] & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{2\alpha_{N_G}\zeta_{N_G}}{\omega_{N_G}}\left[K_V^0\right]\end{bmatrix}$$

$$[K_G]=\begin{bmatrix}[K_e]+\left[K_V^\infty\right] & -\alpha_1\left[K_V^0\right] & \cdots & -\alpha_{N_G}\left[K_V^0\right] \\ -\alpha_1\left[K_V^0\right]^T & \alpha_1\left[K_V^0\right] & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ -\alpha_{N_G}\left[K_V^0\right]^T & \cdots & 0 & \alpha_{N_G}\left[K_V^0\right]\end{bmatrix} \quad \{F_G\}=\begin{Bmatrix}F \\ 0 \\ \vdots \\ 0\end{Bmatrix} \quad \{q_G\}=\begin{Bmatrix}q \\ z_1 \\ \vdots \\ z_{N_G}\end{Bmatrix}=\begin{Bmatrix}q \\ z\end{Bmatrix} \tag{14}$$

The dissipative coordinates $\{z\}$ appears as augmenting state variable which increase considerably the order of the differential equation of motion. In fact, the dimension of the system is at least doubled and the computational time is notably increased. This motivates the use of model reduction methods, as an alternative solution for this problem, which will be presented in the following section.

4. Model reduction methods

Model reduction is necessary to reduce the high order finite element models to a smaller size so that direct dynamic analysis can be performed. Several model reduction methods have commonly been used including Dynamic [16, 17], Guyan [18, 19], Modal and Modal in physical space [20-23] reduction methods. They can be classified either by type of reduction approach, or by type of reduced space coordinates.

The former can be further divided in two categories as elimination dofs approach and modal projection approach. Indeed, elimination dofs approach is based on the partition of the full dofs of the structure into ‘m’ master and‘s’ slave dofs. In the reduction process, master dofs are retained and slave dofs are removed. This category includes notably Dynamic and Guyan reduction methods. For modal reduction approach, a partition of the modal matrix for the associated undamped model is established. The lowest modes are retained and the else are removed. Both modal reduction and modal reduction in physical space belong to this category. The concept of physical, generalized and hybrid coordinates will be clarified in the latter type of reduction.

In fact, based on the type of coordinates retained as the reduced order coordinates, the existing model order reduction methods fall into three basic categories:

- Physical coordinates reduction
- Generalized coordinates reduction
- Hybrid coordinates reduction

In the physical coordinates model method, the reduced model is obtained by removing part of the physical coordinates of the full model. Thus, the coordinates of the reduced model are a subset of the full model. This is the most straightforward model reduction among the three categories. Guyan and Modal in physical space reduction methods belong to this type of coordinates.

In the generalized coordinates model reduction, all the coordinates that are not physical coordinates are generally referred to as generalized coordinates. The Modal reduction method is one of the frequently used generalized coordinates.

The hybrid coordinates model reduction uses a combination of physical and generalized coordinates. Thus, this technique provides a good representation of the dynamic behavior of the sandwich structures. The very useful method belonging of this type is the Dynamic reduction method.

Each method is attached to the definition of a transformation matrix $[T] \in R^{n_G, n_c}$, related the n_G full dofs of the viscoelastic sandwich structure to the n_c reduced dofs where $n_c << n_G$ Thereby, the displacement vector $\{q_G\}$ can be written as a linear combination of the subspace elements presented by the columns of $[T]$ as:

$$\{q_G\} = [T]\{q_c\} \tag{15}$$

Where:

$\{q_G\} \in R^{n_G \times 1}$ is the displacement vector of full GHM model.

$\{q_c\} \in R^{n_c \times 1}$ is the vector of reduced coordinates through the projection on $[T]$.

$[T] \in R^{n_G \times n_c}$ is the transformation matrix.

This transformation takes various forms depending on the used reduction technique.

The equation of motion in full space Eq. (12) is then written in reduced space as follows:

$$\left([T]^T [M_G][T]\right)\{\ddot{q}_c\} + \left([T]^T [D_G][T]\right)\{\dot{q}_c\} + \left([T]^T [K_G][T]\right)\{q_c\} = [T]^T \{F_c\} \tag{16}$$

Hence, the reduced mass, stiffness and damping matrices, as well as the reduced load vector can be written as:

$$\begin{aligned}
[K_c] &= [T]^T [K_G][T] \\
[M_c] &= [T]^T [M_G][T] \\
[D_c] &= [T]^T [D_G][T] \\
\{F_c\} &= [T]^T \{F_G\}
\end{aligned} \tag{17}$$

For each type of reduction process, the transformation from the full space to the reduced space is established through the partition of structural displacement vector $\{q\}$ into two subvectors as follows:

$$\{q\} = \begin{Bmatrix} q^m \\ q^s \end{Bmatrix} \tag{18a}$$

Where the subscript m is related to the master dofs and the subscript s is related to the slave dofs.

Following the master and slave dofs partition and assuming that only the master dofs are loaded, the external load vector $\{F\}$ can be written as:

$$\{F\}=\begin{Bmatrix} F^m \\ 0 \end{Bmatrix} \tag{18b}$$

Consequently, the dynamic equilibrium of the associated undamped system can be expressed as:

$$\left(\begin{bmatrix} K^{mm} & K^{ms} \\ K^{sm} & K^{ss} \end{bmatrix}-\lambda\begin{bmatrix} M^{mm} & M^{ms} \\ M^{sm} & M^{ss} \end{bmatrix}\right)\begin{Bmatrix} q^m \\ q^s \end{Bmatrix}=\begin{Bmatrix} F^m \\ 0 \end{Bmatrix} \tag{19}$$

Thus, the definition of transformation matrices for each type of reduction method is based on the use of the equilibrium equation Eq. (19) as will be shown in the following sections.

4.1. Dynamic reduction method

This method is proposed by Leung [16] then by Petersmann [17]. It uses jointly the modal synthesis method and the dynamic reduction method identically to the substructuring technique proposed by Craig and Bampton [36] in component modal synthesis context.

Using the second part of rows of the Eq. (19), the sub-vector of salve dofs $\{q^s\}$ can be expressed in term of master dofs $\{q^m\}$ as:

$$\{q^s\}=-\left(\left[K^{ss}\right]-\lambda\left[M^{ss}\right]\right)^{-1}\left(\left[K^{sm}\right]-\lambda\left[M^{sm}\right]\right)\{q^m\} \tag{20}$$

It should be noted that this expression is defined when the slave dynamic stiffness $Z(\lambda)=\left(\left[K^{ss}\right]-\lambda\left[M^{ss}\right]\right)$ is nonsingular. Indeed, this condition is satisfactory for each frequency else the eigenfrequencies ($\lambda\neq\sigma_\upsilon,\ \upsilon=1:s$) of the salve problem defined as follows:

$$\left(\left[K^{ss}\right]-\sigma_\upsilon\left[M^{ss}\right]\right)\{\varphi_\upsilon\}=0 \tag{21}$$

Where $\Sigma=diag(\sigma_\upsilon)$ and $\left[\Phi\right]=\left[\cdots\quad\{\varphi_\upsilon\}\quad\cdots\right];\upsilon=1,....,s$ are the spectral and modal matrices respectively.

This leads to define the dynamic contribution of the slave dofs as:

$$\left[T_d(\lambda)\right]=-\left(\left[K^{ss}\right]-\lambda\left[M^{ss}\right]\right)^{-1}\left(\left[K^{sm}\right]-\lambda\left[M^{sm}\right]\right) \tag{22}$$

As can be seen, this relation (22) is an exact dynamic relation which depends strongly on the value of unknown eigenvalue λ. This leads to resolve a non-linear eigenvalue problem. Nevertheless, the viscoelastic behavior of the structure is linearized [35], so it is necessary to approximate this relation to be adequate for the linear viscoelastic problem.

Hence, according to Leung and Petersmann method [16, 17], the hybrid projection coordinates can be expressed as follows:

$$\left\{q^s\right\}=-\left[K^{ss}\right]^{-1}\left[K^{sm}\right]\left\{q^m\right\}+\left[\Phi_p\right]\left\{c\right\} \tag{23}$$

Where $\left[\Phi_p\right]$ is the p truncated modal basis of the slave structure with p<<s.

This hybrid formulation is similar to the Craig-Bampton method applied in the case of sub structuring procedure. Thus, the master dofs are the junction dofs and the slave dofs are the interior dofs.

While this dynamic reduction formulation was developed for undamped systems, a straightforward application of the above developments to the viscoelastic damped sandwich structures yields to the following expression of the slave displacement vector:

$$\left\{q^s\right\}=-\left[K_q^{ss}\right]^{-1}\left[K_q^{sm}\right]\left\{q^m\right\}-\left[K_q^{ss}\right]^{-1}\left[K_{qz}^{sm}\quad K_{qz}^{ss}\right]\left\{z\right\}+\left[\Phi_p\right]\left\{c\right\} \tag{24}$$

Hence, the reduction of the full dofs to the reduced dofs is achieved as follows:

$$\begin{Bmatrix} q^m \\ q^s \\ z \end{Bmatrix}=\underbrace{\begin{bmatrix} I_1 & 0 & 0 \\ t_1 & t_2 & \Phi_p \\ 0 & I_2 & 0 \end{bmatrix}}_{\left[T_{Dyn}\right]}\begin{Bmatrix} q^m \\ z \\ c \end{Bmatrix} \tag{25}$$

Where: $\left[T_{Dyn}\right]$ is the dynamic transformation matrix; $\left[I_1\right]$ and $\left[I_2\right]$ are the identity matrices of appropriate size; $\left[t_1\right]=-\left[K_q^{ss}\right]^{-1}\left[K_q^{sm}\right]$ and $\left[t_2\right]=-\left[K_q^{ss}\right]^{-1}\left[K_{qz}^{sm}\quad K_{qz}^{ss}\right]$ represents the static contribution of the structure and Φ_p basis represents the dynamic contribution of the structure.

For the damped GHM model, the slave problem can be written as follows:

$$\left(\left[K_q^{ss}\right]-\sigma_v\left[M^{ss}\right]\right)\{\varphi_v\}=0 \tag{26}$$

The reduced mass, stiffness and damping matrices can then be written in the form of Eq. (17) using Eq. (25).

This method has a good capacity of prediction of the dynamic behavior of viscoelastic sandwich structures combining static and dynamic contributions through a hybrid reduced coordinates. Nevertheless, it requires the computation of p truncated modes which increase the size order of the system and leads to a few additional CPU time.

4.2. Guyan reduction method

The simplest, yet very useful, model reduction method is the Guyan reduction method, which is introduced by Guyan [18] and Irons [19] in 1965. This method is an approximation of the dynamic reduction method, according to which the inertia associated to the slave coordinates, is neglected. Thus, applying this static reduction procedure for the damped sandwich structures, the relationship between the full dofs and the reduced dofs can be expressed as follows:

$$\begin{Bmatrix} q^m \\ q^s \\ z \end{Bmatrix} = \underbrace{\begin{bmatrix} I_1 & 0 \\ t_1 & t_2 \\ 0 & I_2 \end{bmatrix}}_{[T_{St}]} \begin{Bmatrix} q^m \\ z \end{Bmatrix} \tag{27}$$

Where:

$[T_{St}]$ is the static (or Guyan) transformation matrix $[I_1]$; $[I_2]$; $[t_1]$ and $[t_2]$ are the same quantities as defined for Eq.(25).

Under this form, it appears that Guyan reduction is a particular case of dynamic reduction when no slave modes are taken into account.

The reduced mass, stiffness and damping matrices can then be written in the form of Eq. (17) using Eq. (27).

Validity domain:

This method is valid and useful in an accurate domain. This domain is limited by the cuttoff frequency [24, 25] .It is the smallest frequency determined by the resolution of the eigenvalue problem (26) defined as $f_1 = f_c$. Thereby, in the practice applications, the validity domain of Guyan reduction method is $[0 : f_c / 3]$ which reflects the "effective" frequency band.

Consequently, the quality of Guyan approximation depends on the good selection of master dofs which defines its validity domain. In practice, an optimal selection of master dofs must be based on the maximization of the cuttoff frequency f_c. Out from the validity domain of this method, the accuracy of obtained results is not well controlled.

4.3. Modal reduction method

Modal reduction consists to the derivation of $[\Lambda]$spectral matrix and $[Q]$modal basis corresponding to the eigenvalues and eigenvectors of the associated undamped system Eq. (4). Then, these matrices are divided into two parts as follows:

$$[Q]=[Q_1 \quad Q_2] \qquad ; \qquad [\Lambda]=\begin{bmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{bmatrix} \tag{28}$$

The displacement vector is presented as a combination of the p first eigenvectors contained in $[Q_1]\in R^{N,p}$:

$$\{q\}=[Q_1]\{c\} \tag{29}$$

Applying this procedure to damped viscoelastic sandwich structures, the full model can be reduced through the projection in the generalized space as follows:

$$\begin{Bmatrix} q \\ z \end{Bmatrix} = \underbrace{\begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix}}_{[T_{Mod}]} \begin{Bmatrix} c \\ z \end{Bmatrix} \tag{30}$$

Where $[T_{Mod}]$ is the modal transformation matrix.

The reduced mass, stiffness and damping matrices can then be written in the form of Eq. (17) using Eq. (30).

This method uses non-physical coordinates and the truncation can induces errors in the evaluation of dynamic responses. In practice, the modal reduction method leads to a good accuracy results when the p first modes are chosen typically from 1.5 to 3 times the frequency band of interest.

4.4. Modal reduction in physical space

This method was proposed by O' Callahan [20]. It is based on the modal projection in the physical coordinates. O' Challahan [21, 22] marked that this technique allows, after expansion, to return from the reduced model p exact solutions of the full model.

Indeed, the base $[Q_1]$ is partitioned into m master dofs and s slave dofs as follows:

$$\{q\} = \begin{Bmatrix} q^m \\ q^s \end{Bmatrix} = \begin{bmatrix} Q_{1m} \\ Q_{1s} \end{bmatrix} \{c\} \tag{31}$$

The first line of the Eq. (31) leads to:

$$\{q^m\} = [Q_{1m}]\{c\} \text{ with} [Q_{1m}] \in R^{N \times p} \tag{32}$$

According to the used m master dofs and p retained undamped modes, three cases can be highlighted:

a) $m = p$ and $[Q_{1m}]$ is nonsingular

The Eq. (32) can be solved exactly as:

$$\{c\} = [Q_{1m}]^{-1}\{q^m\} \tag{33a}$$

b) m<p

This case leads to infinity of solutions of $\{c\}$ which is not accurate.

c) m>p and ($rank\left(\left[Q_{1m}\right]\right)=p$) is maximal

The Eq. (32) can be solved in the sense of linear least squares as:

$$\{c\}=\left[Q_{1m}\right]^{+}\left\{q^{m}\right\} \tag{33b}$$

Where $\left[Q_{1m}\right]^{+}=\left(\left[Q_{1m}\right]^{T}\left[Q_{1m}\right]\right)^{-1}\left[Q_{1m}\right]^{T}$ is the Moore Penrose pseudo inverse.

Thus, by substituting Eq. (33b) into Eq. (31), the structural vector dofs can be written:

$$\{q\}=\begin{bmatrix} Q_{1m}Q_{1m}{}^{+} \\ Q_{1s}Q_{1m}{}^{+} \end{bmatrix}\left\{q^{m}\right\} \tag{34}$$

Hence, the relationship between full and reduced dofs through the projection in physical coordinates for viscoelastic sandwich structures can be expressed generally when $m \geq p$ as follows:

$$\begin{Bmatrix} q^{m} \\ q^{s} \\ z \end{Bmatrix}=\underbrace{\begin{bmatrix} Q_{1m}Q_{1m}{}^{+} & 0 \\ Q_{1s}Q_{1m}{}^{+} & 0 \\ 0 & I \end{bmatrix}}_{\left[T_{SP}\right]}\begin{Bmatrix} q^{m} \\ z \end{Bmatrix} \tag{35}$$

Where $\left[T_{SP}\right]$ is the modal transformation matrix defined in physical space.

Consequently, the reduced mass, stiffness and damping matrices can then be written in the form of Eq. (17) using Eq. (35). The definition of a transformation matrix using a number of master dofs either equal or up the number of undamped modes (as mentioned in the cases a or c) leads to the use of maximum rank sub-basis. In fact, the rank of a matrix is defined by the number of rows or columns linearly independent. Numerically, this linear independence is evaluated by the conditioning number.

O' Challahan [20-22] and Friswell [23] shows in their previous studies, that using a sub-basis $[Q_{1m}]$ with high conditioning number (>10^7;10^8) can affect the accuracy of results and can generate erroneous responses. So, the process of selection of master dofs is achieved such that $[Q_{1m}]$ having the minimum conditioning number.

The modal reduction in physical space method allows the derivation of reduced model which is a subset of the original model expressed in physical coordinates. Furthermore, this technique provides an expanded choice of master dofs but it remains limited by the minimum conditioning condition.

5. Numerical Applications

In this section, numerical applications is presented in order to illustrate the finite element procedure used for viscoelastic sandwich beam and plate models and outline the practice interest of proposed reduction strategies. Hence, we consider one mini-oscillator ($N_G = 1$) of viscoelastic sandwich beam and plate which are constituted by two elastic layers (faces) in Aluminum and a viscoelastic layer (core) of the nuance 242F01. The material and geometrical characteristics of the used sandwich structures are shown in Table1.

Table1. Mechanical and geometrical characteristics of the viscoelastic sandwich structures

Elastic Layer (1)	Viscoelastic core	Elastic Layer (2)
$L_{beam} = L_{pltae} = 500mm$	$L_{beam} = L_{pltae} = 500mm$	$L_{beam} = L_{plate} = 500mm$
$b_{beam} = 38mm / b_{pltae} = 400mm$	$b_{beam} = 38mm / b_{pltae} = 400mm$	$b_{beam} = 38mm / b_{plate} = 400mm$
$h_{f_1} = 4.5mm$	$h_c = 0.2mm$	$h_{f_2} = 0.5mm$
$G_{f_1} = 70.3\times10^9 N / m^2$	G_c (GHM modulus)	$G_{f_2} = 70.3\times10^9 N / m^2$
$\rho_{f_1} = 2750Kg / m^3$	$\rho_c = 1099.5Kg / m^3$	$\rho_{f_2} = 2750Kg / m^3$
$\upsilon_{f_1} = 0.3$	$\upsilon_c = 0.5$	$\upsilon_{f_2} = 0.3$

The values of the parameters of the viscoelastic commercially available 242F01, manufactured by3M^{TM} used at 25°C for one mini-oscillator are presented in Table 2 [29].

Table2. Parameters of the GHM viscoelastic model identified for material 242F01 3M™ for one mini-oscillator

Mini-oscillator (i=1)	Value
α_i	1.047
ζ_i	3911.89
$\omega_i\,[rad/s]$	4943.06
$G_0\,[MPa]$	0.079

5.1. Viscoelastic Sandwich beam

The used FE mesh for the viscoelastic sandwich beam involves 2 elements through the width and 20 elements along the length, having a total number of 1600 dofs. The excitation point is selected in the extremity of the beam (Point P, dof of translation u_z) and the responses are depicted in two different points P and K (dof of translation u_z) as shown in Fig.3.

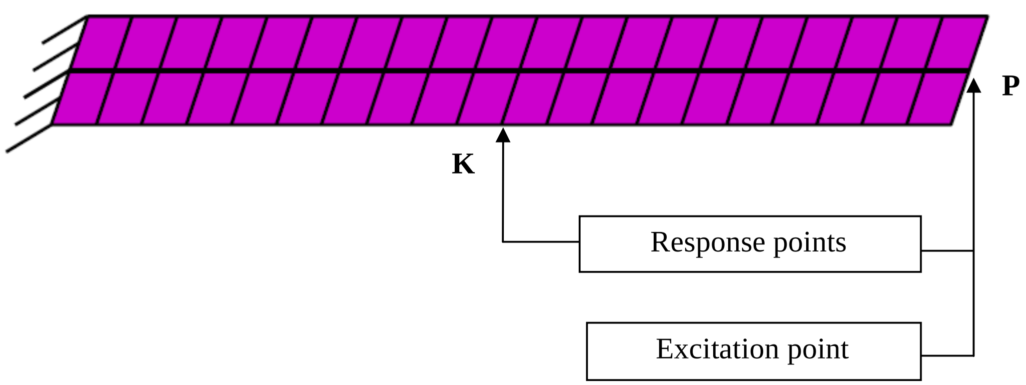


Fig.3. Clamped-Free(C-F) sandwich beam finite element

In the remainder of this section, the results derived from the implementation of the GHM model, as well as the responses of reduced models used the reduction techniques described as above will be presented both in frequency and time domains.

5.1.1. Frequency domain evaluation

The interest herein is focused on the frequency-domain responses for both full and reduced GHM models.

The full GHM model response can be derived directly by using Eq. (11) in order to calculate the frequency response function (FRF) matrix as follows:

$$H(\omega) = \{a\}\left[Z(\omega)\right]^{-1}\{b\}^{T} \tag{36}$$

Where $\left[Z(\omega)\right] = -\omega^2\left[M_G\right] + j\omega\left[D_G\right] + \left[K_G\right]$ is the dynamic stiffness matrix associated to the damped viscoelastic structure; $\{b\}^{T}$ is a column vector which defines among all discretized dofs of the structure the position of the selected excitation degree of freedom; $\{a\}$ is a row vector containing the coordinates where the responses are taken account.

Fig.4 depicts the frequency responses of the full GHM model plotted in the points P and K in the frequency band of interest [0-700] Hz.

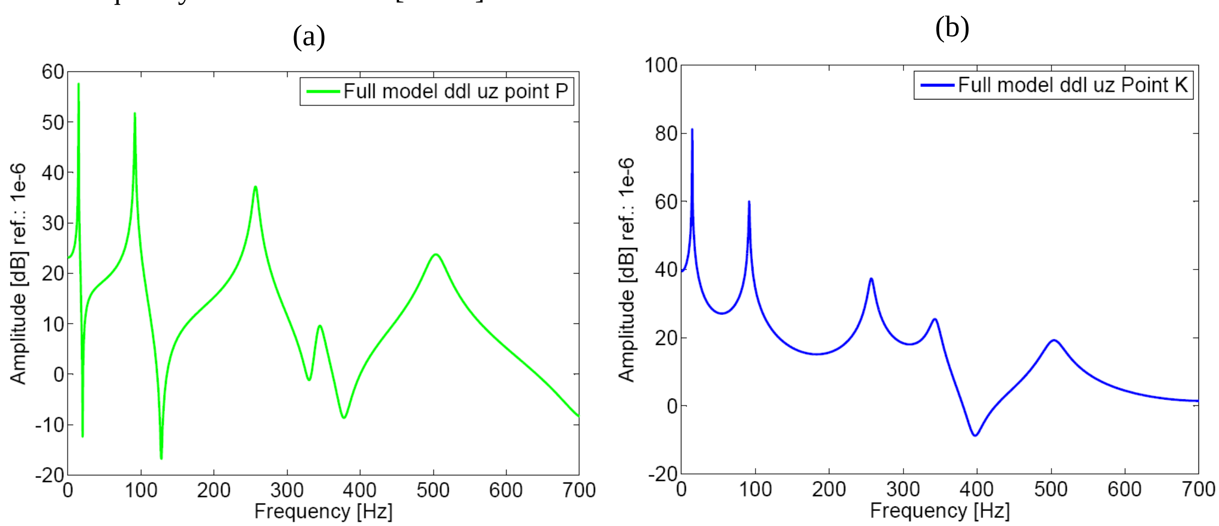


Fig.4. FRFs of the full GHM model plotted in: (a) Point P- (b) Point K

The frequency responses represented by Fig. 4 (a) and (b) are considered as the reference for the full GHM model. Indeed, the FRF amplitudes in [dB] have been computed by using a convenient reference factor through the relation $Amplitude[dB] = 20 \times \log_{10}\left(|H(\omega)| / 1e-6\right)$.

These responses can be determinate by the resolution of Eq. (8) which describes the shear modulus as a rational function as well as by the Eq. (11) which derives the problem as a second order differential equation. In fact, the mathematical development established in order to derive a second order equation of motion Eq.(12) has interest in time domain analysis while the frequency analysis can be carried out directly by the resolution of Eq.(8).

Fig.5 represents the FRFs derived from the resolution of Eqs. (8) and (11) plotted for the point P.

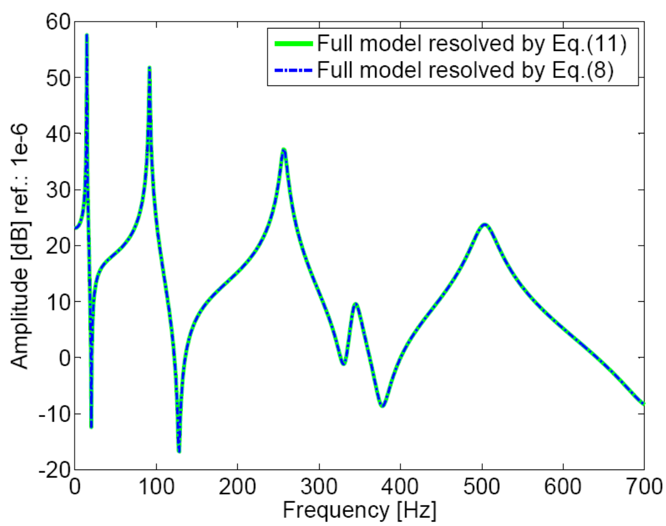


Fig.5. FRFs of full model derived from two mathematical tools

It can be observed that the two frequency responses derived from the resolution of Eqs. (8) and (11) are perfectly identical. This ensures the equivalence of the two equations.

Next step consists of the determination of damped and undamped frequencies of the sandwich beam which is carried out by the resolution of the eigenvalue problem associated to the damped Eq.(11) and the undamped Eq.(4) systems respectively. Table3 represents the five first undamped and damped frequencies of the viscoelastic sandwich beam.

Table3. Undamped and Damped eigenfrequencies of the sandwich beam

Frequency	Undamped eigenfrequencies [Hz]	Damped eigenfrequencies [Hz]
f_1	18.17	14.65
f_2	113.83	91.75
f_3	318.62	256.92
f_4	420.80	343.94
f_5	624.35	503.72

Fig.6 shows the frequency responses corresponding to the damped and undamped systems plotted for point P. This will illustrate the effect of the viscoelastic damping.

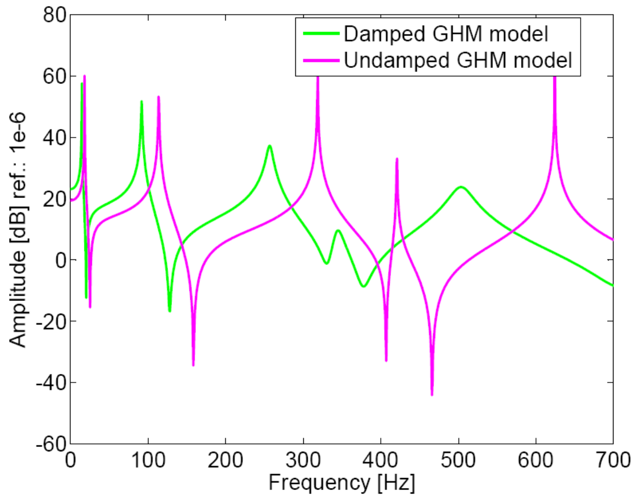


Fig.6. FRFs of the damped and the undamped GHM model for the sandwich beam

- *Elimination dofs reduction approach*

Now, we will compare Dynamic and Guyan reduction methods belonging to the elimination dofs approach with the full model for the viscoelastic sandwich beam.

At the beginning, we choose m=30 master dofs for both reduction methods such that we obtain two transformation matrices having the same dimension: $Dim\left(\left[T_{Dyn}\right]\right) = Dim\left(\left[T_{St}\right]\right)$ and the size of each transformation is equal to (1600×830). For Guyan reduction method, the choice of m=30 master dofs is constituted by the u_z dofs which are the translation dofs normal to the midplane of the sandwich beam. Furthermore, this choice is carried out maximizing the cuttoff frequency. Then, the reduction process is applied and the reduced mass, stiffness and damping matrices as well as the external load vector corresponding to each reduction method are calculated. The reduced dynamic stiffness is evaluated as follows: $\left[Z_c(\omega)\right] = -\omega^2\left[M_c\right] + j\omega\left[D_c\right] + \left[K_c\right]$ and the reduced model corresponding to each reduction method compared to the full model is presented. Fig. 7 (a) and (b) shows the corresponding FRFs plotted for the Point P and K respectively.

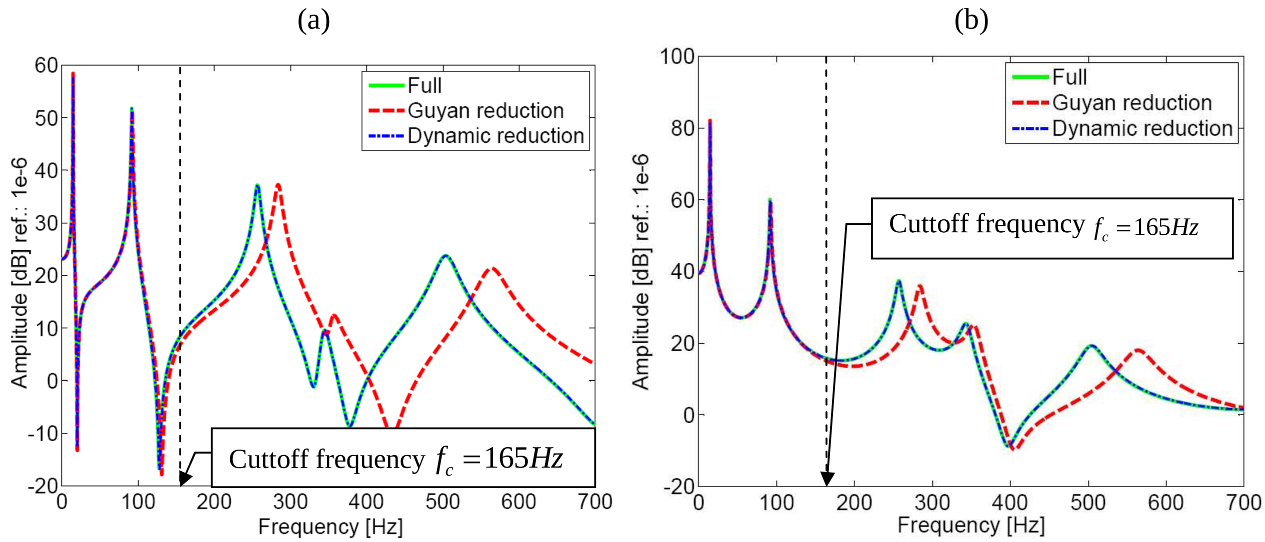


Fig.7. FRFs for full and reduced models: Dynamic/Guyan reduction methods: (a) Point P- (b) Point K of the viscoelastic sandwich beam

From both Fig. 7 (a) and (b), the frequency response for the Dynamic reduced model is identical to this of full model while the Guyan reduced model stick well to the full model for the two first modes of vibration but deviate after the cuttoff frequency ($f_c = 165Hz$) which define the validity domain of the Guyan reduction method. After this cuttoff frequency, the reduced model does not stick with the full model but follows it shape curve. This leads to conclude that Guyan reduction method has the capacity to reproduce the original system but it remains limited by its validity domain. Nevertheless, the Dynamic reduction method enriched by the first slave modes (s=10) gives a very satisfactory agreement with the full model making it a suitable method for prediction of the dynamic behavior of viscoelastic sandwich structures.

Table 4 shows the frequencies values corresponding to the full and the Guyan reduced model. This affirms that the reduced model derived from Guyan reduction method is able to reproduce only the two first modes of the full model and outlines that the validity domain of this method is limited by the cuttoff frequency.

Table4. Full and Guyan reduced eigenfrequencies for the viscoelastic sandwich beam

Frequency	Full frequencies [Hz]	Guyan reduced frequencies [Hz]
f_1	14.65	14.65
f_2	91.75	91.73
f_3	256.92	283.94
f_4	343.94	354.48
f_5	503.72	563.75

- *Modal reduction approach*

Here, the frequency responses derived from modal and modal in physical space reduction methods are compared to the full model. Indeed, we determine firstly the number of modes associated to the undamped structure, which covers 1.5 the frequency band of interest (1.5 f_u =1100 Hz) and (p= 17 modes).Then a projection on the physical coordinates is achieved by the partition of the modal basis into master and slave dofs where m=p=17 (case a in section 4.4) and we inverse directly the modal basis corresponding to the mater dofs contribution such that $[Q_{1m}]$ has the minimum conditionning number. Let's $cond([Q_{1m}])=50$ .Thus, we obtain two transformation matrices having the same size $Dim([T_{Mod}])=Dim([T_{SP}])$ with the dimension of each basis is equal to (1600×817) .

Fig. 8 (a) and (b) depicts the frequency responses for full and reduced models.

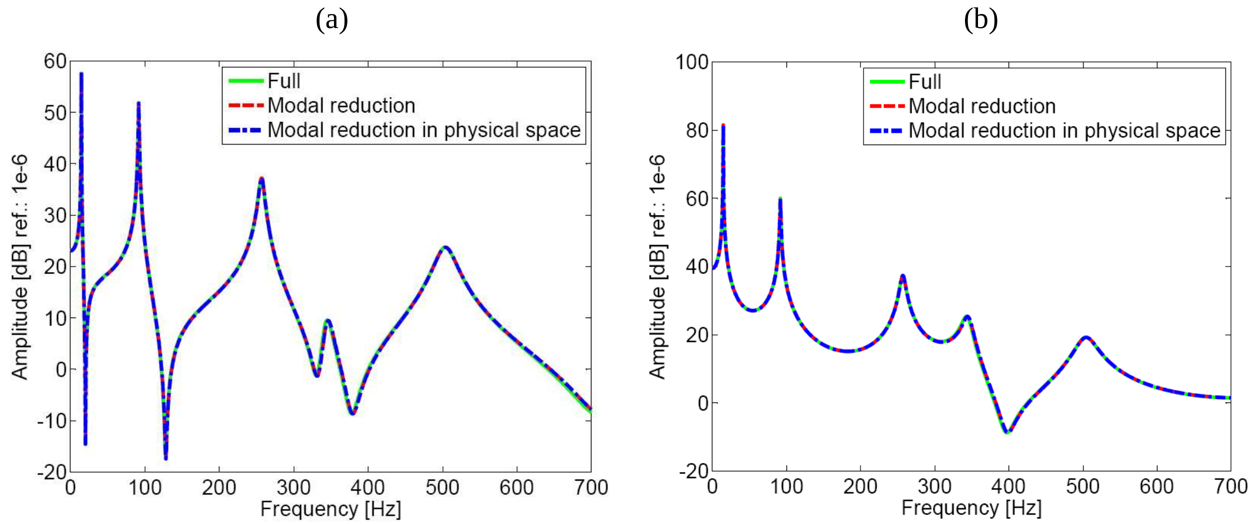


Fig.8. FRFs for full and reduced models: Modal/Modal in physical space reduction methods: (a) Point P-(b) Point K of the viscoelastic sandwich beam

As expected, the frequency responses curves for reduced models and full model are in good agreement for the point P and K as shown by Fig.8 (a) and (b). This leads to conclude that modal reduction projected in generalized or in physical coordinates is a viable method for the prediction of the dynamic behavior of structures incorporating viscoelastic materials.

- *Physical coordinates approach*

An overlap of elimination dofs reduction approach and modal reduction approach is realized through the projection on the physical coordinates leading to compare Guyan reduction method to modal reduction in physical space with the full model. Hence, the two transformation matrices must have the same size to compare them such that the master dofs of Guyan reduction method is equal the master dofs of modal reduction in physical space method. Let's the number of master dofs m=30 which is higher than the number of modes p=17. Consequently, we test the case where m>p (case c in section 4.4) for modal reduction in physical space method. This leads to two transformation matrices as $Dim([T_{SP}]) = Dim([T_{St}])$ with size (1600×830).

Fig. 9 (a) and (b) represents the corresponding frequency responses comparison.

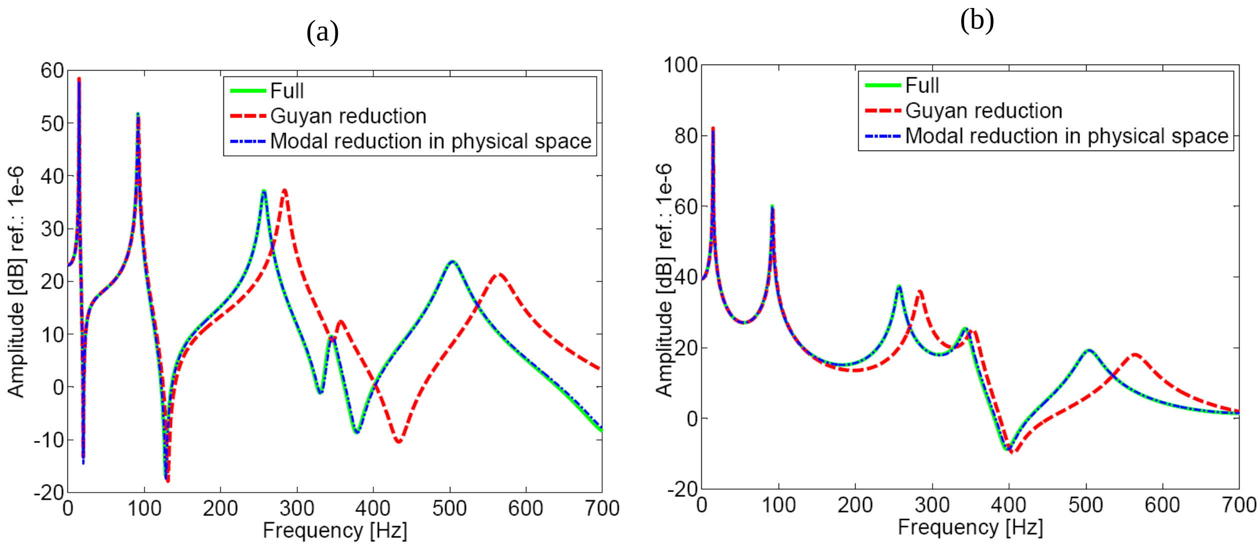


Fig.9. FRFs for full and reduced models: Guyan/Modal in physical space reduction methods: (a) Point P-(b) Point K of the viscoelastic sandwich beam

As can be seen, the frequency response derived from modal reduction method is in good agreement with the reference while the frequency response of Guyan reduction method deviates after the cuttoff frequency both in Fig. 9 (a) and (b). This affirm that Guyan reduction method is limited by its validity domain but it can generally predicts the dynamic behavior of viscoelastic structures with less accuracy than modal reduction in physical space method which needs an additional time of evaluation relative to Guyan method.

Consequently, through the projection on physical coordinates both Guyan and modal reduction in physical space are viable methods which able to reproduce the original model. However, for Guyan reduction method the optimal choice of master dofs is conditioned by the maximum of cuttoff frequency.

The performance of all proposed reduction methods in term of CPU time is shown in Table 5.

Table5. Performance of proposed reduction methods in frequency domain

	Total CPU time [min]				
	Full	Guyan	Dynamic	Modal	Modal in physical space
	263	35	52	65	78
Reduction ratio (%)	-	87	80	75	70

Table 5 shows the total computing time for full and reduced models. This time, evaluated for each reduction method, includes the calculations of the transformation matrix and the FRF response which is obtained by solving linear equations at each frequency point. For the clarity of comparaison, it should be mentioned that all reduced models have the same size 830. Hence, as can be remarked, while the reduction ratio in term of models dimension is about 50%, it is so advantageous in term of CPU time. In fact, this ratio reaches 87% with Guyan reduction method and 80% with dynamic reduction method while it not exceeds 75 % with modal reduction method and 70% with modal reduction in physical space. The additional CPU time for modal reduction method in generalized or physical space can be explained by the calculation requirements of undamped modes and the verification of minimum conditionning number in the case of projection on the physical space. Hence, these reduction methods allow generally a drastic reduction making them a suitable choice to handle both the prohibitive computational effort and the viscoelasticity especially for complex structures with large finite element model or in optimization procedure when the dynamic calculations of such models become more complicated. Consequently, the application of these direct reduction methods in frequency domain able to save time considerably leading to perform the applicability and the efficiency of these methods in time domain.

5.1.2. Time domain evaluation

The interest here is intended to time domain analysis for the viscoelastic sandwich structures. In fact, the prediction of the dynamic behavior of such structures remains until now focused on the frequency analysis more than time analysis. Here both steady state and transient analysis are carried out.

The resolution of temporal equation of motion Eq. (13) is performed using the Newmark's integration technique [37] with an unconditionally stable scheme. This technique is used in order to derive the time responses for both full and reduced models which will be compared for each reduction method. These comparisons are performed through static tools called Temporel prediction indicators.

- *Temporel Prediction indicators*

Results comparison tools are based on the statistic indicator calculations associated to the full and reduced responses. In fact temporal moments are usually used to quantify a temporal signal in order to compare several transient responses [38]. Hence, the i^th order of the temporal moment of a response $y(t)$ is defined as [39]:

$$M_i = \int_{-\infty}^{+\infty} (t - t_s)^i (y(t))^2 \, dt \tag{37}$$

Where t_s represents the temporal shift and *i* the moment index order.

In this case, the temporal moment M_i is defined for $t_s = 0$ and normalized as follows:

$$\begin{cases} E = M_0, \text{Energy (ms}^2) \\ T = \dfrac{M_1}{M_0}, \text{Central time (centroid) (s)} \\ D^2 = \dfrac{M_2}{M_0} - \left(\dfrac{M_1}{M_0}\right)^2, \mathit{Root} \text{ means square duration (s)} \end{cases} \tag{38}$$

Thereby, this triplet of indicators (E, T and D) enable to determine the error generated both in the amplitude and time scales. Indeed, E is used to identify the error in the amplitude of the response; T and D are used to identify the error in the periodicity of the response.

Gerges [40] was proving that a relative error of order of $\pm 4\%$ in energy E, $\pm 2\%$ in central time T and $\pm 4\%$ in Root means square D is admissible in order to validate the reduced model compared to the full model.

In the remainder and for good clearance, it should be noticed that all time responses will be plotted only on the Point P.

- *Steady state analysis*

The sandwich beam is exhibited to a harmonic load of the form $\{F(t)\} = F_0 \sin(\omega t)$ where $F_0 = 1N$ and $\omega = 50\pi rad/s$;(f= 25 Hz). The steady state response is established after 1s of transient response and the oscillations are well stable over a period of time from 2 to 3 s. Therefore, in the following, the time response derived from a harmonic excitation will be presented in the interval of time [2-3] s.

The same procedure of comparaison for reduced and full models presented in section 5.1.1 is also carried out in this section with the time domain analysis. For each type, the size of the reduced models is kept the same as mentioned in the previous sections.

- *Steady state responses for Elimination dofs reduction approach*

The time responses for full and reduced models are presented in Fig.10 (a) and (b).

Fig.10 (a) show that the time responses curves for full model and those of reduced models are in good correlation over a period of time of 1s. Here, Guyan reduction method sticks well with the full model because the excitation frequency in which the sandwich beam subjected is less than the cuttoff frequency ($f < f_c$). So, the excitation covers the validity domain of this method leading to good agreement with the reference.

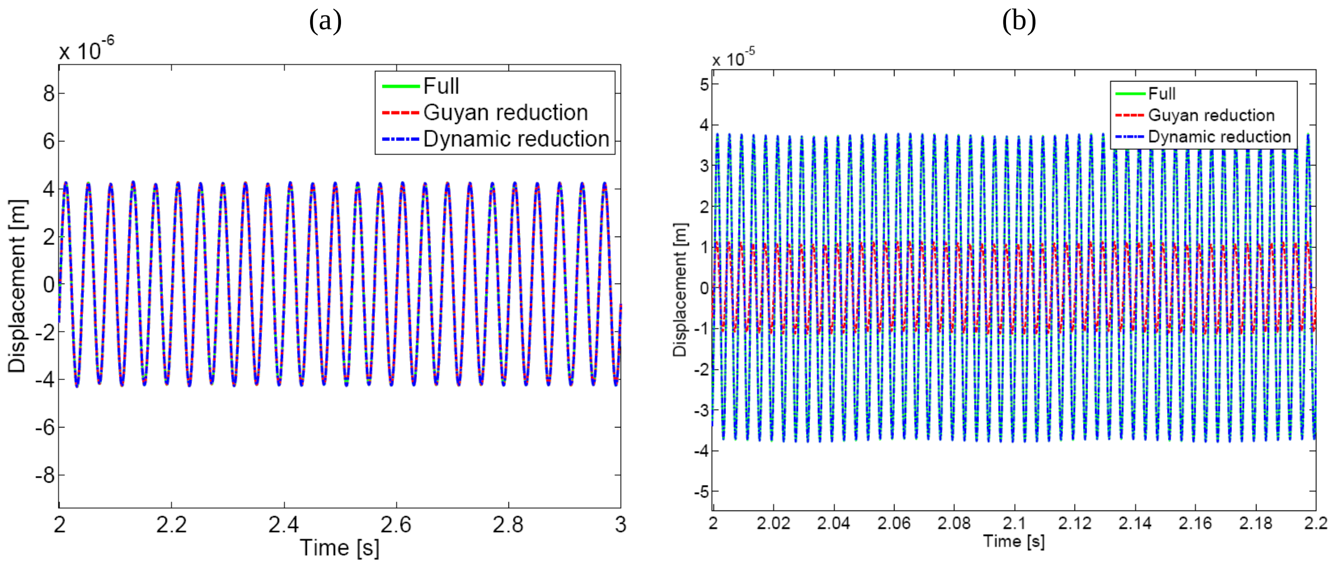


Fig.10. Steady state responses for full and reduced models: Dynamic/Guyan reduction methods of viscoelastic sandwich beam: (a) ($f < f_c$) - (b) ($f > f_c$)

Table 6.1. Temporal moments for the steady state responses of the viscoelastic sandwich beam ($f < f_c$)

	E	T	D
Full	9.3002E-6	1.3804	0.8220
Dynamic	9.3000E-6	1.3804	0.8220
Guyan	9.3586E-6	1.3808	0.8218

Table 6.2. Temporal moments for the steady state responses of the viscoelastic sandwich beam ($f > f_c$)

	E	T	D
Full	4.65001E-6	0.6902	0.4110
Dynamic	4.65000E-6	0.6902	0.4110
Guyan	4.58005E-6	0.6700	0.3889

Furthermore, Table 6.1 shows that the relative error in energy E between the full model and the Guyan reduced model is of order of -0.62% while this error not exceeds 0.002% with the Dynamic reduced model. This confirms the visual impression in amplitudes.

In addition, the relative error in central moment T and means square root D not exceeds 0.04% for Guyan reduced model and it is practically equal to zero for Dynamic reduced model. Consequently, the obtained results in the case where ($f < f_c$) present a satisfactory accuracy compared to the full model enabling to validate the reduced models.

However, when the excitation frequency is higher than the cuttoff frequency ($f > f_c$), the obtained results for full and reduced models of viscoelastic sandwich beam exhibited to harmonic load under a frequency excitation $f = 300Hz$ are depicted in Fig.10 (b).

As can be seen, in this case, the results start to lose its accuracy. Indeed, the reduced Guyan response presents an apparent deviation in both amplitudes and time scales (Table 6.2). The deviation in amplitudes scale is indicated by a relative error which reaches 1.5% in energy E. In the time scale, the relative error has the order of 5% in D and 3 % in T. These values are significant compared to the case where ($f < f_c$) and leads to conclude that Guyan reduction method is limited by its validity domain. Hence, beyond the cuttoff frequency, Guyan reduction method is less accurate. Nevertheless, Dynamic reduction method preserves its capacity to reproduce the full steady state response in both cases leading to affirm the performance of this reduction method in the prediction of the dynamic behavior of viscoelastic sandwich structures.

- *Steady state results for modal reduction approach*

For modal reduction methods the obtained results are shown in Fig.11.

It can be observed in Fig.11 that the time responses of steady state motion for the reduced models are in good agreement with the full model. This is confirmed by the values of the three central temporal moments. In fact, Table 7 shows that the relative error in E does not exceed 0.1% while the relative error in T and D is practically zeros. Thus leads to validate the reduced models which allow a perfect reproduction of the original model in amplitude and time scales.

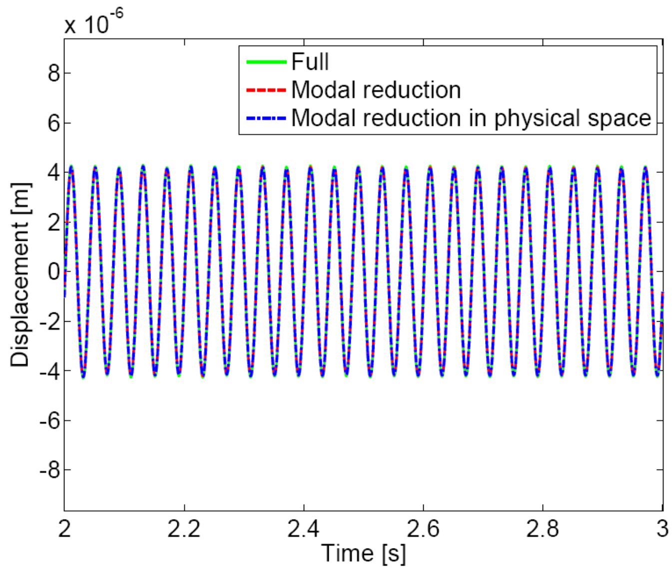


Fig.11. Steady state responses for full and reduced Models: Modal/Modal in physical space reduction methods of the viscoelastic sandwich beam

Table 7. Temporal moments for the steady state responses of modal reduction approach of the viscoelastic sandwich beam

	E	T	D
Full	9.3002E-6	1.3804	0.8220
Modal	9.2904E-6	1.3804	0.8220
Modal in physical space	9.2904E-6	1.3804	0.8220

The comparaison of Guyan reduction method to modal reduction method in physical space for steady state responses is also carried out. The obtained results indicates a relative error in energy E of 0.1% for modal reduction in physical space which reaches to 0.6% for Guyan reduction method. This implies that through the projection on physical space, modal reduction method in physical space has the capacity to reproduce the original model better than the Guyan reduction method.

- *Transient analysis*

In this section, the viscoelastic beam is subjected to an impulse load of duration $T_{impulse} = 2ms$ and amplitude equal to 10N.The same strategy of comparison between the different reduction methods is carried out. Furthermore, as mentioned in the previous sections, for each reduction approach the equality of reduction basis is provided.

- *Transient results for elimination dofs reduction approach*

The comparison between reduced models and full model is shown in Fig.12.

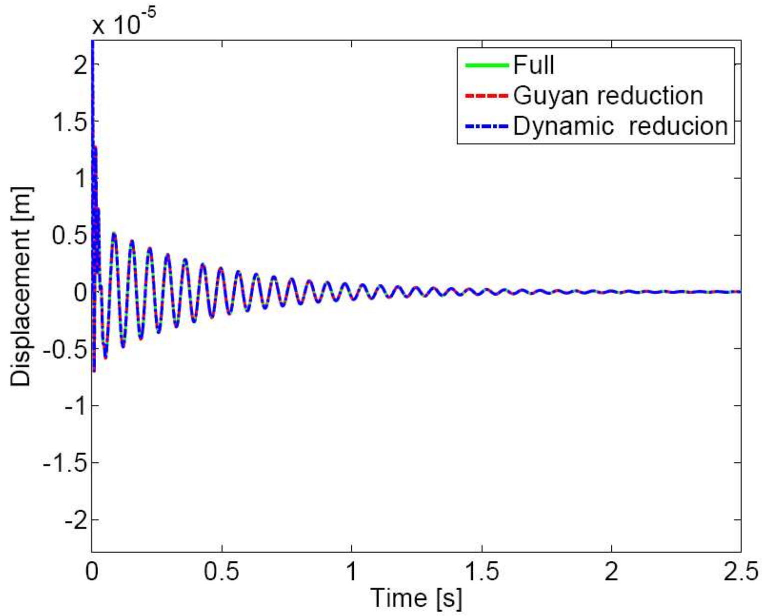


Fig.12. Transient responses for full and reduced Models: Dynamic/Guyan reduction methods of the viscoelastic sandwich beam

Table8. Temporal moments for the transient responses of eliminated dofs reduction approach of the viscoelastic sandwich beam

	E	T	D
Full	1.0387E-6	0.2055	0.0498
Dynamic	1.0387E-6	0.2055	0.0498
Guyan	1.0403E-6	0.2054	0.0498

The time responses to an impulse excitation at the point P of the viscoelastic sandwich beam are well correlated before and after reduction. Besides, the three central moments reflect that

Dynamic reduction method is a viable method which reproduces entirely the original model. Since the frequency spectrum of the impulse excitation covers the validity domain of Guyan reduction method, the reduced response derived from this method stick well with the original while it represents a relative error in E of the order of 0.15% and a relative error in T of the order of 0.04% (Table8). Thus, these values can validate the Guyan reduced model. As result, this method presents a suitable choice in term of simplicity, feasibility of implementation and also satisfactory accurate results.

- *Transient results for modal reduction approach*

The derived transient results for both reduced and full models are presented in Fig.13.

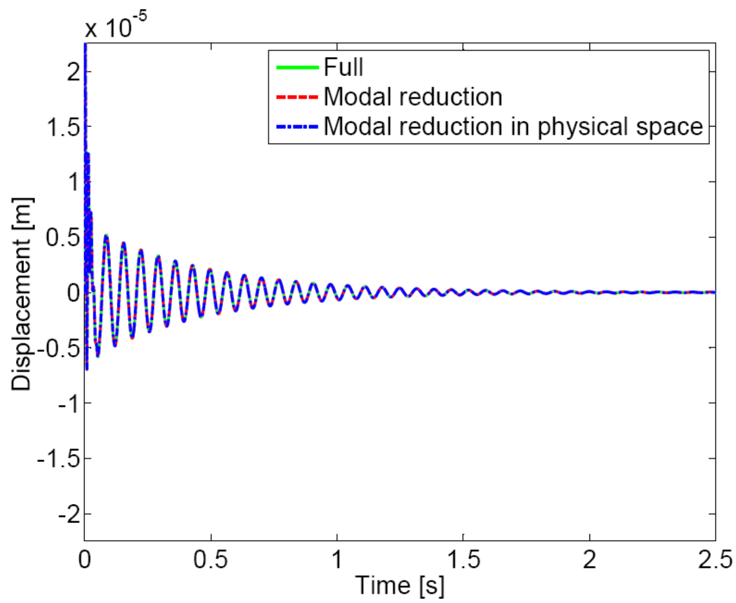


Fig.13. Transient responses for full and reduced Models: Modal/Modal in physical space reduction methods of the viscoelastic beam

Table 9. Temporal moments for the transient responses of modal reduction approach of the viscoelastic sandwich beam

	E	T	D
Full	1.0387E-6	0.2055	0.0498
Modal	1.0369E-6	0.2055	0.0498
Modal in physical space	1.0369E-6	0.2055	0.0498

Fig.13 shows the transient responses for the reduced models derived from modal and modal in physical space reduction methods compared to the full model. It can be observed that these responses are identical. In fact, modal reduction method returns the p first exact modes of the associated undamped model allowing a reproduction of the original model through a generalized coordinates projection while modal reduction in physical space method allows a reproduction of the full model through a projection in physical coordinates. This is affirmed by the three central moments (Table9) which indicated that both reduced models preserve the periodicity of the full response with a relative error in the energy E which does not exceed 0.17% leading to validate these two reduction methods in temporal domain.

Table10. Performance of proposed reduction methods in time domain

	Total CPU time [min]				
	Full	Guyan	Dynamic	Modal	Modal in physical space
	365	40	65	85	92
Reduction ratio (%)	-	88	82	77	75

Table 10 presents the performance of the proposed reduced models compared to full model in time domain. There is a significant reduction ratio in total CPU time required for the evaluation of reduced basis and temporal responses at each iteration which justify the efficiency of these reduction methods in time domain.

5.2. Viscoelastic sandwich plate

In this example, the interest is focused on the validity domain extension of Guyan reduction method. In fact, after meshing the plate into 20×15 finite elements as shown in Fig.14, the master dofs, which are translation dofs u_z, are chosen such as the cuttoff frequency is maximal. Hence, the distribution of the chosen master dofs (m=40) is illustrated in Fig.14.

The plate is clamped on the four sides (C-C-C-C). The FE discretization scheme leads to 8310 dofs in total number. The excitation and the responses are depicted in the point E as presented in Fig.14.

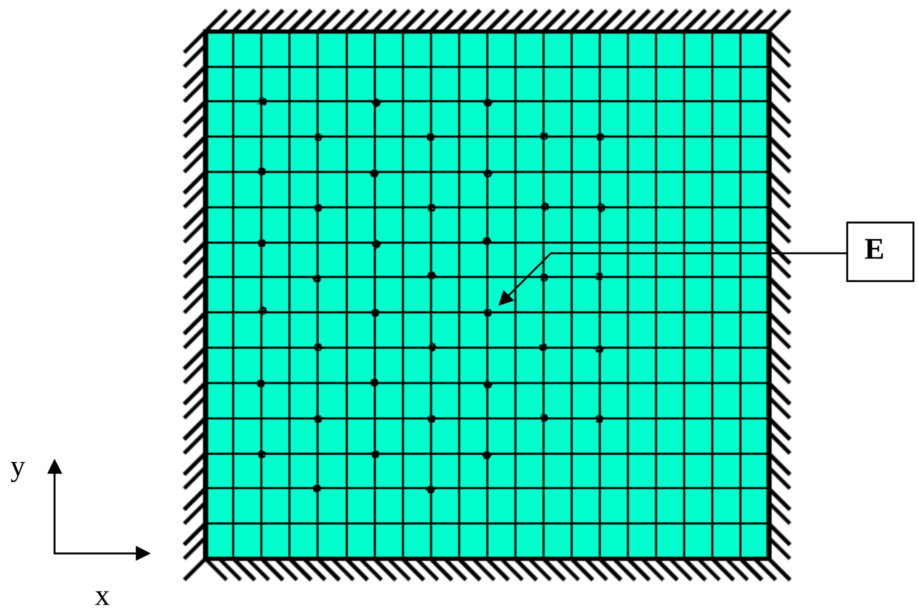


Fig.14. FE model for the viscoelastic sandwich plate with the position of the optimal master dofs (•)

5.2.1. Frequency domain analysis

The frequency analyses for the sandwich plate are carried out as same procedure mentioned in the previous sections. In fact, for each type of reduction, the equality of bases is assured.

For the elimination dofs reduction approach, Guyan and dynamic basis are constructed such as the two bases have the same size (8310×4195). Then, modal and modal in physical space bases belonging to modal reduction approach are also constructed such as the size of each basis is equal to (8310×4180), with p=25 modes which covers 1.5 the frequency band of interest [0-1200] Hz $(1.5 f_u = 1800 Hz)$.

Table11 presents the eight first damped and undamped frequencies of the viscoelastic sandwich plate. Indeed, the difference between the damped and undamped frequencies values indicates the effect of the viscoelastic damping.

Table11. Undamped and Damped frequencies for the viscoelastic sandwich plate

Frequency	Undamped frequencies [Hz]	Damped frequencies [Hz]
f_1	264.26	213.07
f_2	463.78	374.13
f_3	604.83	488.05
f_4	788.08	636.37
f_5	1095.50	885.47
f_6	1125.20	908.92
f_7	1226.53	991.01
f_8	1300.83	1051.70

- *Elimination dofs reduction approach*

The obtained results for the viscoelastic sandwich plate are presented in Fig.15.

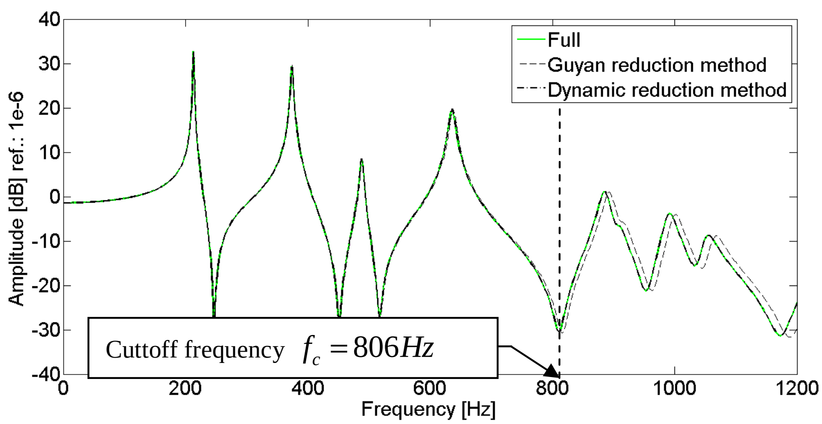


Fig.15. FRFs for full and reduced models: Guyan/Dynamic reduction methods for the viscoelastic sandwich plate

As can be seen, the frequency response derived from the Guyan reduction method has the capacity here to reproduce the frequency response of the full model for the first four modes. Furthermore, beyond the cuttoff frequency which is equal to 806 Hz, the Guyan reduced response follows the shape curve of the full model with small difference. This implies that Guyan reduction method is a viable method for the prediction of the dynamic behavior of viscoelastic sandwich structures, when the choice of master dofs is optimal. So, more the choice is optimal, more the results are accurate. For the dynamic reduction method, its frequency response is in good agreement with response of the full model. This affirms the efficiency of this method in the reproduction of the full model dynamics.

- *Modal reduction approach*

Fig.16 shows the frequency responses for the reduced models derived from modal and modal reduction in physical space methods compared to the full model.

It can be observed that the two reduced frequency responses derived from modal and modal reduction in physical space are identical to the frequency response of full model. This leads to confirm that modal and modal reduction method in physical space has the capacity to reproduce the original coordinates of the sandwich structures through a projection on the generalized coordinates as well as on the physical coordinates with good accuracy.

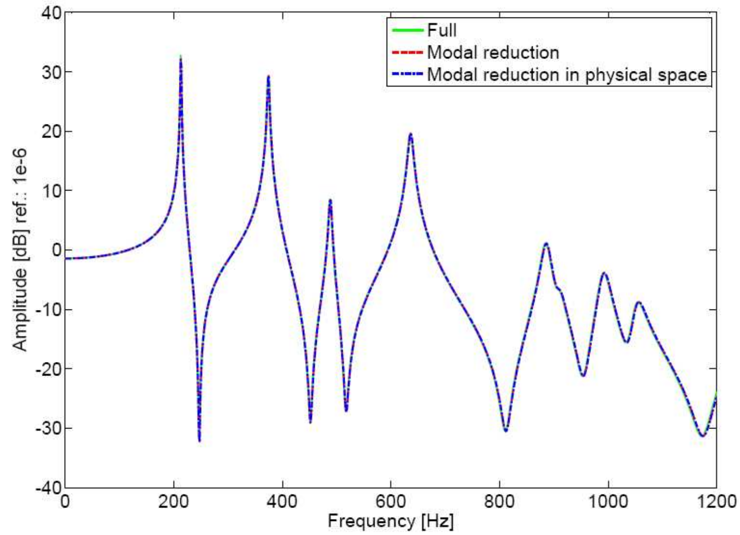


Fig.16. FRFs for full and reduced models: Modal/Modal reduction in physical space methods for the viscoelastic sandwich plate

5.2.2. Time domain analysis

In this section, the time responses are focused on the steady state analysis for the viscoelastic sandwich plate in order to show the performance of Guyan reduction method. Indeed, the viscoelastic sandwich plate is exhibited to a harmonic load of amplitude equal to 1N under an excitation frequency equal to 400 Hz (around the second mode of vibration for the sandwich plate). The steady state response is reached after 0.15s of transient oscillations and where the oscillation becomes more stable, the time responses are plotted.

For Guyan reduction method, two cases are tested: the first one examine the steady state responses of the sandwich plate where it is exhibited to excitation frequency less than the cuttoff frequency and the second shows the else case where the plate is subjected to an excitation frequency higher than the cuttoff frequency.

(a)

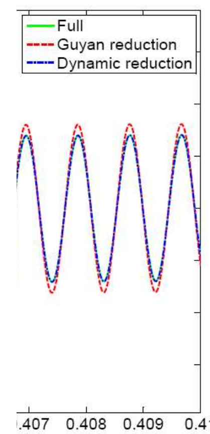


Fig.17. Steady state response for the full and reduced models: Guyan/Dynamic methods of the viscoelastic sandwich plate: (a) ($f < f_c$) - (b) ($f > f_c$)

Table12.1. Temporal moments for the steady state responses ($f < f_c$) of eliminations dofs reduction approach of the viscoelastic sandwich plate

	E	T	D
Full	1.8377E-8	0.2428	0.0218
Dynamic	1.8377E-8	0.2428	0.0218
Guyan	1.8516E-8	0.2428	0.0218

Table12.2. Temporal moments for the steady state responses ($f > f_c$) of eliminations dofs reduction approach of the viscoelastic sandwich plate

	E	T	D
Full	2.1123E-8	0.3578	0.0412
Dynamic	2.1123E-8	0.3578	0.0412
Guyan	2.1128E-8	0.3578	0.0412

- Case1: $f < f_c$

Fig.17 (a) shows the steady state responses of the Guyan and Dynamic reduced models compared to the full model. It can be seen that the dynamic response presents a satisfactory agreement with the full model. This is clarified by the values of the three central moments (E, T, D) (Table12.1) which is identical to those of the full model. Thus dynamic reduction method remains a good choice of reduction methods. For Guyan reduction method, the steady state response reproduces the original response with a relative error in energy E which not exceeds 0.007% as shown in Table12.2 while the central moments T and D, indicators of error in periodicity, are identical to those of the full model. Hence, Guyan reduction method is validate for each frequency excitation less than the cuttoff frequency.

- Case2: $f > f_c$

When the sandwich plate is subjected to a harmonic excitation frequency higher than the cuttoff frequency $\left(f = 1100 Hz \right)$, the steady state response of Guyan reduced model presents a little shift relative to the full model while dynamic reduced response preserve its capacity to reproduce the full response (Fig.17(b)). In fact, the reduced response derived from Guyan reduction method presents a few relative error in energy E of order of 0.02% while its preserve the periodicity of the full model as shown in Fig.17 (b). Consequently, Guyan reduction method can predict with good accuracy the viscoelastic behavior of sandwich structures when the choice of master dofs is optimal. So, compared to the case of sandwich beam where the excitation frequency is high than the cuttoff frequency, Guyan reduction method presents in the case of the plate more satisfactory results.

For modal reduction approach, the obtained results are presented in Fig.18.

The reduced responses obtained from modal and modal reduction in physical space methods are identical to the full model. This is affirmed by the values of the three central moments

(E, T, D) presented in Table13. In fact, the relative error for each moment for the three compared responses is practically equal to zero. This leads to conclude that modal and modal reduction in physical space are a viable methods for the prediction of the dynamic behavior of viscoelastic sandwich plate.

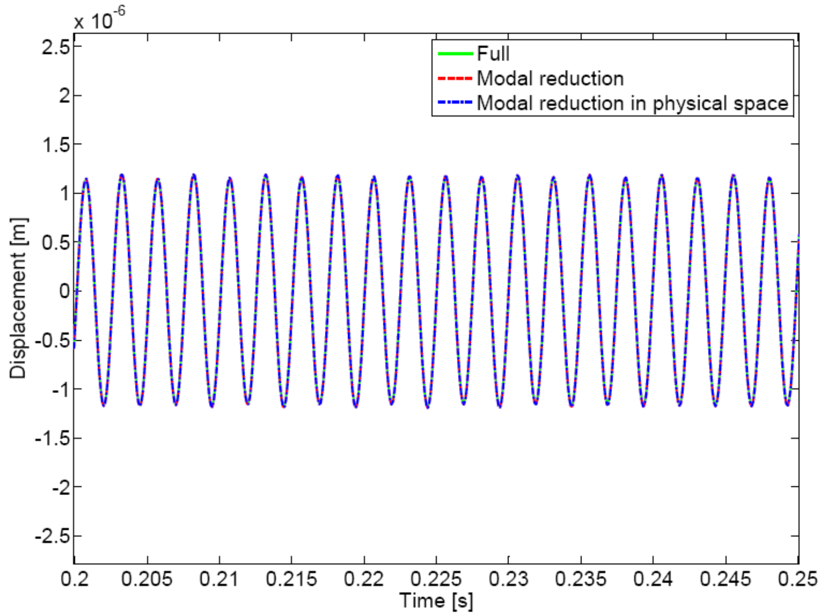


Fig.18. Steady state response for the full and reduced models: Modal/Modal in physical space methods of the viscoelastic sandwich plate

Table 13. Temporal moments for the steady state responses of modal reduction approach of the viscoelastic sandwich plate

	E	T	D
Full	1.8377E-8	0.2428	0.0218
Modal	1.8375E-8	0.2428	0.0218
Modal in physical space	1.8375E-8	0.2428	0.0218

The transient analysis for the viscoelastic sandwich plate subjected to an impulse excitation is also established. Indeed, the transient reduced responses present a good agreement with the full model for each type of reduction. This can be explained for Guyan reduction method by the frequency spectrum of the impulse excitation which covers the validity domain of this method. Hence, the optimal choice of master dofs is an important step in all reduction procedure notably for Guyan reduction method in order to good predicts the dynamic behavior of viscoelastically damped structures.

For this example and for the sake of brevity, only the CPU time evaluated in time domain is illustrated.

Table14. CPU time of the viscoelastic sandwich plate

	Total CPU time [min]				
	Full	Guyan	Dynamic	Modal	Modal in physical space
	1440	168	258	324	356
Reduction ratio (%)	-	88	82	77	75

The dynamic potential of the proposed reduction methods is more highlighted with the viscoelastic plate example. In fact, the saved time required for calculations of full and reduced models increase by increasing the degrees of freedom. Furthermore, these calculations takes into account the evaluation of reduced basis and the iterative procedure generated by the use of Newmark scheme in time domain for each applied reduction method. Hence, these reduction methods constitute an efficient solution to gain time and to handle large finite elements models with viscoelastic components. In other hand, these methods are used in the direct reduction context and they improved their efficiency notably in term of CPU time leading to perform both frequency and temporal analysis. So, when more than one structure is used and taking into account the non-linear behavior of the most structures, the use of model reduction method in the substructuring context or component mode analysis [36] for viscoelastic sandwich structures appears so attractive.

5.3. Temporal analysis with localized nonlinearities in the substructuring context

In this section, attention is focused on assembled viscoelastic sandwich structures. Indeed, the bolted joints are usually modeled by non-linear elements in the junctions of such structures. Therefore, local nonlinearities are introduced to take into account this effect. However, this is done at the price of generation firstly a large systems dimension induced by viscoelastic components and secondly time consuming due to the resolution scheme which become more complicated with the introduction of local nonlinearities. So, it remains challenging to develop an efficient reduction strategy that able to overcome this problem. For that, we propose to combine Guyan reduction method with modal synthesis method for local non-linear viscoelastic structures in the substructuring context. This is done by the addition of a non-linear term in the equation of motion Eq. (13). In fact, the form of this equation as a standard temporal second order equation leads to introduce the local nonlinearities with a simple and soft way. Thereby, the obtained temporal non-linear equation of motion can be written as follows:

$$[M_G]\{\ddot{q}_G\}+[D_G]\{\dot{q}_G\}+[K_G]\{q_G\}+\{f_{nl}(q_G)\}=\{F_G\} \tag{39}$$

Where $\{f_{nl}(q_G)\}$ indicates the added non-linear load which its ith component can be expressed by the Duffing oscillator as follows:

$$\{f_{nl}(q_G)\}_i=\sum_{j=1}^{m}\mu_j\left[(q_G)_i-(q_G)_j\right]^3=\left[K_{nl}(q_G)\right]\{q_G\} \tag{40}$$

m represents the number of attached non-linear springs relied to ith dof; μ_j represents the non-linear stiffness factor for each non-linear spring and $\left[K_{nl}(q_G)\right]$ is the non-linear stiffness matrix contribution.

The application of the proposed reduction strategy leads to the following non-linear reduced model:

$$[M_c]\{\ddot{q}_c\}+[D_c]\{\dot{q}_c\}+[K_c]\{q_c\}+[K_{nlc}]\{q_c\}=\{F_c\} \tag{41}$$

Where: $[M_c]$; $[D_c]$; $[K_c]$; $[K_{nlc}]$ and $\{F_c\}$ represent respectively the reduced mass, damping, linear and non-linear stiffness matrices and the reduced load vector obtained by the application of Guyan transformation matrix which is described in the previous section (4.2) with master (m) and slave (s) dofs expressed for the direct method are replaced respectively by junction (j) and interior (i) dofs for the substructuring procedure.

The FE model of the global viscoelastic sandwich beam is illustrated in Fig.19.

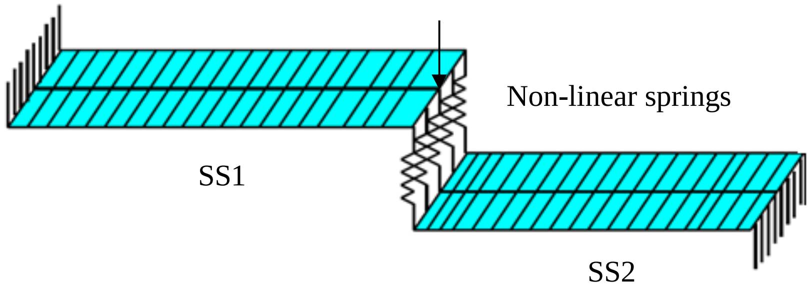


Fig.19. FE model of the global non-linear assembled viscoelastic sandwich beam

The used FE model of the global viscoelastic sandwich beam involves 80 elements with 320 nodes and 5 dofs per node leading to 3200 total dofs. This beam is clamped at its two edges and the mechanical and geometrical properties for each substructure (SS1) or (SS2) are the same as described for the viscoelastic sandwich beam in Table 1. The value of each used non-linear spring coefficient is $\mu = 10^9\, N/m^3$.

First, we start from the knowledge of the dynamic behavior of each substructure (SS1) and (SS2) which are reduced separately by the application of Guyan reduction method.

- Guyan reduction of substructure (SS1)

The displacement vector $\{q_G\}_{(SS1)}$ of the viscoelastic substructure (SS1) is decomposed accordingly to the junction (j) and interior (i) dofs partition as follows:

$$\{q_G\}_{(SS1)} = \begin{Bmatrix} q^j \\ q^i \\ z \end{Bmatrix}_{(SS1)} = [T_{St}] \begin{Bmatrix} q^j \\ z \end{Bmatrix} \tag{42}$$

$[T_{St}]$ is the Guyan transformation matrix as defined in section (4.2).Then, the reduced system is obtained by substituting Eq. (42) into Eq. (17). Thus, the size of the reduced model for the substructure (SS1) is 805 for the studied example. The choice of junction dofs (j=5), which are translation dofs u_z, is carried out on maximizing the cuttoff frequency of the viscoelastic substructure (SS1) which is equal to 165 Hz.

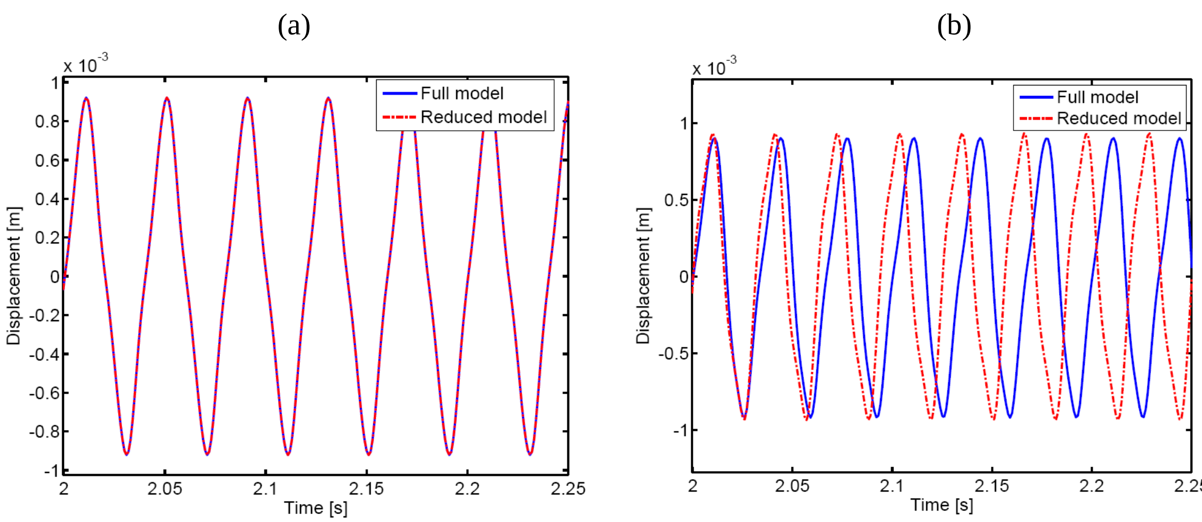
- Guyan reduction of substructure (SS2)

In the same manner, the displacement vector of the second viscoelastic substructure (SS2) is partitioned in term of junction (j) and interior (i) dofs as follows:

$$\{q_G\}_{(SS2)} = \begin{Bmatrix} q^j \\ q^i \\ z \end{Bmatrix}_{(SS2)} = [T_{St}] \begin{Bmatrix} q^j \\ z \end{Bmatrix} \tag{43}$$

The reduced model is obtained in the form of Eq. (17) using Eq. (43). Thereby, its dimension is equal to 805 with j=5 dofs. Furthermore, the cuttoff frequency of the viscoelastic substructure (SS2) is equal to 165Hz.

After that, the reduced matrices are assembled taken into account the localized nonlinearities in the junctions between the two viscoelastic substructures (SS1) and (SS2) leading to a global reduced system of order 1610. The obtained temporal results of the global viscoelastic sandwich beam, which is subjected to a harmonic load in the point A of amplitude 50N to arise effectively the non-linear behavior, in term of displacement and velocity, are presented in Fig.20 (a) and (b).



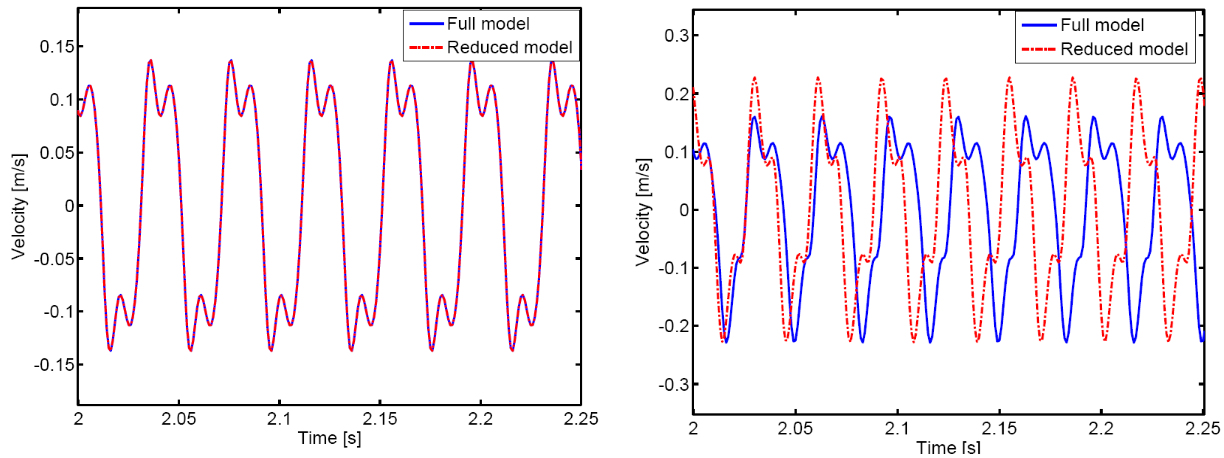


Fig.20. Temporal responses of full and reduced models of the non-linear assembled viscoelastic sandwich beam: (a) f<fc, (b) f>fc

As can be seen, Fig.20 (a) shows that the temporal responses in term of displacement and velocity are in good agreement in the case of a harmonic excitation (f=25Hz) less than the cuttoff frequency of the non-linear assembled viscoelastic (fc=165Hz). Furthermore, the evaluation of the three temporal moments (E, T, D) was proving identical values for both full and reduced models leading to validate the visual correlation. For the case of high excitation (f=300Hz) relative to the cuttoff frequency (fc=165Hz), the full and reduced models present a shift in amplitude and time scales. This shift is about 3% in energy E, 0.1% in T and 0.2% in D for the displacement responses and 5% in E, 0.3% in T and 0.1% in D for the velocity responses. This leads to validate the applicability of the proposed method for non-linear viscoelastic structures in time domain. In other hand, while the reduction ratio in term of systems order is around 50% for such as non-linear example, the accuracy of obtained results in term of displacement and velocity is satisfactory. Hence, this reduction method presents an efficient tool to handle non-linear structures with viscoelastic materials in time domain which enable to perform the frequency analysis with more specific techniques such as harmonic balance method.

The performance of the proposed method in term of CPU time is shown in table 15.

Table15. Performance of the non-linear assembled viscoelastic sandwich beam

	CPU Time [min]	
	Full	Reduced
	684	61
Reduction ratio (%)	92	

There is a significant CPU reduction ratio of 92% leading to conclude that the proposed reduction method for non-linear viscoelastic sandwich structures enable to bring two levels: viscoelasticity and nonlinearity for the compromise good accuracy and time efficiency.

It should be mentioned that from the studied examples of viscoelastic sandwich (beam, plate, assembled beams) which are academic structures, the reduction ratio in term of systems dimension does not exceeding 50% but it can be possible to raise further this ratio with more complex structures.

6. Conclusions

In this paper, finite element procedures are combined to first order shear deformation theory (FSDT) and to GHM model for the modeling of viscoelastic sandwich structures. The introduction of internal variables or dissipation coordinates through a serie of mini-oscillators to take into account the viscoelastic damping is achieved. Unfortunately, this was done at the expense of increasing the model order. Consequently, model reduction methods have been proposed as a convenient alternative for this problem. First, Dynamic reduction method based on the elimination of slave dofs and enrichment of the transformation basis with first slave modes is developed. As result, the reduced model reproduces well the original model with good accuracy and few CPU time making it a best choice of model reduction methods for the compromise accuracy-time gain in direct reduction procedure. Next, Guyan reduction method is expressed by a static basis, neglecting the inertia associated to the slave coordinates. This method allows a simple implementation in the most finite elements codes with a significant reduction ratio in term of CPU time and a good capacity of prediction of the original model especially in substructuring context where the necessity of an efficient reduction method becomes twice reinforced firstly by the large systems dimensions induced by viscoelastic components and secondly by the consuming time generated by the introduction of local nonlinearities. Then, modal reduction method based on the derivation of the first modes associated to the undamped structure is established. This method constitutes a good representation of the original model with reduced CPU time making it a suitable choice for the reduction of sandwich structures incorporating viscoelastic materials. Finally, modal reduction in physical space method is outlined as a projection of modal basis in physical coordinates system. Thereby, the projection on physical space is realized leading to good results. However, this method needs an additional time compared to others reduction methods and requires also to verify the minimum conditionning number condition.

In all reduction procedure, the proposed methods were proving a good accuracy results and a satisfactory agreement with the full model in both frequency and time domains. In other hand, even the reduction ratio in term of systems size was not exceeding 50%; it was reaching 90% in term of CPU time which makes these methods a suitable choice to handle viscoelastic sandwich structures with an accurate and efficient way. Furthermore, the kernel of the idea to use model reduction methods in time domain can be explained by the temporel interest of the GHM model which allows transformation from a frequency rational shear modulus function to a temporal resolved second order equation which improves its importance notably in substructuring context for structures with local nonlinearities.

References:

[1] J. Salençon, Viscoélasticité, Presse des ponts et chaussés, Paris, 1983.

[2] J. D. Ferry, Viscoelastic properties of polymers, New York, John Wiley &Sons; 1980.

[3] F.J. Plantema, Sandwich Construction, New York, John Wiley & Sons, 1966.

[4] E. Reissner, Finite deflections of sandwich plates, AIAA Journal 15(7) (1948) 435-440.

[5] E. Reissner, The effect of transverse shear deformation on the bending elastic plates, Journal of Applied Mechanics 12 (1945) 69-77.

[6] R.D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates, Journal of Applied Mechanics 18 (1951) 31-38.

[7] E. Reissner, On the theory of bending of elastic plates, Journal of Maths and Physics 12 (1944)184-191.

[8] J.N. Reddy, A simple higher order theory for laminated composites plates, Journal of Applied Mechanics 51 (1948) 745-752.

[9] A.J.M .Ferreira, CMC. Roque, M. PALS, Analysis of composite plates using high order shear deformation theory and a finite point formulation based on the multiquadric radial basis function method, Composites Part B 34 (2003) 627-636.

[10] Y.M. Chugal, R.P. Shimpi, A review of Refined Shear Deformation Theories of Isotropic and Anisotropic Laminated Plates, Journal of Reinforced Plastics and Composites 21(9) (2002).

[11] J.N. Reddy, Mechanics of Laminated Composites Plates, Theory and Analysis; 2ed, CRC Press, Florida, 1997.

[12] M. Di Sciuva, Bending, vibration and buckling of simply supported thick multilayered orthotropic plates: An evaluation of a new displacement model. Journal of Sound and Vibration 105 (1986) 425-442.

[13] D.J. Mc-Tavish DJ, P.C. Hughes, Dynamics of viscoelastic structure-a time domain, finite element formulation, Journal of Applied Mechanics 52 (1985) 897-906.

[14] D.J. Mc-Tavish, P.C. Hughes, Modeling of linear viscoelastic space structures, Journal of Applied Mechanics 52 (1993) 103–113.

[15] O.C. Zienkiewciz, R.L. Taylor, The finite element method. Fifth edition. Volume 1: The basis, 2000.

[16]Y.T. Leung, An accurate method of dynamic condensation in structural analysis, International Journal of Numerical Methods in Engineering 14 (1979) 1241-1256.

[17]N. Petersmann, Calculation of eigenvalues using substructuring and dynamic condensation. Southampton: Proceedings of recent international conference in recent advance in structures dynamics, 1984, pp. 211-219.

[18] R.J. Guyan, Reduction of Stiffness and Mass Matrices, AIAA Journal 3(1965)380.

[19] B. Iron, Structural Eigenvalues Problem: Elimination of Unwanted Variables, AIAA Journal, 3(1965) 961-962.

[20] J. O'Callahan, A Procedure for an Improve Reduced System (IRS) Model. Las Vegas: 7th International Modal Analysis Conference, 1989, pp. 17-21.

[21] J. O'Callahan, System Equivalent Reduction Expansion Process (SEREP).Las Vegas: 7th International Modal Analysis Conference, 1989, pp. 27-29.

[22]J. O'Callahan, A Non Smoothing SEREP Process for Modal Expansion. Honolulu (Hawaii):12th International Modal Analysis Conference, 1994, pp. 232-238.

[23] M.I. Friswell, S.D. Garvey, J.E.T. Penny, The convergence of the iterated IRS method, Journal of Sound and Vibration 211(1998) 123-132.

[24] N. Bouhaddi, R. Fillod, Model reduction by a simplified variant of dynamic condensation, Journal of Sound and Vibration 19 (1996) 233-250.

[25] N. Bouhaddi N, R. Fillod, A method for selecting master dof in dynamic substructuring using the Guyan condensation method, Computers and Structures 45 (5/6) (1992) 941-946.

[26] C.H. Park, D.J. Inman, M.J. Lam, Model reduction of Viscoelastic finite element models, Journal of Sound and Vibration 219 (1999) 619-637.

[27] M. Meunier, R.A. Shenoi, Forced response of FRP sandwich panels with viscoelastic materials, Journal of Sound and Vibration 263(2003) 131–151.

[28] M.A. Trindade, A. Benjeddou, R. Ohayon, Modeling of frequency-dependent viscoelastic materials for active-passive vibration damping, Journal of Vibration and Acoustics 122(2) (2000) 169-174.

[29] A.M.G. De Lima, D.A. Rade, Model reduction methods as applied to viscoelastically damped finite element models, Catalao.ufg.br. Brasil, 2010.

[30] C.Y.K. Chee, Static shape control of Laminated Composite Plate smart structure using piezoelectric actuators, Phd Thesis of Sydney University: Department of Aeronautical Engineering, Australia, 2000.

[31] A.W. Faria, Finite element modelling of composites plates: Contribution to the damping, damage and incertitudes, Phd thesis of Uberlandia University, Brasil, 2010.

[32] G.A. Lesieutre, Finite element for dynamic modeling of uniaxial rods with frequency dependent material properties, International Journal of Solids and Structures 29(1992) 1567–1579.

[33] R.L. Bagley, P.J. Torvik, Fractional calculus- a different approach to the analysis of viscoelastically damped structures, AIAA Journal 21(5) (1983) 741–748.

[34] A.M.G. De Lima, A.R. Da Silva, D.A. Rade, N. Bouhaddi, Component mode synthesis combining robust enriched Ritz approach for viscoelastically damped systems, Engineering Structures 32 (2010)1479-1488.

[35] R.M. Christensen, The Theory of Linear Viscoelasticity, An introduction 2nd Ed. Academic Press, New York, 1982.

[36] Jr. R. R. Craig, M.C.C. Bampton, Coupling of substructures for dynamic analysis, AIAA Journal 6 (1968)1913-1919.

[37] N. Newmark, A method of computation for structural dynamics, Journal of the Engineering Mechanics Division 85(7) (1959) 67-94.

[38] F.M. Hemez, S.M. Doebling, From shock response spectrum to temporal moments and Vice-versa, Proceedings of the 21st SEM International Modal Analysis Conference, Kissimmee, 2003.

[39] G. Masson, Robust Modal synthesis adapted for the optimization of High order models, PhD thesis of Franche Comté University, France, 2003.

[40] Y. Gerges, Model reduction methods for non-linear Vibroacoustics, PhD thesis of Franche Comté University, France, 2013.

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Model reduction methods for viscoelastic sandwich structures in frequency and time domains

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Response to reviewer comment

<u>Reviewer 2#</u>

Q1: On page 29, line3:
…and the FRF response which is obtained by a matrix inversion at each frequency point.
Do you really need invert the matrix (General stiffness matrix) in order to calculate the responses? At each frequency point, solving the linear equations by elimination and back substitution (or equivalent algorithm) is more efficient than inverting the matrix and calculating product (matrix and vector).

Response:
The FRF responses are effectively computed by solving a linear system equations in the form Ax=b which is performed in Matlab® software using the cholesky factorization. In fact, the matrix A, symmetric and positive definite, is decomposed as $A=LL^T$ where L is a lower triangular matrix. Then, forward substitution Lz=b and back substitution $L^T x=z$, the desired solution x can be subsequently computed by solving the triangular linear system $L^T x=z$.
Hence, to clarify this idea, the following phrase was rewriting on page 29, line 3 as "and the FRF response which is obtained by solving linear equations at each frequency point".

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