Semi-decentralized Approximation of Optimal Control for Partial Differential Equations in Bounded Domains

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Abstract

We present a computational method for optimal control of linear distributed systems. Its derivation is based on the functional calculus of self-adjoint operators, and on the Dunford-Schwartz representation formula. It is has been devised so that to be implementable on very fine grained computing processors with semi-decentralized coordination. Finally, it is illustrated by an example related to vibration stabilization of a micro-cantilever array. *To cite this article: M. Lenczner, Y. Yakoubi, C. R. Mecanique 337 (2009).*

Résumé

Approximation Semi-décentralisée d'un Contrôle Optimal pour des Equations aux Dérivées Partielles dans un Domaine Borné. Nous présentons une méthode de calcul de contrôle optimal pour des systèmes distribués linéaires. Sa construction repose sur le calcul fonctionnel des opérateurs auto-adjoints et sur la formule de représentation de Dunford-Schwartz. Elle est conçue pour des architectures de calcul à très fine granularité avec coordination semi-décentralisée. Enfin, elle est illustrée par un exemple portant sur la stabilisation des vibrations dans une matrice de micro-cantilevers. *Pour citer cet article : M. Lenczner, Y. Yakoubi, C. R. Mecanique 337 (2009).*

Key words: Optimal Distributed Control ; Decentralized Control ; Multiscale Model ; Micro-cantilever Array.

Mots-clés : Contrôle Optimal Distribué ; Contrôle Décentralisé ; Modèle Multiéchelle ; Matrice de Micro-cantilevers.

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1. Introduction

The method presented in this note is motivated by the emerging field of arrays of microsystems like arrays of atomic force microscopes, micro-mirrors, or micro-membranes. They are, or will be, comprised of a very large number of units subjected to wanted or unwanted interactions (cross-talk effect). Achieving a global control in such system remains a challenging task. Due to computing power and data transmission bottlenecks, it is required to design, in a joint effort, an architecture and dedicated algorithms. With this paper, we propose a computational strategy dedicated to very fine-grained computing processors allowing semi-decentralized exchanges, i.e. between neighbors only. We refer to this concept by using the term *semi-decentralized* architecture or computing. The method is based on a general theory of optimal control for linear infinite dimensional systems, and is illustrated through an example of a two-scale model of micro-cantilever arrays.

Let us consider the Linear Quadratic Regulator (LQR) problem, where we denote by z the state variable and by u the control variable,

$$\begin{aligned} &\frac{dz}{dt}(t) = Az(t) + Bu(t) \text{ for } t > 0 \text{ and } z(0) = z_0, \\ &\min_{u \in U} J(z_0, u) = \min_{u \in U} \int_0^{+\infty} \|Cz\|_Y^2 + (Su, u)_U \, dt, \end{aligned} \tag{1}$$

In this note, we restrict this formulation to bounded input operators, and follow the mathematical setting developed in [1]. So, A is the infinitesimal generator of a continuous semigroup on a separable Hilbert space Z with dense domain $D(A)$, $B \in \mathcal{L}(U; Z)$, $C \in \mathcal{L}(Z; Y)$ and $S \in \mathcal{L}(U; U)$ where U and Y are two Hilbert spaces. We assume that (A, B) is stabilizable and that (A, C) is detectable, in the sense that there exist $K \in \mathcal{L}(Z; U)$ and $F \in \mathcal{L}(Y; Z)$ such that $A - BK$ and $A - FC$ are the infinitesimal generators of two uniformly exponentially stable continuous semigroups. For each $z_0 \in Z$, the LQR problem (1) admits a unique solution

$$u^* = -Kz,$$

where $K = S^{-1}B^*P$, and $P \in \mathcal{L}(Z)$ is the unique self-adjoint nonnegative solution of the operational Riccati equation

$$(A^*P + PA - PBS^{-1}B^*P + C^*C)z = 0, \tag{2}$$

for all $z \in D(A)$. The adjoint A^* of the unbounded operator A is defined from $D(A^*) \subset Z$ to Z by the equality $(A^*z, z')_Z = (z, Az')_Z$ for all $z \in D(A^*)$ and $z' \in D(A)$. The adjoint $B^* \in \mathcal{L}(Z; U)$ of the bounded operator B is defined by $(B^*z, u)_U = (z, Bu)_Z$, the adjoint $C^* \in \mathcal{L}(Y; Z)$ being defined similarly.

2. Example: A model of cantilever array

We refer to the two-scale model [2], of a large one-dimensional cantilever array, see Figure 1(a), derived through an homogenization method dedicated to strongly heterogeneous systems.
The homogenized model is build within three steps. First, a change of variable is introduced so that to formulate the full model in a two-scale referential comprised of micro and macro variables. Then, it is approximated in the sense of large number of cantilevers. Finally, it is mapped back onto the natural referential in which the actual system is described. The present control theory is developed on the model

resulting from the second step, so it is expressed in the two-scale referential, and a large but finite number of cantilevers is approximated by a distribution of an infinite number of cantilevers.

After a number of simplifications, the approximate model expressed in the two-scale referential, appears as posed in a rectangle $\Omega = (0, L_B) \times (0, L_C)$. The parameters L_B and L_C represent respectively the base length in the macroscale direction x, and the scaled cantilever length in the microscale variable y. The base is modelled by the line $\Gamma = \{(x, y) \mid x \in (0, L_B) \text{ and } y = 0\}$, and the rectangle Ω is filled by the distribution of cantilevers. We describe the system motion by its bending displacement only. The cantilevers are oriented in the y-direction, and their motion is governed by an infinite number of Euler-Bernoulli beam equations distributed along the x-direction. Each of them is subjected to a control force $u_C(t, x)$ taken independent of y for simplicity. This simplistic choice does not affect the method presented hereafter, so it can be replaced by any other realistic force distribution. Denoting by $w_C(t, x, y)$, ρ^C, ℓ_C and R^C the bending displacements, the mass per unit length, the width and the bending coefficient, the governing equation is

$$\rho^C \partial^2_{tt} w_C + R^C \partial^4_{y\cdots y} w_C = u_C \text{ in } \Omega,$$

endowed with the boundary conditions $w_C = w_B$, $\partial_y w_C = 0$ at $y = 0$ and $\partial^2_{yy} w_C = \partial^3_{yyy} w_C = 0$ at $y = L_C$ representing an end clamped in the base, and a free end. The base is governed by an Euler-Bernoulli beam equation with two kind of distributed forces, one exerted by the attached cantilevers and the other, denoted by $u_B(t, x)$, originates from an actuator distribution. The bending displacement, the mass per unit length and the bending coefficient being denoted by $w_B(t, x)$, ρ^B and R^B, the base governing equation states as

$$\rho^B \partial^2_{tt} w_B + R^B \partial^4_{x\cdots x} w_B = -\ell_{\mathbf{C}} R^C \partial^3_{yyy} w_C + u_B.$$

The base is assumed to be clamped, so the boundary conditions are $w_B = \partial_x w_B = 0$ at both ends. Finally, both equations are completed with initial conditions on displacements and velocities, $w_B = w_{B,0}$, $\partial_t w_B = w_{B,1}$, $w_C = w_{C,0}$, and $\partial_t w_C = w_{C,1}$. The LQR problem, corresponding to a vibration stabilization problem, is set for the control variables $(u_B, u_C) \in U = L^2(\Gamma)^2$ and for the cost functional

$$\mathcal{J}\left(w_{B,0}, w_{B,1}, w_{C,0}, w_{C,1}; u_B, u_C\right) = \int_0^\infty \left\|\partial^2_{xx} w_B\right\|^2_{L^2(\Gamma)} + \left\|\partial^2_{yy} w_C\right\|^2_{L^2(\Omega)} + \|u_B\|^2_{L^2(\Gamma)} + \|u_C\|^2_{L^2(\Gamma)}\, dt.$$

3. Matrices of functions of a self-adjoint operator

Since the approximation of K is based on the concept of matrices of functions of a self-adjoint operator, this section is devoted to their definition. Let Λ be a self-adjoint operator on a separable Hilbert space X with domain $D(\Lambda)$, we denote by $\sigma(\Lambda)$ its spectrum and by $I_\sigma = (\sigma_{\min}, \sigma_{\max})$ an open interval that includes $\sigma(\Lambda)$.

We recall that if Λ is compact then $\sigma(\Lambda)$ is bounded and is constituted of real eigenvalues λ_k. They are the solutions of the eigenvalue problem $\Lambda\phi_k = \lambda_k\phi_k$, where ϕ_k is an eigenvector associated to λ_k chosen normed in X, i.e. such that $||\phi_k||_X = 1$. For a given real valued function f, continuous on I_σ, $f(\Lambda)$ is the linear self-adjoint operator on X defined by $f(\Lambda)z = \sum_k f(\lambda_k) z_k \phi_k$ where $z_k = (z, \phi_k)_X$, with domain $D\left(f\left(\Lambda\right)\right) = \{z \in X | \sum_k |f(\lambda_k) z_k|^2 < \infty\}$. Then, if f is a $n_1 \times n_2$ matrix of real valued functions f_{ij}, continuous on I_σ, $f(\Lambda)$ is a matrix of linear operators $f_{ij}(\Lambda)$ with domain $D\left(f\left(\Lambda\right)\right) = \{z \in X^{n_2} | \sum_k \sum_{j=1}^{n_2} \left|f_{ij}\left(\lambda_k\right)\left(z_j\right)_k\right|^2 < \infty \forall i = 1..n_1\}$.

In the general case, where Λ is not compact and where f is a continuous function, the self-adjoint operator $f(\Lambda)$ is defined on X by the Stieltjes integral $f(\Lambda) = \int_{-\infty}^{+\infty} f(\lambda)\, dE_\lambda$, and its domain is $D\left(f\left(\Lambda\right)\right) = \{z \in X | \int_{-\infty}^{+\infty} |f(\lambda)|^2 d||E_\lambda z||_X^2 < \infty\}$ where E_λ is the spectral family associated to Λ, see [3]. When f is a matrix, $f(\Lambda)$ is a matrix of linear operators with entries defined by the above formula and with domain $D(f(\Lambda)) = \{z \in X^{n_2} | \int_{-\infty}^{+\infty} \sum_{j=1}^{n_2} |f_{ij}(\lambda)|^2 d||E_\lambda z_j||_X^2 < \infty \forall i = 1..n_1\}$.

4. Factorization of K by a Matrix of Functions of Λ

The semi-decentralized control law derivation starts with factoring K under the form of a product of a function of Λ with two operators admitting a natural semi-decentralized approximation. To do so, we introduce the following assumptions.

There exist three integers n_Z, n_U and $n_Y \in \mathbb{N}^*$, three isomorphisms $\Phi_Z \in \mathcal{L}(X^{n_Z}; Z)$, $\Phi_U \in \mathcal{L}(X^{n_U}; U)$ and $\Phi_Y \in \mathcal{L}(X^{n_Y}; Y)$ and four matrices of functions $a(\lambda) \in \mathbb{R}^{n_Z \times n_Z}$, $b(\lambda) \in \mathbb{R}^{n_Z \times n_U}$, $c(\lambda) \in \mathbb{R}^{n_Y \times n_Z}$ and $s(\lambda) \in \mathbb{R}^{n_U \times n_U}$ continuous on I_σ such that $A = \Phi_Z a(\Lambda)\Phi_Z^{-1}$, $B = \Phi_Z b(\Lambda)\Phi_U^{-1}$, $C = \Phi_Y c(\Lambda)\Phi_Z^{-1}$ and $S = \Phi_U s(\Lambda)\Phi_U^{-1}$. We also impose that Z, U and Y are endowed with the inner products $(z, z')_Z = \left(\Phi_Z^{-1} z, \Phi_Z^{-1} z'\right)_{X^{n_Z}}$, $(u, u')_U = \left(\Phi_U^{-1} u, \Phi_U^{-1} u'\right)_{X^{n_U}}$, and $(y, y')_Y = \left(\Phi_Y^{-1} y, \Phi_Y^{-1} y'\right)_{X^{n_Y}}$.

Proposition 1 : *Under the above assumptions, the controller K admits the factorization*

$$K = \Phi_U k\left(\Lambda\right) \Phi_Z^{-1},$$

where $k\left(\lambda\right) = s^{-1}\left(\lambda\right) b^T\left(\lambda\right) p\left(\lambda\right)$, and where for all $\lambda \in \sigma(\Lambda)$, $p(\lambda)$ is the unique self-adjoint nonnegative matrix solving the algebraic Riccati equation

$$a^T\left(\lambda\right) p + p a\left(\lambda\right) - p b\left(\lambda\right) s^{-1}\left(\lambda\right) b^T\left(\lambda\right) p + c^T\left(\lambda\right) c\left(\lambda\right) = 0.$$

Sketch of the proof The algebraic Riccati equation can be found after replacing A, B, C and S by their decomposition in the Riccatti equation (2).

5. Approximation of $k(\Lambda)$

We present an approximation, when I_σ is bounded, of $k(\Lambda)$ by an other function of Λ which is easily discretized and implemented in a semi-decentralized architecture. The strategy must be general, and in the same time the approximation must be accurate. A simple choice would be to adopt a polynomial or a rational approximation, but their discretization would yield high errors when discretizing high powers of Λ. This is avoided when using the Dunford-Schwartz formula, see [4], representing a function of an operator. Indeed, it involves only the operator $(\zeta I - \Lambda)^{-1}$ which may be simply, and accurately approximated. However, this formula requires the function to be holomorphic in an open vicinity of $\sigma(\Lambda)$. Since the explicit expression of the function is generally unknown, this set cannot be easily determined. So the function is replaced by a highly accurate rational approximation, and the Dunford-Schwarz formula can be applied with a path surrounding I_σ but no pole. We implicitly assume that the poles of the rational approximation are far from I_σ. Let $k_N = (k_{Nij})_{i,j}$ be a matrix of rational approximations of the entries k_{ij} over the bounded interval I_σ,

$$k_N\left(\lambda\right) = \sum_{m=0}^{N^N} d_m \lambda^m / \sum_{m'=0}^{N^D} d'_{m'} \lambda^{m'},$$

where d_m, $d'_{m'}$ are matrices of coefficients, and $N = \left(N^N, N^D\right)$ is the couple comprised of the matrix N^N of numerator polynomial degrees, and of the matrix N^D of denominator polynomial degrees. The path $\mathcal{C}$, in the Dunford-Schwartz formula, $k_N(\Lambda) = \frac{1}{2i\pi}\int_{\mathcal{C}} k_N(\zeta)\left(\zeta I - \Lambda\right)^{-1} d\zeta$, is chosen to be an ellipse parameterized by $\zeta(\theta) = \zeta_1(\theta) + i\zeta_2(\theta)$, with $\theta \in [0, 2\pi]$. The parametrization is used as a change of variable, so the integral is rewritten on the form $I(g) = \int_0^{2\pi} g(\theta)\, d\theta$, and may be approximated by a quadrature formula $I_M(g) = \sum_{l=1}^{M} g(\theta_l)\, w_l$ involving M nodes $(\theta_l)_{l=1,..,M} \in [0, 2\pi]$ and M weights $(w_l)_{l=1,..,M}$. For each $z \in X^{n_Z}$ and $\zeta \in \mathcal{C}$, we introduce the n_Z-vector field $v^\zeta = -i\zeta' k_N(\zeta)\left(\zeta I - \Lambda\right)^{-1} z$. Decomposing v^ζ into its real part v_1^ζ and its imaginary part v_2^ζ, the couple (v_1^ζ, v_2^ζ) is solution of the system

$$\begin{cases} \zeta_1 v_1^\zeta - \zeta_2 v_2^\zeta - \Lambda v_1^\zeta = Re\left(-i\zeta' k_N(\zeta)\right) z, \\ \zeta_2 v_1^\zeta + \zeta_1 v_2^\zeta - \Lambda v_2^\zeta = Im\left(-i\zeta' k_N(\zeta)\right) z. \end{cases} \tag{3}$$

Thus, combining the rational approximation k_N and the quadrature formula yields an approximate realization $k_{N,M}(\Lambda)\, z = \frac{1}{2\pi}\sum_{l=1}^{M} v_1^{\zeta(\theta_l)} w_l$ of $k(\Lambda)\, z$.

Remark: For real-time realization, computing $k_{N,M}(\Lambda)\, z$ requires solving M systems like (3) corresponding to the M nodes $\zeta(\theta_l)$. The matrices $k_N(\zeta(\theta_l))$ can be computed off-line once and for all, and stored in memory, so their determination does not penalize a rapid real-time computation. In short, the only parameter governing accuracy in a real-time computation, apart from spatial discretization discussed in next Section, is the number M of quadrature points.

6. Spatial Discretization

The method is not complete until Λ^{-1} has been discretized by an operator Λ_h^{-1} yielding a spatial discretization of the Equation (3)

$$\begin{cases} \zeta_1 v_{1,h}^\zeta - \zeta_2 v_{2,h}^\zeta - \Lambda_h v_{1,h}^\zeta = Re\left(-i\zeta' k_N(\zeta)\right) z_h \\ \zeta_2 v_{1,h}^\zeta + \zeta_1 v_{2,h}^\zeta - \Lambda_h v_{2,h}^\zeta = Im\left(-i\zeta' k_N(\zeta)\right) z_h, \end{cases}$$

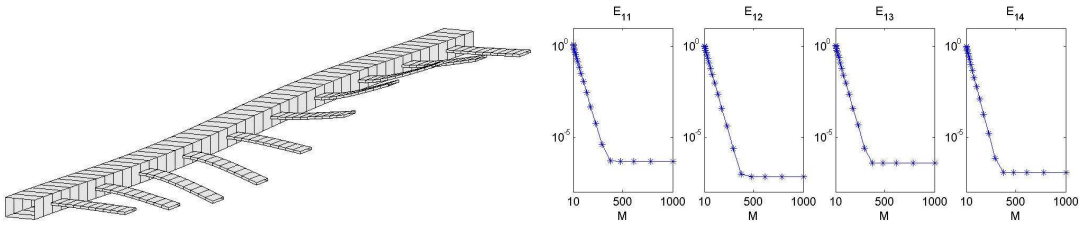
and the final semi-decentralized approximation $k_{N,M,h} z_h = \frac{1}{2\pi}\sum_{l=1}^{M} v_{1,h}^{\zeta_l} w_l$ of the realization $k(\Lambda)z$.

7. Application

We set $\bar{w}_C = w_C - w_B$ and introduce the basis of normalized eigenfunctions $(\psi_k)_k$ solutions of the eigenvalue problem $\partial^4_{y\cdots y}\psi = \lambda^C\psi$ in $(0, L_C)$ with boundary conditions $\psi(0) = \partial_y\psi(0) = 0$, $\partial^2_{yy}\psi(L_C) = \partial^3_{yyy}\psi(L_C) = 0$, and the normality condition $\|\psi_k\|_{L^2(0,L_C)} = 1$. In practical applications, a very small number of cantilever modes is sufficient to describe properly the system. We take into account only the first one, keeping in mind that the method can handle more than one mode. Therefore, we adopt the approximation $\bar{w}_C(t,x,y) \simeq \bar{w}_C^1(t,x)\,\psi_1(y)$, where $\bar{w}_C^1$ is the coefficient of the first mode ψ_1 in the modal decomposition of $\bar{w}_C$. Introducing $\bar{\psi}_1 = \int_0^{L_C}\psi_1 dy$, $u_C^1 = \int_0^{L_C} u_C\psi_1 dy$, $\widetilde{w}_C = \bar{w}_C^1 + \bar{\psi}_1 w_B$ and $c_1 = \partial_y^3\psi_1(0)$, the couple $(w_B, \widetilde{w}_C)$ is solution of the system of equations posed on Γ,

$$\begin{cases} \rho^B\partial^2_{tt} w_B + R^B\partial^4_{x\cdots x} w_B + \ell_C R^C c_1 \widetilde{w}_C = u_B \text{ in } \Gamma, \\ \rho^C\partial^2_{tt}\widetilde{w}_C + R^C\lambda_1^C\widetilde{w}_C - R^C\lambda_1^C\bar{\psi}_1 w_B = u_C^1 \quad \text{in } \Gamma, \end{cases}$$

with the boundary conditions $w_B = \partial_x w_B = 0$ at both ends. The cost functional is simplified accordingly, $\mathcal{J} \simeq \int_0^\infty \left\|\partial_{xx}^2 w_B(t,x)\right\|_{L^2(\Gamma)}^2 + \left\|\lambda_1^C \widetilde{w}_C(t,x)\right\|_{L^2(\Gamma)}^2 + \|u_B\|_{L^2(\Gamma)}^2 + \left\|u_C^1\right\|_{L^2(\Gamma)}^2 dt$. We set $z^T = \left[\, w_B \;\; \widetilde{w}_C \;\; \partial_t w_B \;\; \partial_t \widetilde{w}_C \,\right]$, $u^T = \left[\, u_B \;\; u_C^1 \,\right]$, $A = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -R^B \partial^4_{x\cdots x}/\rho^B & -\ell_C R^C c_1/\rho^B & 0 & 0 \\ R^C \lambda_1^C \tilde{\psi}_1/\rho^C & -R^C \lambda_1^C/\rho^C & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I/\rho^B & 0 \\ 0 & I/\rho^C \end{bmatrix}$, $C = \begin{bmatrix} \partial_{xx}^2 & 0 & 0 & 0 \\ 0 & \lambda_1^C I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $S = I$. Here, A is the infinitesimal generator of a continuous semigroup on the separable Hilbert space $Z = H_0^2(\Gamma) \times L^2(\Gamma)^3$ with dense domain $D(A) = H^4(\Gamma) \cap H_0^2(\Gamma) \times L^2(\Gamma) \times H_0^2(\Gamma) \times L^2(\Gamma)$. It is known that $B \in \mathcal{L}(U;Z)$, $C \in \mathcal{L}(Z;Y)$, and $S \in \mathcal{L}(U;U)$, where $Y = L^2(\Gamma)^4$. We also know that (A,B) is stabilizable and that (A,C) is detectable. For the isomorphisms, we choose $\Phi_Z = \begin{bmatrix} \Lambda^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, $\Phi_U = I$, and $\Phi_Y = \begin{bmatrix} \partial_{xx}^2 \Lambda^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ which yields $a(\lambda) = \begin{bmatrix} 0 & 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & 0 & 1 \\ -R^B/\rho^B \lambda^{-1/2} & -\ell_C R^C c_1/\rho^B & 0 & 0 \\ R^C \lambda_1^C \tilde{\psi}_1/\rho^C \lambda^{1/2} & -R^C \lambda_1^C/\rho^C & 0 & 0 \end{bmatrix}$, $b(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I/\rho^B & 0 \\ 0 & I/\rho^C \end{bmatrix}$, $c(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1^C & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $s(\lambda) = 1$. In a numerical experiment, we have discretized $\Lambda^{-1} = \partial_{xxxx}^4$ by a finite differences scheme and have set all coefficients R^B, ρ^B, ℓ_C, R^C, ρ^C, L_C to one, and L_B to 4.73. Thus, all eigenvalues of Λ turn to be included in $(0,1)$. It is observed that the functions $k_{ij}(\lambda)$ are singular at 0, so their rational approximation has been build on the interval $J_\sigma = \left(10^{-2}, 1\right)$ at a precision of 10^{-7}. This is equivalent to truncate high frequencies. Numerical integrations have been performed with a standard trapezoidal quadrature rule. Four relative errors $E_{ij} = ||k_{ij,N,M} - k_{ij}||_{L^2(J_\sigma)} / ||k_{ij}||_{L^2(J_\sigma)}$, between the exact functions and their final approximation, are reported in Figure 1(b), in logarithmic scale, where M varies from 10 to 10^3. The errors decrease exponentially until some limits corresponding to the limited precision of the rational approximations.



(a) (b)

Figure 1. (a) Array of Cantilevers. (b) Errors between k and $k_{N,M}$.

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